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# STABILITY OF UNCONDITIONAL SCHAUDER DECOMPOSITIONS IN $\ell_p$ SPACES

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**Abstract**

We use the best constants in the Khintchine inequality to generalise a theorem of Kato [‘Similarity for sequences of projections’, *Bull. Amer. Math. Soc.* **73**(6) (1967), 904–905] on similarity for sequences of projections in Hilbert spaces to the case of unconditional Schauder decompositions in $\ell_p$ spaces. We also sharpen a stability theorem of Vizitei [‘On the stability of bases of subspaces in a Banach space’, in: *Studies on Algebra and Mathematical Analysis*, Moldova Academy of Sciences (Kartja Moldovenjaska, Chişinău, 1965), 32–44; (in Russian)] in the case of unconditional Schauder decompositions in any Banach space.

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## 1. Introduction

The stability theory for Schauder bases in Banach spaces originates in 1940 with the work of Krein *et al.* [18] on the well-known basis problem for a Banach space. The main theorem from [18] has the following important consequence: in any Banach space with a basis, a basis may be chosen from an arbitrary dense set. This theorem, the Krein–Milman–Rutman stability theorem, has many generalisations, analogues and applications (see, for example, [19, 24]).

Throughout what follows, $E$ denotes a Banach space, $H$ denotes a Hilbert space and $\mathbb{Z}_+$ is the set of nonnegative integers. In the 1960s, Marcus, Vizitei and Kato investigated the stability property for Schauder decompositions in connection with the problems of spectral theory in $H$. The results of Marcus and Vizitei were applied to the spectral analysis of dissipative operators [10, 21] and slightly perturbed normal operators [27]. Djakov and Mityagin applied a stability theorem from [21] to show that spectral decompositions corresponding to Hill operators with singular potentials [8], as well as decompositions associated with one-dimensional

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periodic Dirac operators [9], converge unconditionally. It follows that there exists an unconditional Schauder decomposition consisting of invariant subspaces. In 2010, Wyss [29] used a modification of a lemma from [22] to prove the same property for $p$-subordinate perturbations of normal operators.

In the same year, Zwart [30] applied a lemma from [28] to obtain the following remarkable spectral theorem for the generator $A$ of a $C_0$-group on $H$: if the eigenvalues $\{\lambda_n\}$ of $A$ (counted with multiplicity) can be decomposed into $K$ sets $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \dots, \{\lambda_{n,K}\}$ with $\inf_{n\neq m} |\lambda_{n,k} - \lambda_{m,k}| > 0$, $k = 1, \dots, K$, and the span of the generalised eigenvectors of $A$ is dense, then there are spectral projections $\{J_n\}_{n=0}^{\infty}$ of $A$ such that $\{J_n H\}_{n=0}^{\infty}$ forms an unconditional Schauder decomposition in $H$ with $\max_n \dim J_n H = K$.

The present paper focuses on the generalisation of the following theorem.

Theorem 1.1 [16]. *Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of nonzero selfadjoint projections in $H$ satisfying $\sum_{n=0}^{\infty} P_n = I$, $P_n P_m = \delta_n^m P_n$ for $n, m \in \mathbb{Z}_+$, and let $\{J_n\}_{n=0}^{\infty}$ be a sequence of nonzero projections in $H$ such that $J_n J_m = \delta_n^m J_n$ for $n, m \in \mathbb{Z}_+$. Furthermore, suppose that*

$$\dim P_0 = \dim J_0 = m < \infty, \tag{1.1}$$

$$\sum_{n=1}^{\infty} \|P_n(J_n - P_n)x\|^2 \le c^2 \|x\|^2 \quad \text{for all } x \in H,$$

*where $c$ is a constant such that $0 \le c < 1$. Then $\{J_n\}_{n=0}^{\infty}$ is similar to $\{P_n\}_{n=0}^{\infty}$, that is, there exists an isomorphism $S$ such that $J_n = S P_n S^{-1}$ for $n \in \mathbb{Z}_+$.*

This theorem provides an effective tool for spectral analysis of various perturbations of operators in $H$ (see, for example, [7, 12, 17]). Recently, Adduci and Mityagin [1] applied Theorem 1.1 to show that the eigensystem of the perturbed harmonic oscillator $-d^2/dt^2 + t^2 + B$, with $B = b(t)$ and domain in $L_2(\mathbb{R})$, is an unconditional basis. In [2], they also applied Theorem 1.1 to the spectral analysis of the perturbation of a selfadjoint operator with discrete spectrum.

The main goal of this paper is to generalise Theorem 1.1 to the case of $\ell_p$ spaces, where $1 \le p < \infty$. For this purpose, we consider unconditional Schauder decompositions in $\ell_p$ instead of orthogonal Schauder decompositions in $H$. Moreover, we use intrinsic geometric properties of $\ell_p$ and the best constants in the Khintchine inequality. As immediate consequences of the main result, we obtain some stability theorems for unconditional and symmetric bases in $\ell_p$. The paper [20] also studies the interplay between the problem of stability for Schauder decompositions in a Banach space and the intrinsic geometric properties of a Banach space. The main object in [20] is $\ell_\Psi$-Hilbertian Schauder decomposition. In the present paper, we focus on the most interesting and informative case of stability for unconditional Schauder decompositions in $\ell_p$ space. Moreover, we consider some applications concerning symmetric bases in $\ell_p$. Note that $\ell_\infty$ has no Schauder decomposition [25], although there are nonseparable Banach spaces with Schauder decompositions [6, 23] and separable Banach spaces without them [3]. For more on Schauder decompositions we refer to [4, 10, 13, 19, 20, 25].

The paper has the following structure. The next section presents an auxiliary lemma (Lemma 2.3) which describes the dependence of the properties of unconditional Schauder decompositions in $E$ on the geometry of $E$. In Section 3, we use Lemma 2.3 to sharpen a stability theorem of Vizitei in the case of unconditional Schauder decompositions in $E$. Section 4 is devoted to the main result concerning stability of unconditional Schauder decompositions in $\ell_p$ spaces (Theorems 4.1 and 4.2). It also provides some results on stability of unconditional and symmetric bases in $\ell_p$ (Theorems 4.4 and 4.5). Finally, in Section 5 we discuss some applications and show that certain sequences are symmetric bases of $\ell_p$.

## 2. The auxiliary lemma

We will need the following definitions.

Definition 2.1 [25]. Let $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ be a Schauder decomposition in $E$. Then the sequence of continuous linear projections $\{P_n\}_{n=0}^{\infty}$ on $E$, defined by $P_n x = x_n$, $n \in \mathbb{Z}_+$, where $x = \sum_{n=0}^{\infty} x_n$, $x_n \in \mathfrak{M}_n$, is called the sequence of coordinate projections associated to the decomposition $\{\mathfrak{M}_n\}_{n=0}^{\infty}$, or, for short, the associated sequence of coordinate projections (a.s.c.p.).

Definition 2.2. A Schauder decomposition $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ in $E$ is said to be unconditional with constant $M$ provided there exists $M \geq 1$ such that

$$\Big\|\sum_{i=0}^{n} \delta_i y_i\Big\| \leq M \Big\|\sum_{i=0}^{n} y_i\Big\| \quad \text{for all } n \in \mathbb{Z}_+,\ y_i \in \mathfrak{M}_i,\ \{\delta_i\}_{i=0}^{n} \subset \{0,1\}.$$

One trivially notices that every orthogonal Schauder decomposition in $H$ is unconditional with constant 1. The following auxiliary lemma shows how the properties of unconditional Schauder decompositions in $E$ depend on the inner geometry of $E$ and it will be useful throughout the paper.

Lemma 2.3. *Let $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ be an unconditional Schauder decomposition in $E$ with constant $M$ and the a.s.c.p. $\{P_n\}_{n=0}^{\infty}$. Also assume that $E$ has type or infratype $p$ and that $E$ has cotype or $M$-cotype $q$. Then there exist constants $T = T(p, M) > 0$ and $C = C(q, M) > 0$ such that for every $x \in E$*

$$C\Big(\sum_{n=0}^{\infty} \|P_n x\|^q\Big)^{1/q} \leq \|x\| \leq T\Big(\sum_{n=0}^{\infty} \|P_n x\|^p\Big)^{1/p}. \tag{2.1}$$

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PROOF. Since $E$ has type or infratype $p$, for each $x \in E$ and for every finite set $\{P_j x\}_{j=0}^n \subset E$ there exists a set $\{\underline{\epsilon}_j\}_{j=0}^n \subset \{-1, 1\}$ such that

$$\begin{aligned}
\Big\|\sum_{j=0}^n \underline{\epsilon}_j P_j x\Big\| &= \Big(\min_{\epsilon_j=\pm 1}\Big\|\sum_{j=0}^n \epsilon_j P_j x\Big\|^2\Big)^{1/2} \\
&\le \begin{cases} \Big(\mathbf{E}\Big\|\sum_{j=0}^n \epsilon_j P_j x\Big\|^2\Big)^{1/2} \le T_p(E)\Big(\sum_{j=0}^n \|P_j x\|^p\Big)^{1/p} & \text{when } E \text{ has type } p, \\ I_p(E)\Big(\sum_{j=0}^n \|P_j x\|^p\Big)^{1/p} & \text{when } E \text{ has infratype } p \end{cases} \\
&\le T(p)\Big(\sum_{j=0}^n \|P_j x\|^p\Big)^{1/p} \quad \Big(\text{here } \mathbf{E}\Big\|\sum_{j=0}^n \epsilon_j x_j\Big\|^2 = \frac{1}{2^{n+1}} \sum_{\epsilon_j=\pm 1}\Big\|\sum_{j=0}^n \epsilon_j x_j\Big\|^2\Big).
\end{aligned}$$

Define the operators $P_n^+ = \sum_{j:\underline{\epsilon}_j=1} P_j$, $P_n^- = \sum_{j:\underline{\epsilon}_j=-1} P_j$ on $E$. Then

$$\|x\| = \lim_{n\to\infty} \|(P_n^+ - P_n^-)^2 x\| \le 2M \lim_{n\to\infty} \|(P_n^+ - P_n^-)x\| \le 2MT(p)\Big(\sum_{n=0}^\infty \|P_n x\|^p\Big)^{1/p}.$$

Hence, the right-hand side of (2.1) is proved with $T = 2MT(p)$.

Further, since $E$ has cotype or $M$-cotype $q$, for each $x \in E$ and for every finite set $\{P_j x\}_{j=0}^n \subset E$ there exists a set $\{\overline{\epsilon}_j\}_{j=0}^n \subset \{-1, 1\}$ such that

$$\begin{aligned}
\Big\|\sum_{j=0}^n \overline{\epsilon}_j P_j x\Big\| &= \Big(\max_{\epsilon_j=\pm 1}\Big\|\sum_{j=0}^n \epsilon_j P_j x\Big\|^2\Big)^{1/2} \\
&\ge \begin{cases} \Big(\mathbf{E}\Big\|\sum_{j=0}^n \epsilon_j P_j x\Big\|^2\Big)^{1/2} \ge C_q(E)\Big(\sum_{j=0}^n \|P_j x\|^q\Big)^{1/q} & \text{when } E \text{ has cotype } q, \\ M_q(E)\Big(\sum_{j=0}^n \|P_j x\|^q\Big)^{1/q} & \text{when } E \text{ has } M\text{-cotype } q \end{cases} \\
&\ge C(q)\Big(\sum_{j=0}^n \|P_j x\|^q\Big)^{1/q}.
\end{aligned}$$

Observe that for each set $\{\overline{\epsilon}_j\}_{j=0}^n \subset \{-1, 1\}$ there exist two sets $\{\delta_j^+\}_{j=0}^n \subset \{0, 1\}$ and $\{\delta_j^-\}_{j=0}^n \subset \{0, 1\}$ such that

$$\Big\|\sum_{j=0}^n \overline{\epsilon}_j P_j x\Big\| = \Big\|\sum_{j=0}^n \delta_j^+ P_j x - \sum_{j=0}^n \delta_j^- P_j x\Big\| \le 2M\|x\|.$$

Therefore, $C(q)(\sum_{j=0}^n \|P_j x\|^q)^{1/q} \le 2M\|x\|$, and the left-hand side of (2.1) is proved with $C = (2M)^{-1}C(q)$. □

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Lemma 2.3 is a generalisation of a lemma from [28]. Since every Banach space has (trivially) type 1 and cotype $\infty$, Lemma 2.3 is valid for unconditional Schauder decompositions in any Banach space. Note that $(\sum_{n=0}^{\infty} \|P_n x\|^q)^{1/q}$ turns into $\sup_{n\in\mathbb{Z}_+} \|P_n x\|$ when $q=\infty$. Concerning type, infratype, cotype and $M$-cotype, see, for example, [13–15, 19, 20].

In the case of unconditional Schauder decompositions in $L_p(\mu)$ spaces where $1 \le p < \infty$, we have the following corollary of Lemma 2.3.

Corollary 2.4. *Let $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ be an unconditional Schauder decomposition in $L_p(\mu)$, $1 \le p < \infty$, with constant $M$ and the a.s.c.p. $\{P_n\}_{n=0}^{\infty}$. Denote by $p_0$ the solution of*

$$\Gamma\Big(\frac{p+1}{2}\Big) = \frac{\sqrt{\pi}}{2}$$

*in the interval $[1,2]$, where $\Gamma$ is the Euler gamma function ($p_0 \approx 1.84742$). Then for each $x \in L_p(\mu)$ we have the following two-sided inequalities:*

$$\frac{2^{-1/2-1/p}}{M}\Big(\sum_{n=0}^{\infty} \|P_n x\|^2\Big)^{1/2} \le \|x\| \le 2M\Big(\sum_{n=0}^{\infty} \|P_n x\|^p\Big)^{1/p} \quad \textit{for } 1 \le p \le p_0,$$

$$\frac{2^{-1/2}}{M}\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{1/p}\Big(\sum_{n=0}^{\infty} \|P_n x\|^2\Big)^{1/2} \le \|x\| \le 2M\Big(\sum_{n=0}^{\infty} \|P_n x\|^p\Big)^{1/p} \quad \textit{for } p_0 \le p \le 2,$$

$$\frac{1}{2M}\Big(\sum_{n=0}^{\infty} \|P_n x\|^p\Big)^{1/p} \le \|x\| \le \sqrt{8}M\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{1/p}\Big(\sum_{n=0}^{\infty} \|P_n x\|^2\Big)^{1/2} \quad \textit{for } p \ge 2.$$

Proof. It is known [13] that $L_p(\mu)$ has the best possible type $\min\{2,p\}$ and the best possible cotype $\max\{2,p\}$. The best constants associated to these values of type and cotype may be estimated as

$$T_p(L_p(\mu)) = 1, \quad C_2(L_p(\mu)) \le A_p^{-1} \quad \text{for } 1 \le p \le 2,$$
$$T_2(L_p(\mu)) \le B_p, \quad C_p(L_p(\mu)) = 1 \quad \text{for } p \ge 2,$$

where $A_p$ and $B_p$ are the best constants in the Khintchine inequality [13], namely,

$$A_p = \begin{cases} 2^{1/2-1/p} & \text{for } 0 < p \le p_0, \\ 2^{1/2}\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{1/p} & \text{for } p_0 \le p \le 2, \\ 1 & \text{for } p \ge 2, \end{cases}$$

$$B_p = \begin{cases} 1 & \text{for } 0 < p \le 2, \\ 2^{1/2}\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{1/p} & \text{for } p \ge 2, \end{cases}$$

(see [11]). Now the corollary follows from Lemma 2.3 and its proof. □

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## 3. A sharpening of the theorem of Vizitei

We will use the terminology from [25] in this section. The theorem of Vizitei [26] provides some stability properties of $p$-Besselian Schauder decompositions in $E$. It may be characterised as a stability theorem of geometric type (see also [25, Theorem 15.17]. By virtue of Lemma 2.3, any unconditional Schauder decomposition in a Banach space $E$, which has cotype or $M$-cotype $q$, is $q$-Besselian. Therefore, we can sharpen the theorem of Vizitei as follows.

THEOREM 3.1. *Suppose $E$ has cotype or $M$-cotype $q$ with $2 \le q \le \infty$ and define $p$ by $1/p + 1/q = 1$ (if $q = \infty$, then $p = 1$). Also suppose that $E$ has an unconditional Schauder decomposition $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ with constant $M$ and the a.s.c.p. $\{P_n\}_{n=0}^{\infty}$. Then the following assertions hold.*

(1) *There is a constant $\lambda \in (0, 1)$ such that every sequence of subspaces $\{\mathfrak{N}_n\}_{n=0}^{\infty}$ of $E$ satisfying*

$$\Big(\sum_{n=0}^{\infty} \theta(\mathfrak{M}_n, \mathfrak{N}_n)^p\Big)^{1/p} \le \lambda$$

*where*

$$\theta(\mathfrak{M}, \mathfrak{N}) = \max\Big\{\sup_{x\in\mathfrak{M},\|x\|=1} \operatorname{dist}(x, \mathfrak{N}), \sup_{y\in\mathfrak{N},\|y\|=1} \operatorname{dist}(y, \mathfrak{M})\Big\}$$

*is itself an unconditional Schauder decomposition of $E$, isomorphic to $\{\mathfrak{M}_n\}_{n=0}^{\infty}$, with constant $M\|S\|\,\|S^{-1}\|$, where $\mathfrak{N}_n = S\mathfrak{M}_n$, $n \in \mathbb{Z}_+$. Note that the constant $\lambda$ may be chosen as $\lambda = (4\sup_{0\le n<\infty}\|\sum_{j=0}^{n} P_j\|(1 + \sup_{0\le n<\infty}\|P_n\|)^2)^{-1}$.*

(2) *Every sequence of subspaces $\{\mathfrak{N}_n\}_{n=0}^{\infty}$ of $E$ satisfying*

$$\sum_{n=0}^{\infty} \theta(\mathfrak{M}_n, \mathfrak{N}_n)^p < \infty, \tag{3.1}$$

*and admitting a sequence $\{J_n\}_{n=0}^{\infty}$ such that $(\{\mathfrak{N}_n\}_{n=0}^{\infty}, \{J_n\}_{n=0}^{\infty})$ is an $E$-complete generalised biorthogonal system, is a $q$-Besselian Schauder decomposition of $E$. If, additionally, $\dim \mathfrak{M}_n < \infty$ for all $n \in \mathbb{Z}_+$, then the same conclusion holds for every $\omega$-linearly independent sequence of subspaces $\{\mathfrak{N}_n\}_{n=0}^{\infty}$ satisfying (3.1).*

Note that, since every Banach space has (trivially) cotype $\infty$, Theorem 3.1 is valid in arbitrary Banach spaces possessing unconditional Schauder decompositions.

## 4. The main result

The main result of the paper is formulated as follows.

THEOREM 4.1. *Let $\{\mathfrak{N}_n\}_{n=0}^{\infty}$ be a Schauder decomposition of the space $\ell_p$, $1 \le p < \infty$, with the a.s.c.p. $\{F_n\}_{n=0}^{\infty}$ such that $\dim F_0 < \infty$ and*

$$\sum_{n=0}^{\infty} \|F_n x\|^p = \|x\|^p \quad \textit{for each } x \in \ell_p. \tag{4.1}$$

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*Assume that $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ is an unconditional Schauder decomposition of $\ell_p$ with constant $M$ and the a.s.c.p. $\{P_n\}_{n=0}^{\infty}$, where $P_0 = F_0$. Also suppose that $\{J_n\}_{n=0}^{\infty}$ is a sequence of nonzero projections in $\ell_p$ satisfying (1.1) such that $J_nJ_m = \delta_n^m J_n$ for $n, m \in \mathbb{Z}_+$. Suppose that, for each $x \in \ell_p$,*

$$\Big(\sum_{n=1}^{\infty} \|P_n(J_n - P_n)x\|^p\Big)^{1/p} \le \varsigma_1\|x\| \quad \textit{where } 0 \le \varsigma_1 < \frac{1}{2M} \tag{4.2}$$

*when $1 \le p \le 2$, or*

$$\Big(\sum_{n=1}^{\infty} \|P_n(J_n - P_n)x\|^2\Big)^{1/2} \le \varsigma_2(p)\|x\| \quad \textit{where } 0 \le \varsigma_2(p) < \frac{1}{\sqrt{8}M}\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{-1/p} \tag{4.3}$$

*when $p \ge 2$. Then $\{J_n(\ell_p)\}_{n=0}^{\infty}$ is also an unconditional Schauder decomposition of $\ell_p$, isomorphic to $\{\mathfrak{M}_n\}_{n=0}^{\infty}$.*

Proof. To prove the theorem we use the method of Kato, as proposed in [16], and apply Corollary 2.4. Define the operator $S$ on $\ell_p$ by

$$S = \sum_{n=0}^{\infty} P_nJ_n.$$

To show that $S$ exists in the strong sense, we will prove that

$$\sum_{n=0}^{\infty}(P_n - P_nJ_n) = \sum_{n=0}^{\infty} P_n(P_n - J_n)$$

is strongly convergent. Since $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ is an unconditional Schauder decomposition in $\ell_p$ with constant $M$ and the a.s.c.p. $\{P_n\}_{n=0}^{\infty}$, Corollary 2.4 yields the following two assertions. For $1 \le p \le 2$, $x \in \ell_p$ and $N \in \mathbb{Z}_+$, by (4.2),

$$\begin{aligned}\Big\|\sum_{n=k}^{k+N} P_n(P_n - J_n)x\Big\|^p &\le (2M)^p \sum_{j=0}^{\infty}\Big\|P_j\Big(\sum_{n=k}^{k+N} P_n(P_n - J_n)x\Big)\Big\|^p\\ &= (2M)^p \sum_{n=k}^{k+N} \|P_n(P_n - J_n)x\|^p \to 0\end{aligned}$$

as $k \to \infty$. Analogously, using (4.3), for $p \ge 2$, $x \in \ell_p$ and $N \in \mathbb{Z}_+$,

$$\begin{aligned}\Big\|\sum_{n=k}^{k+N} P_n(P_n - J_n)x\Big\|^2 &\le 8M^2\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{2/p} \sum_{j=0}^{\infty}\Big\|P_j\Big(\sum_{n=k}^{k+N} P_n(P_n - J_n)x\Big)\Big\|^2\\ &= 8M^2\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{2/p} \sum_{n=k}^{k+N} \|P_n(P_n - J_n)x\|^2 \to 0\end{aligned}$$

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as $k \to \infty$. Thus, $\sum_{n=0}^{\infty} P_n(P_n - J_n)x$ converges and, consequently, the series

$$\sum_{n=0}^{\infty} P_n J_n x = \sum_{n=0}^{\infty} P_n x - \sum_{n=0}^{\infty} P_n(P_n - J_n)x$$

also converges. Now consider the operator

$$R = \sum_{n=1}^{\infty} P_n(P_n - J_n) = I - P_0 - \sum_{n=1}^{\infty} P_n J_n.$$

Observe that, on the one hand, for $1 \leq p \leq 2$ and each $x \in \ell_p$,

$$\begin{aligned}
\|Rx\|^p &= \Big\|\sum_{n=1}^{\infty} P_n(P_n - J_n)x\Big\|^p \\
&\leq (2M)^p \sum_{j=0}^{\infty} \Big\|P_j\Big(\sum_{n=1}^{\infty} P_n(P_n - J_n)x\Big)\Big\|^p \quad \text{by Corollary 2.4} \\
&= (2M)^p \sum_{n=1}^{\infty} \|P_n(P_n - J_n)x\|^p \leq (2M)^p \varsigma_1^p \|x\|^p \quad \text{by (4.2).}
\end{aligned}$$

On the other hand, for $p \geq 2$ and each $x \in \ell_p$,

$$\begin{aligned}
\|Rx\|^2 &= \Big\|\sum_{n=1}^{\infty} P_n(P_n - J_n)x\Big\|^2 \\
&\leq 8M^2\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{2/p} \sum_{j=0}^{\infty} \Big\|P_j\Big(\sum_{n=1}^{\infty} P_n(P_n - J_n)x\Big)\Big\|^2 \quad \text{by Corollary 2.4} \\
&= 8M^2\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{2/p} \sum_{n=1}^{\infty} \|P_n(P_n - J_n)x\|^2 \leq 8M^2\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{2/p} \varsigma_2^2(p)\|x\|^2 \quad \text{by (4.3).}
\end{aligned}$$

It follows that $\|R\| < 1$. Further, we observe that, since $S = P_0 J_0 + I - P_0 - R$,

$$\|S\| < \|J_0\| + 3 < \infty.$$

Thus, the theorem will be proved if we show that $S$ is continuously invertible. To prove this, consider the operator

$$\widetilde{S} = \sum_{n=1}^{\infty} P_n J_n = I - P_0 - R.$$

Since $\dim P_0 = m < \infty$ by the definition of the projection $P_0$, we see that $(I - P_0)$ is a Fredholm operator with

$$\operatorname{nul}(I - P_0) = m, \quad \operatorname{ind}(I - P_0) = 0, \quad \gamma(I - P_0) = 1,$$

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where $\operatorname{nul} T$ denotes the nullity, $\operatorname{ind} T$ the index and $\gamma(T)$ the reduced minimum modulus of the operator $T$ (for these notions, see, for example, [17, Ch. IV, Section 5.1]). Indeed, first we note that $\operatorname{nul}(I - P_0) = \dim P_0 = m$,

$$\operatorname{def}(I - P_0) = \dim \ell_p|_{Im(I-P_0)} = \dim \operatorname{coker}(I - P_0) = \dim(\operatorname{Im}(I - P_0))^{\perp} = m,$$

and $\operatorname{ind}(I - P_0) = \operatorname{nul}(I - P_0) - \operatorname{def}(I - P_0) = 0$, where $\operatorname{def} T$ denotes the deficiency of $T$ (see, for example, [4, 17]). Second, by virtue of (4.1) we observe that for each $x \in \ell_p$,

$$\begin{aligned}\inf_{v\in\ker(I-P_0)} \|x - v\| &= \inf_{v\in\operatorname{Im} F_0} \Big(\sum_{n=0}^{\infty} \|F_n(x - v)\|^p\Big)^{1/p} = \Big(\sum_{n=1}^{\infty} \|F_n(x - F_0 x)\|^p\Big)^{1/p} \\ &= \Big(\sum_{n=0}^{\infty} \|F_n(x - F_0 x)\|^p\Big)^{1/p} = \|(I - P_0)x\|.\end{aligned}$$

Consequently,

$$\gamma(I - P_0) = \sup\Big\{\gamma : \|(I - P_0)x\| \geq \gamma \inf_{v\in\ker(I-P_0)} \|x - v\|, x \in D(I - P_0) = \ell_p\Big\} = 1.$$

Furthermore, since $\|R\| < 1 = \gamma(I - P_0)$, $\widetilde{S} = (I - P_0) - R$ is also Fredholm with

$$\operatorname{nul} \widetilde{S} \leq \operatorname{nul}(I - P_0) = m, \quad \operatorname{ind} \widetilde{S} = \operatorname{ind}(I - P_0) = 0 \tag{4.4}$$

(see [17, Ch. IV, Theorem 5.22]). Since $S = P_0 J_0 + \widetilde{S}$, where $P_0 J_0$ is compact, $S$ is also Fredholm and $\operatorname{ind} S = \operatorname{ind} \widetilde{S} = 0$ (see [17, Ch. IV, Theorem 5.26]). Therefore, $\operatorname{nul} S = \operatorname{def} S$ and $S$ will be invertible if and only if $\operatorname{nul} S = \operatorname{def} S = 0$. Thus it is sufficient to show that $\operatorname{nul} S = 0$. To this end, we first prove that

$$\ker \widetilde{S} = \operatorname{Im} J_0. \tag{4.5}$$

If $x \in \operatorname{Im} J_0$, that is, $x = J_0 y$, then $\widetilde{S} x = \widetilde{S} J_0 y = \sum_{n=1}^{\infty} P_n J_n J_0 y = 0$ and, consequently, $x \in \ker \widetilde{S}$. On the other hand, $\ker \widetilde{S} \subset \operatorname{Im} J_0$, since $\ker \widetilde{S}$ and $\operatorname{Im} J_0$ are linear subspaces, $\dim \operatorname{Im} J_0 = m$ and $\dim \ker \widetilde{S} \leq m$ by (4.4). Now suppose that $x \in \ker S$. Then,

$$0 = P_0 S x = P_0 \sum_{n=0}^{\infty} P_n J_n x = P_0 J_0 x$$

and $\widetilde{S} x = S x - P_0 J_0 x = 0$. Hence, $x \in \ker \widetilde{S}$, $x = J_0 y$ by (4.5) and, therefore,

$$P_0 x = P_0 J_0 y = P_0 \sum_{n=0}^{\infty} P_n J_n J_0 y = P_0 \sum_{n=0}^{\infty} P_n J_n x = 0.$$

As a result, $(I - R)x = (\widetilde{S} + P_0)x = 0$, and, since $\|R\| < 1$, we obtain $x = 0$. Thus, $\ker S = \{0\}$, $\operatorname{nul} S = 0$ and $S$ is continuously invertible. Finally, we note that $J_n = S^{-1} P_n S$, $n \in \mathbb{Z}_+$, implies $\mathfrak{M}_n = S J_n \ell_p$, $n \in \mathbb{Z}_+$, which completes the proof. □

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Note that any Schauder decomposition $\{\mathfrak{N}_n\}_{n=0}^{\infty}$ with the a.s.c.p. $\{F_n\}_{n=0}^{\infty}$ satisfying (4.1) is called a Schauder–Orlicz decomposition in [20] with Orlicz function $\Phi(t) = t^p$. For decompositions of this kind we have the following result.

THEOREM 4.2. *Let $\{\mathfrak{M}_n\}_{n=0}^{\infty}$ be a Schauder–Orlicz decomposition of $\ell_p$, $1 \le p < \infty$, with Orlicz function $\Phi(t) = t^p$ and the a.s.c.p. $\{P_n\}_{n=0}^{\infty}$, where $\dim P_0 < \infty$. Suppose that $\{J_n\}_{n=0}^{\infty}$ is a sequence of nonzero projections in $\ell_p$ satisfying (1.1) such that $J_nJ_m = \delta_n^m J_n$ for $n, m \in \mathbb{Z}_+$. If for each $x \in \ell_p$ one has*

$$\Big(\sum_{n=1}^{\infty} \|P_n(J_n - P_n)x\|^p\Big)^{1/p} \le \varsigma\|x\| \quad \textit{where } 0 \le \varsigma < 1,$$

*then $\{J_n(\ell_p)\}_{n=0}^{\infty}$ is an unconditional Schauder decomposition of $\ell_p$, isomorphic to $\{\mathfrak{M}_n\}_{n=0}^{\infty}$.*

To formulate some stability results for unconditional and symmetric bases in $\ell_p$ spaces we propose the following definition based on Definition 2.2.

DEFINITION 4.3. We call a sequence $\{\phi_n\}_{n=0}^{\infty} \subset E$ an unconditional basis of $E$ with constant $M$ if the sequence of corresponding one-dimensional subspaces $\{\mathrm{Lin}\,\{\phi_n\}\}_{n=0}^{\infty}$ forms an unconditional Schauder decomposition of $E$ with constant $M$.

For example, every orthonormal basis in $H$ is unconditional with constant $M = 1$. In the case $\dim \mathfrak{M}_n = 1, n \in \mathbb{Z}_+$, Theorem 4.1 leads to the following theorem on stability of unconditional bases in $\ell_p$.

THEOREM 4.4. *Let $\{\phi_n\}_{n=0}^{\infty}$ be a bounded unconditional basis of $\ell_p$, $1 \le p < \infty$, with constant $M$ and the associated sequence of coordinate functionals $\{\phi_n^*\}_{n=0}^{\infty}$ such that, for all $x \in \ell_p$, $\langle \phi_0^*, x\rangle\phi_0 = F_0x$, where $\{F_n\}_{n=0}^{\infty}$ is the a.s.c.p. of the Schauder–Orlicz decomposition in $\ell_p$ with Orlicz function $\Phi(t) = t^p$. Also assume that $(\{\psi_n\}_{n=0}^{\infty}, \{\psi_n^*\}_{n=0}^{\infty})$ is a biorthogonal system in $\ell_p$ satisfying*

$$0 < \inf_n \|\psi_n\| \le \sup_n \|\psi_n\| < \infty.$$

*Suppose that, for every $x \in \ell_p$,*

$$\Big(\sum_{n=1}^{\infty} |\langle\psi_n^*, x\rangle\langle\phi_n^*, \psi_n\rangle - \langle\phi_n^*, x\rangle|^p \|\phi_n\|^p\Big)^{1/p} \le \varsigma_1\|x\| \quad \textit{where } 0 \le \varsigma_1 < \frac{1}{2M},$$

*in the case $1 \le p \le 2$, or*

$$\Big(\sum_{n=1}^{\infty} |\langle\psi_n^*, x\rangle\langle\phi_n^*, \psi_n\rangle - \langle\phi_n^*, x\rangle|^2 \|\phi_n\|^2\Big)^{1/2} \le \varsigma_2\|x\| \quad \textit{where } 0 \le \varsigma_2 < \frac{1}{\sqrt{8}M}\Big(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\Big)^{-1/p},$$

*in the case $p \ge 2$ ($\varsigma_2 = \varsigma_2(p)$). Then $\{\psi_n\}_{n=0}^{\infty}$ is also an unconditional basis of $\ell_p$, equivalent to $\{\phi_n\}_{n=0}^{\infty}$.*

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In what follows, we denote by $\{e_n\}_{n=0}^{\infty}$ the natural basis of $\ell_p$, that is, $e_n = (\delta_i^n)$, $n \in \mathbb{Z}_+$. One easily observes that $\{\mathrm{Lin}\,\{e_n\}\}_{n=0}^{\infty}$ is a Schauder–Orlicz decomposition of $\ell_p$ with Orlicz function $\Phi(t) = t^p$. Moreover, it is known that $\ell_p$ spaces, $1 \le p < \infty$, have a unique, up to equivalence, symmetric basis [19, Proposition 3.b.5]. Combining these facts with Theorem 4.2, we obtain the following result.

Theorem 4.5. *Let $(\{\psi_n\}_{n=0}^{\infty}, \{\psi_n^*\}_{n=0}^{\infty})$ be a biorthogonal system in $\ell_p$, $1 \le p < \infty$, satisfying*

$$0 < \inf_n \|\psi_n\| \le \sup_n \|\psi_n\| < \infty.$$

*If, for every $x \in \ell_p$,*

$$\Big(\sum_{n=1}^{\infty} |\langle \psi_n^*, x\rangle\langle e_n^*, \psi_n\rangle - \langle e_n^*, x\rangle|^p\Big)^{1/p} \le \varsigma\|x\| \quad \text{where } 0 \le \varsigma < 1,$$

*then $\{\psi_n\}_{n=0}^{\infty}$ is a symmetric basis of $\ell_p$.*

## 5. Applications

Just as Theorem 1.1 plays a special role in the analysis of spectral properties of nonselfadjoint and unbounded operators in $H$ (see, for example, [1, 2, 7, 12, 17]), Theorems 3.1, 4.1 and 4.2 may be very useful in the study of spectral properties of perturbations of unbounded operators acting on $E$ or $\ell_p$. For this purpose, we consider perturbations of operators generating unconditional spectral Schauder decompositions in $E$ or $\ell_p$, instead of perturbations of selfadjoint operators, which generate orthogonal spectral Schauder decompositions in $H$.

It is known that if $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $H$ then $\{\varphi_n + (1/n)\varphi_{n+1}\}_{n=1}^{\infty}$ is a Riesz basis of $H$; see, for example, [5]. Note that this fact also follows from Theorem 1.1 of Kato. We will show that some similar facts are true for symmetric bases of $\ell_p$. Set $e_{-j} = 0$ for $j \in \mathbb{N}$ and consider the following systems:

$$\psi_n = e_n - \theta_{n-1}e_{n-1} + \theta_{n-1}\theta_{n-2}e_{n-2} + \cdots + (-1)^{n+1}\prod_{k=0}^{n-1}\theta_k \cdot e_0 \quad n \in \mathbb{Z}_+,$$
$$\psi_n^* = e_n + \theta_n e_{n+1} \quad n \in \mathbb{Z}_+.$$

Proposition 5.1. *If $|\theta_n| \le c$, $n \in \mathbb{N}$, where $c < 1$, then $\{\psi_n\}_{n=0}^{\infty}$ and $\{\psi_n^*\}_{n=0}^{\infty}$ are symmetric bases of $\ell_p$, $1 \le p < \infty$.*

Proof. A trivial computation shows that $(\{\psi_n\}_{n=0}^{\infty}, \{\psi_n^*\}_{n=0}^{\infty})$ is a biorthogonal system in $\ell_p$, $1 \le p < \infty$, such that $0 < \inf_n \|\psi_n\| \le \sup_n \|\psi_n\| < \infty$. Obviously,

$$\langle e_n^*, \psi_n\rangle = 1, \quad \langle \psi_n^*, x\rangle = \langle e_n^*, x\rangle + \overline{\theta}_n\langle e_{n+1}^*, x\rangle \quad n \in \mathbb{Z}_+.$$

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Hence,

$$\Big(\sum_{n=1}^{\infty} |\langle \psi_n^*, x\rangle\langle e_n^*, \psi_n\rangle - \langle e_n^*, x\rangle|^p\Big)^{1/p} = \Big(\sum_{n=1}^{\infty} |\overline{\theta}_n \langle e_{n+1}^*, x\rangle|^p\Big)^{1/p} \le c\Big(\sum_{n=1}^{\infty} |\langle e_{n+1}^*, x\rangle|^p\Big)^{1/p}$$
$$\le c\Big(\sum_{n=0}^{\infty} |\langle e_n^*, x\rangle|^p\Big)^{1/p} = c\|x\|.$$

By virtue of Theorem 4.5, $\{\psi_n\}_{n=0}^{\infty}$ is a symmetric basis of $\ell_p$, $1 \le p < \infty$. Consequently, $\{\psi_n^*\}_{n=0}^{\infty}$ is a symmetric basis of $\overline{\mathrm{Lin}}\,\{\psi_n^*\}_{n=0}^{\infty}$ in $\ell_q$, where $1/p + 1/q = 1$ (see [24, Proposition 22.5]). Since every basis in a reflexive space is shrinking (see, for example, [24, page 278, Example 4.3]), $\{\psi_n^*\}_{n=0}^{\infty}$ is a symmetric basis of the whole of $\ell_q$, where $1 < q < \infty$. To prove that $\{\psi_n^*\}_{n=0}^{\infty}$ is a symmetric basis of $\ell_1$, we observe that $\{\psi_n^*\}_{n=0}^{\infty}$ is a bounded unconditional basis of $\ell_1$. Therefore, by [24, Theorem 18.2], $\{\psi_n^*\}_{n=0}^{\infty}$ is equivalent to the natural basis of $\ell_1$ and, by [19, Proposition 3.b.5], $\{\psi_n^*\}_{n=0}^{\infty}$ is a symmetric basis of $\ell_1$. □

In particular, from Proposition 5.1, $\{e_0 + e_1\} \cup \{e_n + (1/(n+1))e_{n+1}\}_{n=1}^{\infty}$ forms a symmetric basis of $\ell_p$, $1 \le p < \infty$.

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