ERGODIC THEOREMS

By U. KRENGEL (*with a supplement by* A. BRUNEL): pp. viii+357. Walter de Gruyter, Berlin and New York, 1985. ISBN 3-11-008478-3. (≃US$42.70)

Imagine a review article that surveyed all of the literature on ergodic theorems produced since the original papers of von Neumann and Birkhoff in 1931-32. There have been several survey articles on various parts of ergodic theory, but to my knowledge such a comprehensive description of this subject, in its current rich and advanced stage of development, has not been attempted. Imagine further that the article not only mentioned most of the relevant work, but went on to put it into context; to relate different results to one another; in some cases to judge the relative importance of results; to sketch the historical development of particular lines of investigation; to give proofs of the most interesting and important theorems and to sketch the ideas behind the proofs of most of the others; to indicate which questions seem promising for research and which others may be overworked or unappealing at this time. Surely such a task would be beyond a review article, or even a book, or series of books; yet, amazingly enough, much of this useful job has been accomplished by the book under review in a comfortable span of 357 pages.

Of course an actual book cannot hope to achieve such an impossible goal, and in order even to begin to approach it, it must enlist the aid of its readers. First of all, the book is written at a rather sophisticated level. The reader will need not only background in measure theory and functional analysis, but also a taste for the abstract, a good library, and perhaps a few knowledgeable friends. Many proofs are not really carried out in detail, but rather the reader is told how to do them (sometimes only with the help of a few code words, like 'approximation argument'), and even the ones that are produced are very concise. Motivation for the many abstractions and generalizations that appear is usually hard to come by, although occasionally there are some extremely interesting examples and applications, as for subadditive ergodic theorems (to percolation and the ranges of random walks, p. 39), and multiparameter processes (to random lines in the plane and random graphs, p. 202). Many fancy topics (Bochner integrals, von Neumann algebras) appear and disappear extremely rapidly. Intriguing terms ('superharmonic function', 'potential', 'state', ...) are introduced, but whether or not the associated mathematical objects have any relation to our usual ideas about such terms is never discussed. Some of the basic ideas of ergodic theory are buried in the centres of proofs or discussions of other topics, rather than being set out in plain view. This book is not a work of pedagogy intended for the novice; it would not serve as a textbook for an introductory course.

Someone with adequate background, however, will find it easy to learn the essentials of any one of a large number of topics quickly and efficiently just by studying the appropriate part of this book. Moreover, with careful reading he can discover the significance of particular results, their role in the development of the subject, and the need for particular hypotheses, and can follow the development of a single theorem through time and through different contexts. For example, on

p. 197 the function spaces introduced by Fava arise naturally as part of a continuing story, whereas encountered alone they might seem curious or artificial. It is possible to follow a single convergence idea, such as the local ergodic theorem, through a variety of metamorphoses, seeing how it extends or adapts (as new hypotheses are needed or the conclusion changes) when faced with continuous or multiple parameters, vector values, different underlying operators on different kinds of Banach spaces, and so on. Often when we arrive in new territory the author mentions or even quickly constructs the defining counterexamples.

The author's emphasis tends towards the probabilistic and functional-analytic. Thus while all of the standard ergodic theorems appear, the heart of the book deals with the ergodic theory of operators on Banach spaces of functions. There is not much more to say about the Chacon–Ornstein and Akcoglu theorems and all the related work on the ergodic theory of operators on L^p than appears here. The book is true to its name, in that it is devoted to the study of convergence results in themselves rather than to the various properties and examples of systems in which they can be applied. Subadditive ergodic theorems (including Oseledec's theorem), mean ergodic theory, continuous-parameter, multiparameter, and even amenable group actions, and subsequence and weighted convergence theorems all receive detailed treatments. Many other topics, such as Furstenberg's multiple recurrence theory and the theory of entropy, are sketched. A final section includes pointers to farther-out topics like set-valued stationary processes and convergence of non-linear combinations of $f, fT, \ldots, fT^{n-1}$. An appendix by Antoine Brunel presents an elegant exposition of the theory of Harris operators, through the Zero-Two Law.

This book provides an overview of the subject in the unique way that can be accomplished only by scholarship that is both thoughtful and painstaking; now that all of these results have been assembled and organized in this systematic way, we have available a valuable tool for the further advancement of the subject. Now one can more easily see the relationships among various types of questions and different approaches to them. It becomes possible to judge which methods have been superseded for certain purposes and which may be adapted for new purposes. Browsing in this book does not just produce lots of new information (and it is astoundingly full of information – you will find out about many interesting circles of ideas that you've never heard of before; the bibliography lists over one thousand items) – but it is also a rich source of new ideas. For as you consider the relationships of ideas independently of time, as is now possible, new problems suggest themselves. The introductory remarks to sections and subsections and the detailed notes (describing the historical development, mentioning further results, or occasionally pointing out errors and omissions in the literature) contribute significantly to the development of this wide and deep perspective. Any mathematician whose work impinges on ergodic theory should keep this book handy both for reference and for inspiration.

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