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The Saddle-Point Method and the Li Coefficients

Kamel Mazhouda

Abstract. In this paper, we apply the saddle-point method in conjunction with the theory of the Nörlund–Rice integrals to derive precise asymptotic formula for the generalized Li coefficients established by Omar and Mazhouda. Actually, for any function F in the Selberg class $\mathcal{S}$ and under the Generalized Riemann Hypothesis, we have

$$\lambda_F(n) = \frac{d_F}{2} n \log n + c_F n + O(\sqrt{n} \log n),$$

with

$$c_F = \frac{d_F}{2}(\gamma - 1) + \frac{1}{2}\log(\lambda Q_F^2), \quad \lambda = \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$

where γ is the Euler's constant and the notation is as below.

1 Introduction

Let us consider the xi-function $\xi(s) = s(s-1)\Gamma(s/2)\pi^{-s/2}\zeta(s)$ and the Li coefficients $(\lambda_n)_{n\geq 1}$ defined by

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} \Big[s^{n-1} \log \xi(s) \Big]_{s=1}.$$

Then the Li criterion says that the Riemann Hypothesis holds if and only if the coefficients (λ_n) are positive numbers. Bombieri and Lagarias [2] obtained an arithmetic expression for the Li coefficients λ_n and gave an asymptotic formula as $n \to \infty$. More recently, Maslanka [10] computed λ_n for $1 \leq n \leq 3300$ and empirically studied the growth behavior of the Li coefficients. Coffey [3, 4] studied the arithmetic formula and established a lower bound for the Archimedean prime contribution by means of series rearrangements using the Euler-Maclaurin summation. In [11], a generalization of the Li criterion for functions F in the Selberg class was given, and in [13] an explicit formula for the Li coefficients associated to F was established.

The object of this paper is to derive a precise asymptotic formula for the generalized Li coefficients using the saddle-point method.

The Selberg class $\mathcal{S}$ consists of Dirichlet series

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}, \quad \Re(s) > 1$$

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satisfying the following hypotheses.

- **Analytic continuation:** there exists a non-negative integer m such that $(s-1)^m F(s)$ is an entire function of finite order. We denote by m_F the smallest integer m that satisfies this condition.
- **Functional equation:** for $1 \le j \le r$, there are positive real numbers $Q_F,\ \lambda_j$ and there are complex numbers $\mu_j,\ \omega$ with $\Re(\mu_j) \ge 0$ and $|\omega| = 1$, such that

  $$\phi_F(s) = \omega\overline{\phi_F(1-\bar{s})}$$

  where
  $$\phi_F(s) = F(s)Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j).$$
- **Ramanujan hypothesis:** $a(n) = O(n^\epsilon)$.
- **Euler product:** $F(s)$ satisfies

  $$F(s) = \prod_p \exp\Big(\sum_{k=1}^{+\infty} \frac{b(p^k)}{p^{ks}}\Big)$$

  with suitable coefficients $b(p^k)$ satisfying $b(p^k) = O(p^{k\theta})$ for some $\theta < \frac{1}{2}$.

It is expected that for every function in the Selberg class the analogue of the Riemann hypothesis holds, *i.e,* that all non trivial (non-real) zeros lie on the critical line $\Re(s) = \frac{1}{2}$. The degree of $F \in \mathcal{S}$ is defined by

$$d_F = 2\sum_{j=1}^{r} \lambda_j.$$

The degree is well defined (although the functional equation is not unique by Legendre's duplication formula). The logarithmic derivative of $F(s)$ also has the Dirichlet series expression

$$-\frac{F'}{F}(s) = \sum_{n=1}^{+\infty} \Lambda_F(n)n^{-s}, \quad \Re(s) > 1,$$

where $\Lambda_F(n) = b(n)\log n$ is an analogue of the Von Mongoltd function $\Lambda(n)$ defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } k \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $N_F(T)$ counts the number of zeros of $F(s) \in \mathcal{S}$ in the rectangle $0 \le \Re(s) \le 1$, $|\Im(s)| \le T$ (according to multiplicities), one can show by standard contour integration the formula

$$N_F(T) = \frac{d_F}{2\pi} T\log T + c_F T + O(\log T),$$

in analogy to the Riemann–Von Mangoldt formula for Riemann's zeta-function $\zeta(s)$, the prototype of an element in $\mathcal{S}$. For more details concerning the Selberg class we refer to the survey of Kaczorowski and Perelli [6].

2 The Li Criterion

Let F be a function in the Selberg class non-vanishing at $s = 1$ and let us define the xi-function $\xi_F(s)$ by $\xi_F(s) = s^{m_F}(s-1)^{m_F}\phi_F(s)$. The function $\xi_F(s)$ satisfies the functional equation $\xi_F(s) = \omega\overline{\xi_F(1-\bar{s})}$. The function ξ_F is an entire function of order 1. Therefore by the Hadamard product, it can be written as

$$\xi_F(s) = \xi_F(0)\prod_{\rho}\Big(1 - \frac{s}{\rho}\Big),$$

where the product is over all zeros of $\xi_F(s)$ in the order given by $|\Im(\rho)| < T$ for $T \to \infty$. Let $\lambda_F(n)$, $n \in \mathbb{Z}$, be a sequence of numbers defined by a sum over the non-trivial zeros of $F(s)$ as

$$\lambda_F(n) = \sum_{\rho}\Big[1 - \Big(1 - \frac{1}{\rho}\Big)^n\Big],$$

where the sum over ρ is

$$\sum_{\rho} = \lim_{T\mapsto\infty}\sum_{|\Im\rho|\leq T}.$$

These coefficients are expressible in terms of power-series coefficients of functions constructed from the ξ_F-function. For $n \leq -1$, the Li coefficients $\lambda_F(n)$ correspond to the following Taylor expansion at the point $s = 1$

$$\frac{d}{dz}\log\xi_F\left(\frac{1}{1-z}\right) = \sum_{n=0}^{+\infty}\lambda_F(-n-1)z^n,$$

and for $n \geq 1$, they correspond to the Taylor expansion at $s = 0$

$$\frac{d}{dz}\log\xi_F\Big(\frac{-z}{1-z}\Big) = \sum_{n=0}^{+\infty}\lambda_F(n+1)z^n.$$

Let Z be the multi-set of zeros of $\xi_F(s)$ (counted with multiplicity). The multi-set Z is invariant under the map $\rho \longmapsto 1 - \bar{\rho}$. We have

$$1 - \Big(1 - \frac{1}{\rho}\Big)^{-n} = 1 - \Big(\frac{\rho-1}{\rho}\Big)^{-n} = 1 - \Big(\frac{-\rho}{1-\rho}\Big)^n = 1 - \overline{\Big(1 - \frac{1}{1-\bar{\rho}}\Big)^n}$$

and this gives the symmetry $\lambda_F(-n) = \overline{\lambda_F(n)}$. Using the corollary in [2, Theorem 1], we get the following generalization of the Li criterion for the Riemann hypothesis.

Theorem 2.1 *Let $F(s)$ be a function in the Selberg class $\mathcal{S}$ non-vanishing at $s = 1$. All non-trivial zeros of $F(s)$ lie in the line $\Re e(s) = 1/2$ if and only if $\Re(\lambda_F(n)) > 0$ for $n = 1, 2, \dots$.*

Next, we recall the following explicit formula for the coefficients $\lambda_F(n)$. Let consider the following hypothesis:

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$\mathcal{H}$: there exists a constant $c > 0$ such that $F(s)$ is non-vanishing in the region:

$$\Big\{ s = \sigma + it;\ \sigma \geq 1 - \frac{c}{\log(Q_F + 1 + |t|)} \Big\}.$$

Theorem 2.2 *Let $F(s)$ be a function in the Selberg class $\mathcal{S}$ satisfying $\mathcal{H}$. Then we have*

$$(2.1)\ \lambda_F(-n) = m_F + n\Big(\log Q_F - \frac{d_F}{2}\gamma\Big)$$
$$- \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \lim_{X \longrightarrow +\infty} \Big\{ \sum_{k \leq X} \frac{\Lambda_F(k)}{k} (\log k)^{l-1} - \frac{m_F}{l} (\log X)^l \Big\}$$
$$+ n \sum_{j=1}^{r} \lambda_j \Big(- \frac{1}{\lambda_j + \mu_j} + \sum_{l=1}^{+\infty} \frac{\lambda_j + \mu_j}{l(l + \lambda_j + \mu_j)} \Big)$$
$$- \sum_{j=1}^{r} \sum_{k=2}^{n} \binom{n}{k} (-\lambda_j)^k \sum_{l=0}^{+\infty} \Big(\frac{1}{l + \lambda_j + \mu_j} \Big)^k,$$

where γ is the Euler constant.

Examples

- In the case of the Riemann zeta function, $m_\zeta = 1$, $Q_\zeta = \pi^{-1/2}$, $r = 1$, $\lambda_1 = \frac{1}{2}$, and $\mu_1 = 0$. With the equality

$$(-1)^k \sum_{l=0}^{+\infty} \Big(\frac{1}{2l + 1} \Big)^k = (-1)^k \Big(1 - \frac{1}{2^k} \Big) \zeta(k),$$

  we find λ_ζ, which was established by Bombieri and Lagarias [2, p. 281].
- For the Hecke L-functions, $Q_F = \frac{\sqrt{N}}{2\pi}$, $m_F = 0$, $\lambda_1 = 1$, and $\mu_1 = \frac{1}{2}$, we find $\lambda_E(n)$, which was established by X.-J. Li [9, p. 496].

3 Saddle-Point Method and the Nörlund–Rice Integrals

Given a complex integral with a contour traversing simple saddle-point, the saddle-point corresponds locally to a maximum of the integrand along the path. It is then natural to expect that a small neighborhood of the saddle-point might provide the dominant contribution to the integral. The saddle-point method is applicable precisely when this is the case and when this dominant contribution can be estimated by means of local expansions. The method then constitutes the complex-analytic counterpart of Laplace's method for evaluating real integrals depending on a large parameter, and we can regard it as being

$$\text{Saddle-point method} = \text{Choice of contour} + \text{Laplace's method}.$$

To estimate $\int_A^B F(z)dz$, it is convenient to set $F(z) = e^{f(z)}$, where $f(z) \equiv f_n(z)$, involves some large parameter n. We chose a contour $\mathcal{C}$ through a saddle-point η such

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that $f'(\eta) = 0$. Next, we split the contour as $\mathcal{C} = \mathcal{C}^{(0)} \cup \mathcal{C}^{(1)}$, and the following conditions are to be verified.

(i) On the contour $\mathcal{C}^{(1)}$ the tails integral $\int_{\mathcal{C}^{(1)}}$ is negligible

$$\int_{\mathcal{C}^{(1)}} F(z)dz = O\Big(\int_{\mathcal{C}} F(z)dz\Big).$$

(ii) Along $\mathcal{C}^{(0)}$, a quadratic expansion,

$$f(z) = f(\eta) + \tfrac{1}{2} f''(\eta)(z-\eta)^2 + O(\phi_n)$$

is valid, with $\phi_n \to 0$ as $n \to \infty$, uniformly with respect to $z \in \mathcal{C}^{(0)}$.

(iii) The incomplete Gaussian integral taken over the central range is asymptotically equivalent to a complete Gaussian integral with ($\epsilon = \pm 1$):

$$\int_{\mathcal{C}^{(0)}} e^{\frac{1}{2} f''(\eta)(z-\eta)^2} dz \sim \epsilon i \int_{-\infty}^{+\infty} e^{-|f''(\eta)|\frac{x^2}{2}} dx \equiv \epsilon i \sqrt{\frac{2\pi}{|f''(\eta)|}}.$$

Assuming (i), (ii), and (iii), one has, with $\epsilon = \pm 1$

$$\frac{1}{2\pi} \int_A^B e^{f(z)} dz \sim \epsilon \frac{e^{f(\eta)}}{\sqrt{2\pi f''(\eta)}}.$$

This method is the main tool to prove our result. We finish this section by reviewing the definition of the Nörlund–Rice integral.

Lemma 3.1 *Let $f(s)$ be holomorphic in the half-plane $\Re(s) \geq \eta_0 - \frac{1}{2}$. Then the finite differences of the sequence $(f(k))$ admit the integral representation*

$$\sum_{k=n_0}^{n} \binom{n}{k} (-1)^k f(k) = \frac{(-1)^n}{2i\pi} \int_{\mathcal{C}} f(s) \frac{n!}{s(s-1)\cdots(s-n)} ds,$$

where the contour of integration $\mathcal{C}$ encircles the integers $\{n_0, \dots, n\}$ in a positive direction and is contained in $\Re(s) \geq \eta_0 - \frac{1}{2}$.

Proof The integral on the right is the sum of its residues at $s = n_0, \dots, n$, which precisely equals the sum on the left. ■

4 Asymptotic Formula for the Li Coefficients

A natural problem is to determine the asymptotic behavior of the numbers $\lambda_F(n)$. Our main result in this paper is stated in the following theorem.

Theorem 4.1 *Let $F(s)$ be a function in the Selberg class $\mathcal{S}$. Then, under the Generalized Riemann Hypothesis, we have*

$$\lambda_F(n) = \frac{d_F}{2} n \log n + c_F n + O\big(\sqrt{n} \log n\big),$$

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where

$$c_F = \frac{d_F}{2}(\gamma - 1) + \frac{1}{2}\log(\lambda Q_F^2), \quad \lambda = \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$$

and γ *is the Euler constant.*

Remark 4.2 We conjecture that the asymptotic formula for the numbers $\lambda_F(n)$ in Theorem 4.1 holds for any function in the Selberg class without any assumption.

For our purpose, it is sufficient to study sums of the form

$$H_n(m,k) = \sum_{l=2}^{n} (-1)^l \binom{n}{l} \frac{\zeta(l, \frac{m}{k})}{k^l}, \tag{4.1}$$

where $\zeta(s,q)$ is the Hurwitz zeta function given by

$$\zeta(s,q) = \sum_{n=0}^{+\infty} \frac{1}{(n+q)^s}.$$

Proposition 4.3 $H_n(m,k)$*, defined by* (4.1)*, satisfy the estimate*

$$H_n(m,k) = \Big(\frac{m}{k} - \frac{1}{2}\Big) - \frac{n}{k}\Big(\psi\Big(\frac{m}{k}\Big) + \log k + 1 - h_{n-1}\Big) + a_n(m,k),$$

where the $a_n(m,k)$ *are exponentially small:*

$$a_n(m,k) =$$
$$\frac{1}{k}\Big(\frac{2n}{\pi k}\Big)^{1/4} \exp\Big(-\sqrt{\frac{4\pi n}{k}}\Big) \cos\Big(\sqrt{\frac{4\pi n}{k}} - \frac{5\pi}{8} - \frac{2\pi m}{k}\Big) + O\Big(n^{-1/4} e^{-2\sqrt{\frac{\pi n}{k}}}\Big).$$

Here, $h_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ *is a harmonic number, and* $\psi(x)$ *is the logarithm derivative of the Gamma function.*

Proof Convert the sum to the Nörlund–Rice integral, and extend the contour to the half-circle at positive infinity. The half-circle does not contribute to the integral. One obtains

$$H_n(m,k) = \frac{(-1)^n}{2i\pi} n! \int_{3/2-i\infty}^{3/2+i\infty} \frac{\zeta(s, \frac{m}{k})}{k^l s(s-1)\cdots(s-n)} ds.$$

Moving the integral to the left, one encounters a single pole at $s = 0$ and a pole at $s = 1$. The residue of the pole at $s = 0$ is

$$Res(s=0) = \zeta(0, \frac{m}{k}) = -\frac{1}{k\pi} \sum_{l=1}^{k} \sin\Big(\frac{2\pi l m}{k}\Big)\psi\Big(\frac{l}{k}\Big) = -B_1\Big(\frac{l}{k}\Big) = \frac{1}{2} - \frac{m}{k},$$

where ψ is the digamma function, B_1 is the Bernoulli polynomial of order 1, and

$$Res(s=1)=\frac{n}{k}\Big(\psi\Big(\frac{m}{k}\Big)+\log k+1-h_{n-1}\Big).$$

Then we obtain

$$H_n(m,k)=\Big(\frac{m}{k}-\frac{1}{2}\Big)-\frac{n}{k}\Big(\psi\Big(\frac{m}{k}\Big)+\log k+1-h_{n-1}\Big)+a_n(m,k),$$

where

$$a_n(m,k)=O\Big(e^{-\sqrt{Kn}}\Big)$$

for a constant K of order m/k. Indeed we have

$$a_n(m,k)=\frac{(-1)^n}{2i\pi}n!\int_{-1/2-i\infty}^{-1/2+i\infty}\frac{\zeta(s,\frac{m}{k})}{k^l s(s-1)\cdots(s-n)}.$$

Recall that the Hurwitz zeta function satisfies the following functional equation

$$\zeta\Big(1-s,\frac{m}{k}\Big)=\frac{2\Gamma(s)}{(k\pi k)^s}\sum_{l=1}^{k}\cos\Big(\frac{\pi s}{2}-\frac{2\pi lm}{k}\Big)\zeta\Big(s,\frac{l}{k}\Big).$$

Therefore,

(4.2)

$$\begin{aligned}a_n(m,k)&=-\frac{n!}{2ki\pi}\sum_{l=1}^{k}\int_{3/2-i\infty}^{3/2+i\infty}\frac{1}{(2\pi)^s}\frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)}\cos\Big(\frac{\pi s}{2}-\frac{2\pi lm}{k}\Big)\zeta\Big(s,\frac{l}{k}\Big)\,ds\\&=-\frac{n!}{2ki\pi}\sum_{l=1}^{k}e^{i\frac{2\pi lm}{k}}\int_{3/2-i\infty}^{3/2+i\infty}\frac{1}{(2\pi)^s}\frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)}e^{-i\frac{\pi s}{2}}\zeta\Big(s,\frac{l}{k}\Big)\,ds\\&\quad-\frac{n!}{2ki\pi}\sum_{l=1}^{k}e^{-i\frac{2\pi lm}{k}}\int_{3/2-i\infty}^{3/2+i\infty}\frac{1}{(2\pi)^s}\frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)}e^{i\frac{\pi s}{2}}\zeta\Big(s,\frac{l}{k}\Big)\,ds.\end{aligned}$$

For large values of n, those integrals will be evaluated by means of the saddle-point method. Note that the integrand in (4.2) has a minimum, on the real axis, near $s=\sigma_0=\sqrt{2ln/k}$, and so the appropriate parameter is $z=s/\sqrt{n}$. Change s by z, and take z constant and n large. Then

$$a_n(m,k)=-\frac{1}{2i\pi}\sum_{l=1}k\Big\{e^{i\frac{2\pi lm}{k}}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}e^{f(z)}dz+e^{-i\frac{2\pi lm}{k}}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}e^{\bar{f}(z)}dz\Big\}.\tag{4.3}$$

We have

$$f(z)=\log n!+\tfrac{1}{2}\log n+\phi(z\sqrt{n}),$$

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with

$$\phi(s) = -s\log\Big(\frac{2\pi l}{k}\Big) - i\frac{\pi s}{2} + \log\Big(\frac{\Gamma(s)\Gamma(s-1)}{\Gamma(s+n)}\Big) + O\Big(\Big(\frac{l}{k+l}\Big)^s\Big),$$

using the approximation

$$\zeta(s, l/k) = (k/l)^s + O\Big(\Big(\frac{l}{k+l}\Big)^s\Big)$$

for large s. Furthermore,

$$\log\zeta(s) = \sum_{n=2}^{+\infty}\frac{\Lambda(n)}{n^s\log n},$$

where $\Lambda(n)$ is the Von-Mangoldt function. The asymptotic expansion for the Gamma function is given by the Stirling expansion

$$\log\Gamma(x) = \Big(x-\frac{1}{2}\Big)\log x - x + \frac{1}{2}\log(2\pi) + \sum_{j=1}^{+\infty}\frac{B_{2j}}{2j(2j-1)x^{2j-1}},$$

where B_k are the Bernoulli numbers. Expanding to $O(1/n)$ and collecting terms, we deduce

$$\begin{aligned} f(z) = \frac{1}{2}\log n - z\sqrt{n}\Big(\log\Big(\frac{2\pi l}{k}\Big) + i\frac{\pi}{2} + 2 - 2\log z\Big) \\ + \log(2\pi) - 2\log z - \frac{z^2}{2} + \frac{1}{6z\sqrt{n}}(10+z^2) \\ + \frac{1}{2n}\Big(1 - \frac{z^2}{2} - \frac{z^4}{6} + \frac{73}{72z^2}\Big) + O(n^{-3/2}). \end{aligned}$$

The saddle-point is obtained by solving the equation $f'(z) = 0$, and we have

$$z_0 = (1+i)\sqrt{\frac{\pi l}{k}}.$$

We need $f''(z) = 2\sqrt{n}/z + O(1)$ to use the saddle-point formula. Substituting, we obtain

$$\text{(4.4)}\quad \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{f(z)}dz = \Big(\frac{2\pi^3 ln}{k}\Big)^{1/4} e^{\frac{i\pi}{8}}\exp\Big(-(1+i)\sqrt{\frac{4\pi ln}{k}}\Big) + O\Big(n^{-1/4}e^{-2\sqrt{\frac{\pi ln}{k}}}\Big).$$

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The integral for $\overline{f}$ is the complex conjugate of (4.4) (having a saddle-point at the complex conjugate $\overline{z_0}$). Finally, equations (4.3) and (4.4) together give

$$a_n(m,k) = \frac{1}{k}\Big(\frac{2n}{\pi}\Big)^{1/4}\sum_{l=1}^{k}\Big(\frac{l}{k}\Big)^{1/4}\exp\Big(-\sqrt{\frac{4\pi ln}{k}}\Big)\ \cos\Big(\sqrt{\frac{4\pi ln}{k}}-\frac{5\pi}{8}-\frac{2\pi lm}{k}\Big) + O\Big(n^{-1/4}e^{-2\sqrt{\frac{\pi ln}{k}}}\Big).$$

For large n, only the $l = 1$ term contributes significantly, and so

$$a_n(m,k) = \frac{1}{k}\Big(\frac{2n}{\pi k}\Big)^{1/4}\exp\Big(-\sqrt{\frac{4\pi n}{k}}\Big)\cos\Big(\sqrt{\frac{4\pi n}{k}}-\frac{5\pi}{8}-\frac{2\pi m}{k}\Big)+O\Big(n^{-1/4}e^{-2\sqrt{\frac{\pi n}{k}}}\Big),$$

which means that the terms a_n are exponentially small. ■

Proof of Theorem 4.1 Without loss of generality, we assume that μ_j is a real number. First, write the arithmetic formula of $\lambda_F(-n)$ (equation (2.1)) as

$$\lambda_F(-n) = m_F + n\Big(\log Q_F - \frac{d_F}{2}\gamma\Big) - \sum_{l=1}^{n}\tbinom{n}{l}\eta_F(l-1) + n\sum_{j=1}^{r}\lambda_j\Big(-\frac{1}{\lambda_j+\mu_j}+\sum_{l=1}^{+\infty}\frac{\lambda_j+\mu_j}{l(l+\lambda_j+\mu_j)}\Big) - \sum_{j=1}^{r}I_j, \tag{4.5}$$

where

$$\eta_F(l) = \frac{(-1)^l}{l!}\lim_{X\longrightarrow+\infty}\Big\{\sum_{k\le X}\frac{\Lambda_F(k)}{k}(\log k)^l - \frac{m_F}{l+1}(\log X)^{l+1}\Big\}$$

are the generalized Stieltjes constants and

$$I_j = \sum_{k=2}^{n}\tbinom{n}{k}(-\lambda_j)^k\sum_{l=0}^{+\infty}\Big(\frac{1}{l+\lambda_j+\mu_j}\Big)^k.$$

Note that

$$I_j^{(1)} = \sum_{k=2}^{n}\tbinom{n}{k}(-\lambda_j)^k\sum_{l=0}^{+\infty}\frac{1}{(l+\lambda_j+\mu_j)^k} = \sum_{k=2}^{n}\tbinom{n}{k}(-1)^k\frac{\zeta(k,\lambda_j+\mu_j)}{(\lambda_j^{-1})^k},$$

which, with the above notation of $H_n(m,k)$ (equation (4.1)), is equal to

$$I_j^{(1)} = H_n\Big(1+\frac{\mu_j}{\lambda_j},\lambda_j^{-1}\Big).$$

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Applying Proposition 4.3 with $m = 1 + \frac{\mu_j}{\lambda_j}$ and $k = \lambda_j^{-1}$, we deduce

(4.6)
$$I_j = \Big(\lambda_j + \mu_j - \frac{1}{2}\Big) - n\lambda_j\Big(\psi(\lambda_j+\mu_j) + \log(\lambda_j^{-1}) + 1 - h_{n-1}\Big) + a_n\Big(1 + \frac{\mu_j}{\lambda_j}, \lambda_j^{-1}\Big),$$

where

$$a_n\Big(1+\frac{\mu_j}{\lambda_j}, \lambda_j^{-1}\Big) = \lambda_j\Big(\frac{2n}{\pi}\lambda_j\Big)^{1/4}\exp(-\sqrt{4\pi n\lambda_j})\,\cos\Big(\sqrt{4\pi n\lambda_j} - \frac{5\pi}{8} - 2\pi(\lambda_j+\mu_j)\Big) + O\Big(n^{-1/4}e^{-2\sqrt{\pi n\lambda_j}}\Big).$$

The a_n are exponentially small, then

$$a_n\Big(1+\frac{\mu_j}{\lambda_j}, \lambda_j^{-1}\Big) = O(1). \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$I_j = \Big(\lambda_j + \mu_j - \frac{1}{2}\Big) - n\lambda_j\Big\{\psi(\lambda_j+\mu_j) + \log(\lambda_j^{-1}) + 1 - h_{n-1}\Big\} + O(n). \tag{4.8}$$

Summing (4.8) over j, we get

$$\sum_{j=1}^{r} I_j = \sum_{j=1}^{r}\Big(\lambda_j + \mu_j - \frac{1}{2}\Big) - n\sum_{j=1}^{r}\lambda_j\Big\{\psi(\lambda_j+\mu_j) + \log(\lambda_j^{-1}) + 1 - h_{n-1}\Big\} + O(n). \tag{4.9}$$

Using the expression

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{l=1}^{+\infty}\frac{z}{l(l+z)},$$

where γ is the Euler constant, and the estimate

$$h_n = \log n - \gamma + \frac{1}{2n} + O\Big(\frac{1}{2n^2}\Big),$$

we deduce from (4.5) and (4.9) that

$$\lambda_F(-n) = \Big(\sum_{j=1}^{r}\lambda_j\Big)\, n\log n + \Big\{\Big(\sum_{j=1}^{r}\lambda_j\Big)(\gamma-1) + \log Q_F + \sum_{j=1}^{r}\lambda_j\log\lambda_j\Big\}\, n - \sum_{l=1}^{n}\tbinom{n}{l}\eta_F(l-1) + O(n).$$

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Recalling that $d_F = \sum_{j=1}^{r} \lambda_j$ and noting that $\lambda = \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$, we have

$$(4.10)\quad \lambda_F(-n) = \frac{d_F}{2} n \log n + \Big\{ \frac{d_F}{2}(\gamma - 1) + \frac{1}{2}\log\Big(\lambda Q_F^2\Big)\Big\} n - \sum_{l=1}^{n} \binom{n}{l} \eta_F(l-1) + O(n).$$

Now, we obtain a bound for $S_F(n) = -\sum_{l=1}^{n} \binom{n}{l} \eta_F(l-1)$ in terms of

$$\lambda_F(-n, T) := \sum_{\rho;\ |\Im\rho| \le T} 1 - \Big(1 - \frac{1}{\rho}\Big)^n,$$

where T is a parameter.

Lemma 4.4 *If the Generalized Riemann Hypothesis holds for $F \in \mathcal{S}$, then*

$$S_F(n) = O(\sqrt{n} \log n).$$

Proof The proof is analogous to the argument used by Lagarias in [7]. We use a contour integral argument, and we introduce the kernel function

$$k_n := \Big(1 + \frac{1}{s}\Big)^n - 1 = \sum_{l=1}^{n} \binom{n}{l} \Big(\frac{1}{s}\Big)^l.$$

If C is a contour enclosing the point $s = 0$ counterclockwise on a circle of small enough positive radius, the residue theorem gives

$$I(n) = \frac{1}{2i\pi} \int_C k_n(s) \Big(-\frac{F'}{F}(s+1)\Big)\, ds = \sum_{l=1}^{n} \binom{n}{l} \eta_{l-1} = S_F(n).$$

We deform the contour to the counterclockwise oriented rectangular contour C' consisting of vertical lines with real part $\Re(s) = \sigma_0$ and $\Re(s) = \sigma_1$, where we will choose $-3 < \sigma_0 < -2$, $\sigma_1 = 2\sqrt{n}$ and the horizontal lines at $\Im(s) = \pm T$, where we will choose $T = \sqrt{n} + \epsilon_n$ for some $0 < \epsilon_n < 1$. The residue theorem gives

$$\begin{aligned} I'(n) &= \frac{1}{2i\pi} \int_{C'} k_n(s) \Big(-\frac{F'}{F}(s+1)\Big)\, ds \\ &= S_F(n) + \sum_{\rho;\ |\Im\rho| \le T} \Big(1 + \frac{1}{\rho - 1}\Big)^n - 1 + O(1). \end{aligned}$$

The term $O(1)$ evaluates the residues coming from the trivial zeros of $F(s)$. Using the symmetry $\rho \mapsto 1 - \overline{\rho}$, we can write

$$\Big(\frac{1 - \overline{\rho}}{-\overline{\rho}}\Big)^n - 1 = \Big(\frac{\overline{\rho} - 1}{\overline{\rho}}\Big)^n - 1.$$

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Then

$$I'(n) = S_F(n) - \lambda_F(-n, T) + O(1).$$

We have

$$|\lambda_F(-n, \sqrt{n}) - \lambda_F(-n, T)| = O(\log n).$$

This follows from the observation that $|T - \sqrt{n}| < 1$, that there are $O(\log n)$ zeros in an interval of length one at this height, and that for each zero $\rho = \beta + i\gamma$ with $\sqrt{n} \le |\Im(\rho)| < \sqrt{n} + 1$ there holds

$$\Big| \Big(\frac{\rho - 1}{\rho} \Big) \Big| \le \Big| 1 + \frac{1}{n} \Big|^{n/2} \le 2.$$

We now choose the parameters σ_0 and T appropriately to avoid the poles of the integrand. We may choose σ_0 so that the contour avoids any trivial zero and $T = \sqrt{n} + \epsilon_n$ with $0 \le \epsilon_n \le 1$ so that the horizontal lines do not approach closer than $O(\log n)$ to any zero of $F(s)$. Recall from [16] that for $-2 < \Re(s) < 2$ there holds

$$\frac{F'}{F}(s) = \sum_{\{\rho;\ |\Im(\rho - s)| < 1\}} \frac{1}{s - \rho} + O\Big(\log(Q_F(1 + |s|)) \Big).$$

Then on the horizontal line in the interval $-2 \le \Re(s) \le 2$, we have

$$\Big| \frac{F'}{F}(s+1) \Big| = O(\log^2 T).$$

The Euler product for $F(s)$ converges absolutely for $\Re(s) > 1$, hence the Dirichlet series for $\frac{F'}{F}(s)$ converges absolutely for $\Re(s) > 1$. More precisely, for $\sigma = \Re(s) > 1$

$$\Big| \frac{F'}{F} \Big| (\sigma) < \infty.$$

For $\sigma = \Re(s) > 2$, we obtain the bound

$$\Big| \frac{F'}{F}(s) \Big| \le \Big| \frac{F'}{F} \Big| (\sigma) \le 2^{-(\sigma - 2)}.$$

Consider the integral $I'(n)$ on the vertical segment (L_1) having $\sigma_1 = 2\sqrt{n}$. We have

$$\Big| \Big(1 - \frac{1}{s} \Big)^n - 1 \Big| \le \Big(1 + \frac{1}{\sigma_1} \Big)^n + 1 \le \Big(1 + \frac{1}{2\sqrt{n}} \Big)^n \le \exp(\sqrt{n}/2) < 2^{\sqrt{n}}.$$

Then

$$\Big| \frac{F'}{F}(s) \Big| \le C_0 2^{-2(\sqrt{n}+2)}.$$

Furthermore, the length of the contour is $O(\frac{n}{\log n})$, and we obtain $|I'_{L_1}| = O(1)$. Let $s = \sigma + it$ be a point on one of the two horizontal segments. We have $T \ge \sqrt{n}$, so that

$$\Big| 1 + \frac{1}{s} \Big| \le 1 + \frac{\sigma + 1}{\sigma^2 + T^2}.$$

By hypothesis $T^2 \geq n$, so for $-2 \leq \sigma \leq 2$, we have

$$|k_n(s)| \leq \left(1 + \frac{3}{4+n}\right)^n + 1 = O(1)$$

and

$$\left|\frac{F'}{F}(s)\right| = O(\log^2 T) = O(\log^2 n),$$

since we have chosen the ordinate T to stay away from zeros of $F(s)$. We step across the interval (L_2) toward the right, in segments of length 1, starting from $\sigma = 2$. Furthermore,

$$\left|\frac{k_n(s+1)+1}{k_n(s)+1}\right| \leq \left(1 + \frac{1}{T^2}\right)^n \leq e,$$

and we obtain an upper bound for $|k_n(s)\frac{F'}{F}(s)|$ that decreases geometrically at each step. After $O(\log n)$ steps it becomes $O(1)$, and the upper bound is

$$|I'_{L_2,L_4}(n)| = O(\log^2 n + \sqrt{n}) = O(\sqrt{n}).$$

For the vertical segment (L_3) with $\Re(s) = \sigma_0$, we have $|k_n(s)| = O(1)$ and $|\frac{F'}{F}(s)| = O[Q_F(\log(|s|+1))]$. Since the segment (L_3) has length $O(\sqrt{n})$, we obtain

$$|I'_{L_3}| = O(\sqrt{n}\log n).$$

Totalling the above bounds gives

$$S_F(n) = \lambda_F(-n, T) + O(\sqrt{n}\log n),$$

with $T = \sqrt{n} + \epsilon_n$. If the Generalized Riemann Hypothesis holds for $F(s)$, then we have $|1 - \frac{1}{\rho-1}| = 1$. Since each zero contributes a term of absolute value at most 2 to $\lambda_F(-n, T)$, we obtain using the zero density estimate $(N_F(T) \sim T\log T)$

$$\lambda_F(-n, T) = O(T\log T + 1).$$

Therefore $\lambda_F(-n, \sqrt{n}) = O(\sqrt{n}\log n)$, and Lemma 4.4 follows. ■

Using Lemma 4.4 and the expression (4.10) of $\lambda_F(-n)$ and $\lambda_F(-n) = \overline{\lambda_F(n)}$, we obtain

$$\lambda_F(n) = \frac{d_F}{2} n\log n + \left\{\frac{d_F}{2}(\gamma - 1) + \frac{1}{2}\log(\lambda Q_F^2)\right\} n + O(\sqrt{n}\log n),$$

which concludes the proof of Theorem 4.1. ■

Examples

- In the case of the Riemann zeta function, we have $d_\zeta = 1$, $Q_\zeta = \pi^{-1/2}$, and $\lambda = \frac{1}{2}$. This proves again under the Riemann Hypothesis the asymptotic formula established by A. Voros in [17, equation (17), p. 59].

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- Also, in the case of the principal L-function $L(s,\pi)$ attached to an irreducible cuspidal unitary automorpohic representation of $GL(N)$, as in Rudnick and Sarnak [14, §2], we have $D_L = N,\ Q_L = Q(\pi)\pi^{-N/2}$, and $\lambda = 2^{-n}$. We find under the Generalized Riemann Hypothesis the asymptotic formula for $\lambda_n(\pi)$ established by Lagarias in [7, equations (1.12) and (1.13), p. 4].

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Faculté des sciences de Monastir, Département de mathématiques, 5000 Monastir, Tunisia
e-mail: kamel.mazhouda@fsm.rnu.tn