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PROPERTIES OF THE TRAJECTORIES OF SET-VALUED INTEGRALS IN BANACH SPACES

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Let $F: T \to 2^X \setminus \{\emptyset\}$ be a closed-valued multifunction into a separable Banach space X. We define the sets $K(t) = \int_0^t F(s)ds \subseteq X$ and $C_F = \{x(\cdot) \in C(T,X): x(t) = \int_0^t f(s)ds,\ t \in T,\ f \in S_F^1\}$. We prove various convergence theorems for those two sets using the Hausdorff metric and the Kuratowski–Mosco convergence of sets. Then we prove a density theorem of C_F in$C_{\overline{\text{conv}}F}$ and a corresponding convexity theorem for $F(\cdot)$. Finally we study the "differentiability" properties of $K(\cdot)$. Our work extends and improves earlier ones by Artstein, Bridgland, Hermes and Papageorgiou.

1. INTRODUCTION

The purpose of this paper is to study the indefinite set-valued integral $K(t) = \int_0^t F(s)ds$ of a multifunction $F: T = [0,b] \to 2^X \setminus \{\emptyset\}$ and the corresponding set of continuous functions $C_F = \{x(\cdot) \in C(T,X): x(t) = \int_0^t f(s)ds,\ t \in T,\ f \in S_F^1\}$, where S_F^1 denotes the set of integrable selectors of $F(\cdot)$. It is clear that the distinction between C_F and $K(t)$ is essentially that between a function and the value of the function at t.

The importance of those sets lies on the fact that, up to an appropriate translation, they are the attainable set at time t of a semilinear control system (for $K(t)$) and the set of trajectories of the system (for C_F). Also the study of the properties of those two sets can be helpful in the solution of differential equations with multivalued right hand side (differential inclusions). Such generalised differential equations arise naturally in control theory, where, by taking the union over all admissible controls of the corresponding control vector field, we transform the differential equation into a differential inclusion in which the control does not appear explicitly (deparametrisation). Furthermore, the study of the sets $K(t)$ and C_F, complements the study of the set-valued integral (also known in the literature as the "Aumann integral"), which is an important analytical tool in control theory (see Aubin and Cellina [2]), in mathematical economics (see Klein and Thompson [9]) and in statistics (see Richter [13]). Finally in this work, we also make some contributions to the "differentiation" theory of multifunctions.

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Earlier work on these topics was done by Hermes [**7**], Bridgland [**3**], Artstein [**1**] and Papageorgiou [**10**]. The first three dealt with $\mathbf{R}^n$-valued multifunctions, while the fourth considered also Banach space valued multifunctions. Here we continue the work started in [**10**], we improve some of the results in that paper and we also obtain new ones.

2. Preliminaries

Let $(\Omega, \sum)$ be a measurable space and X a separable Banach space. Throughout this work we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X\colon \text{ nonempty, closed, (convex)}\}$$

and

$$P_{(w)k(c)}(X) = \{A \subseteq X\colon \text{ nonempty, } (w-) \text{ compact, (convex)}\}.$$

A multifunction (set-valued function) $F\colon \Omega \to P_f(X)$ is said to be *measurable* if it satisfies one of the following two equivalent conditions:

(i) for every $z \in X$, $\omega \to d(z, F(\omega)) = \inf\{\|z - x\| : x \in F(\omega)\}$ is measurable;

(ii) there exist $f_n\colon \Omega \to X$ measurable functions such that $F(\omega) = cl\{f_n(\omega)\}_{n \geqslant 1}$ for all $\omega \in \Omega$.

If $(\Omega, \sum)$ admits a σ-finite measure $\mu(\cdot)$ with respect to which $\sum$ is complete, then (i) and (ii) above are equivalent to

(iii) $GrF = \{(\omega, x) \in \Omega \times X\colon x \in F(\omega)\} \in \sum \times B(X)$, where $B(X)$ is the Borel σ-field of X (graph measurability).

Now let $\mu(\cdot)$ be a complete σ-finite measure on $(\Omega, \sum)$. By S_F^1 we will denote the set of integrable selectors of the multifunction $F(\cdot)$ that is $S_F^1 = \{f \in L^1(X)\colon f(\omega) \in F(\omega)\mu\text{–a.e.}\}$. This set may be empty. Using Auman's selection theorem (see Wagner [**16**]), we can show that S_F^1 is nonempty if and only if $\inf\{\|x\| : x \in F(\omega)\} \in L_+^1$. This is the case if $\sup\{\|x\| : x \in F(\omega)\} \in L_+^1$. Such a multifunction is said to be *integrably bounded*. Using S_F^1 we can define a set-valued integral for $F(\cdot)$ by setting $\int_\Omega F(\omega)d\mu(\omega) = \{\int_\Omega f(\omega)d\mu(\omega)\colon f \in S_F^1\}$.

On $P_f(X)$ we can define a (generalised) metric, known as the *Hausdorff metric*, by setting

$$h(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

It is well-known that $P_f(X)$ and $P_{fc}(X)$ are complete metric spaces with the Hausdorff metric $h(\cdot, \cdot)$. We say that a sequence $\{A_n\}_{n \geqslant 1} \subseteq P_f(X)$ *h-converges to* A, denoted by $A_n \xrightarrow{h} A$, if and only if $h(A_n, A) \to 0$. On $2^X \setminus \{\emptyset\}$, $h(\cdot, \cdot)$ is a pseudometric.

Another notion of set convergence that we will need in this paper is the so-called *Kuratowski–Mosco convergence* of sets ($K-M$ convergence). Let $\{A_n\}_{n\geqslant 1} \subseteq 2^X \setminus \{\emptyset\}$. We define

$$w-\overline{\lim} A_n = \{x = w - \lim x_{n_k} : x_{n_k} \in A_{n_k} \& n_1 < n_2 < \ldots < n_k < \ldots\}$$
$$\text{and } s-\underline{\lim} A_n = \{x = s - \lim x_n : x_n \in A_n \& n \geqslant 1\}.$$

Here w stands for the weak topology on X and s for the strong topology on X. We say that the A_n's converge to A in the Kuratowski–Mosco sense, denoted by $A_n \xrightarrow{K-M} A$, if and only if $w-\overline{\lim} A_n \subseteq A \subseteq s-\underline{\lim} A_n$. Given that we always have $s-\underline{\lim} A_n \subseteq w-\overline{\lim} A_n$, we can say that $A_n \xrightarrow{K-M} A$ if and only if $w-\overline{\lim} A_n = A = s-\underline{\lim} A_n$.

Let $A \in 2^X \setminus \{\emptyset\}$. The *support function of* A is the function $\sigma_A \colon X^* \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by $\sigma_A(x^*) = \sup\{(x^*, a) \colon a \in A\}$, $x^* \in X^*$. A set-valued function $M \colon \sum \to P_f(X)$ is said to be a *multimeasure*, if for all $x^* \in X^* A \to \sigma_{M(A)}(x^*)$ is a signed measure. Finally if $A \in 2^X \setminus \{\emptyset\}$, we write $|A| = \sup\{\|a\| : a \in A\}$ (the *norm* of A).

3. Convergence results

In this section we prove some convergence results for the sets $K(t)$ and C_F, that extend the finite-dimensional ones of Bridgland [**3**] and the infinite-dimensional ones in [**10**], where the assumptions are much stronger (the underlying state space is separable, reflexive, while the multifunctions have compact, convex values).

So let $T = [0, b]$ and let X be a separable Banach space.

Theorem 3.1. *If F_n, $F \colon T \to P_f(X)$ are integrably bounded multifunctions be such that $\{|F_n(\cdot)|\}_{n\geqslant 1}$ is uniformly integrable and $F_n(t) \xrightarrow{h} F(t)$ a.e., then $C_{F_n} \xrightarrow{h} C_F$.*

Proof: Let $y \in C_F$. Then by definition (see Section 1) we have $y(t) = \int_0^t f(s)ds$, $t \in T$, $f \in S_F^1$. We have:

$$d(y, C_{F_n}) = \inf\{\|y - y_n\|_\infty : y_n \in C_{F_n}\}.$$

Note that $y_n(t) = \int_0^t f_n(s)ds, t \in T\, f_n \in S_{F_n}^1$. We have:

$$\|y - y_n\| = \sup_{t\in T} \left\| \int_0^t f(s)ds - \int_0^t f_n(s)ds \right\| \leqslant \sup_{t\in T} \int_0^t \|f(s) - f_n(s)\|\, ds$$
$$= \int_0^b \|f(s) - f_n(s)\|\, ds.$$

Then using Theorem 2.2 of Hiai–Umegaki [**8**], we get

(1)
$$\begin{aligned}\inf\{\|y-y_n\|_\infty : y_n \in C_{F_n}\} &\leqslant \inf\{\int_0^b \|f(s)-f_n(s)\|\,ds\colon f_n \in S^1_{F_n}\}\\ &= \int_0^b \inf\{\|f(s)-z\| : z \in F_n(s)\}ds\\ &= \int_0^b d(f(s),\, F_n(s))ds \leqslant \int_0^b h(F_n(s),\, F(s))ds.\\ &\Longrightarrow d(y, C_{F_n}) \leqslant \int_0^b h(F_n(s),\, F(s))ds.\end{aligned}$$

Similarly, by interchanging the rôles of $F(\cdot)$ and $F_n(\cdot)$ in the above argument, we get that for all $y_n \in C_{F_n}$

(2)
$$d(y_n,\, C_F) \leqslant \int_0^b h(F_n(s),\, F(s))ds.$$

From (1) and (2) above, we deduce that

$$h(C_{F_n},\, C_F) \leqslant \int_0^b h(F_n(s),\, F(s))ds.$$

Note that $h(F_n(s),\, F(s)) \leqslant |F_n(s)| + |F(s)|$ and by hypothesis $\{|F_n(\cdot)|\}_{n\geqslant 1}$ is uniformly integrable, while $h(F_n(s),\, F(s)) \to 0$ a.e. as $n \to \infty$. So from the extended dominated convergence theorem we get that $\int_0^b h(F_n(s),\, F(s))ds \to 0$ as $n \to \infty$ $\Longrightarrow h(C_{F_n},\, C_F) \to 0$ as $n \to \infty$. ∎

REMARKS. (a) The sets C_{F_n}, C_F are not in general closed. They will be if the multifunctions $F_n(\cdot)$ and $F(\cdot)$ are $P_{wkc}(X)$-valued and integrably bounded. We will show this for C_F, the proof being the same for C_{F_n}, $n \geqslant 1$. So let $\{y_m\}_{m\geqslant 1} \subseteq C_F$ be such that $y_m \to y$ in $C(T, X)$. Then by definition $y_m(t) = \int_0^t f_m(s)ds$, $t \in T$, $f_m \in S^1_F$. From Proposition 3.1 of [**11**], we know that S^1_F is w-compact in $L^1(X)$ and, by the Eberlein–Smulian theorem, sequentially w-compact. So, by passing to a subsequence if necessary, we may assume that $f_m \xrightarrow{w} f$ in $L^1(X)$. So $\int_0^t f_m(s)ds \xrightarrow{w} \int_0^t f(s)ds$ for all $t \in T \Longrightarrow y(t) = \int_0^t f(s)ds$, $t \in T$, $f \in S^1_F \Longrightarrow y \in C_F \Longrightarrow C_F$ is closed. Also Theorem 3.1 of [**10**] tells us that C_F is sequentially compact in $C(T,\, X_w)$.

(b) Our result extends Theorem 3.2 of Bridgland [**3**]. Note that even when $X = \mathbb{R}^n$, our result is more general than that of Bridgland since we do not require the F_n's to be uniformly pointwise bounded. Furthermore in Bridgland [**3**], $F_n(\cdot)$ and $F(\cdot)$ are convex-valued.

Let $K_n(t) = \int_0^t F_n(s)ds$ and $K(t) = \int_0^t F(s)ds$ for $t \in T$.

Theorem 3.2. *If the hypotheses of Theorem 3.2 hold, then for every* $t \in T$, $K_n(t) \xrightarrow{h} K(t)$ as $n \to \infty$.

Proof: Let $e_t(\cdot)\colon C(T,X) \to X$ be the evaluation map at t. Then $K_n(t) = e_t(C_{F_n})$ and $K(t) = e_t(C_F)$. From the properties of the Hausdorff metric we have

$$h(K_n(t),\, K(t)) \leqslant \|e_t\|\, h(C_{F_n},\, C_F) \to 0 \text{ as } n \to \infty.$$

$\square$

Next we will derive similar convergence results using the $K-M$ convergence of sets.

Theorem 3.3. *If* $F_n,\, F\colon T \to P_{fc}(X)$ *are measurable multifunctions such that* $F_n(t) \subseteq G(t)$ *a.e. with* $G\colon T \to P_{wkc}(X)$ *integrably bounded and* $F_n(t) \xrightarrow{K-M} F(t)$ *a.e., then* $C_{F_n} \xrightarrow{K-M} C_F$.

Proof: Let $y \in w-\overline{\lim}\, C_{F_n}$. Then by definition (see Section 2), we can find $y_k \in C_{F_{n_k}}$ such that $y_k \xrightarrow{w} y$ in $C(T,X)$. Hence for every $t \in T$, $y_k(t) \xrightarrow{w} y(t)$ in X. For every $k \geqslant 1$ we have $y_k(t) = \int_0^t f_k(s)ds$, $t \in T$, $f_k \in S^1_{F_{n_k}}$. Since $S^1_{F_{n_k}} \subseteq S^1_G$ and the latter is sequentially w-compact in $L^1(X)$, by passing to a subsequence if necessary, we may assume that $f_k \xrightarrow{w} f$ in $L^1(X)$. Invoking Theorem 4.4 of [**12**], we get $f \in S^1_F$. So $y(t) = \int_0^t f(s)ds$, $t \in T$, $f \in S^1_F \Longrightarrow y \in C_F$. Therefore we have:

$$w-\overline{\lim}\, C_{F_n} \subseteq C_F. \tag{1}$$

Next let $y \in C_F$. Then $y(t) = \int_0^t f(s)ds$, $t \in T$, $f \in S^1_F$. Once again Theorem 4.4 of [**12**] tells us that we can find $f_n(\cdot) \in S^1_{F_n}$ such that $f_n \xrightarrow{s} f$ in $L^1(X)$. Hence if $y_n(t) = \int_0^t f_n(s)ds\, t \in T$, we have $\|y_n(t) - y(t)\| \leqslant \int_0^b \|f_n(s) - f(s)\|\, ds$ for all $t \in T \Longrightarrow y_n \to y$ in $C(T,X)$ with $y_n \in C_{F_n}$. Thus we get that

$$C_F \subseteq s-\underline{\lim}\, C_{F_n}. \tag{2}$$

From (1) and (2) above we conclude that $C_{F_n} \xrightarrow{K-M} C_F$. $\square$

We have a corresponding convergence result for the indefinite set-valued integrals $K_n(\cdot)$ and $K(\cdot)$.

Theorem 3.4. *If the hypotheses of Theorem 3.3 hold, then for every* $t \in T$, $K_n(t) \xrightarrow{K-M} K(t)$ *as* $n \to \infty$.

Proof: Let $z \in w-\overline{\lim}\, K_n(t)$. Then there exist $z_k \in K_{n_k}(t)$ such that $z_k \xrightarrow{w} z$. We have:

$$z_k = \int_0^t f_k(s)ds,\ f_k \in S^1_{F_{n_k}}.$$

By passing to a subsequence if necessary, we may assume that $f_k \xrightarrow{w} f$ in $L^1(X)$. From Theorem 4.4 of [**12**], we have $f \in S_F^1$. Then $z_k = \int_0^t f_k(s)ds \xrightarrow{w} \int_0^t f(s)ds = z \in K(t)$. So we have

$$w - \overline{\lim} K_n(t) \subseteq K(t). \tag{1}$$

Next let $z \in K(t)$. Then $z = e_t(y)$, $y \in C_F$. From Theorem 3.3 we know that we can find $y_n \in C_{F_n}$ such that $y_n \to y$ in $C(T, X) \Longrightarrow e_t(y_n) = z_n \to z = e_t(y)$ and $z_n \in K_n(t)$. So we deduce that

$$K(t) \subseteq s - \underline{\lim} K_n(t). \tag{2}$$

From (1) and (2) above we conclude that $K_n(t) \xrightarrow{K-M} K(t)$ as $n \to \infty$. ∎

REMARK. Our result generalises Corollary 3.3 of Bridgland [**3**].

4. Convexity theorems

In this Section we present two theorems that can be useful in control theory, in particular in connection with the "bang-bang" principle.

So we have:

Theorem 4.1. *If $F: T \to P_f(X)$ is a measurable multifunction such that $S_F^1 \neq \emptyset$, then $\overline{C}_F = \overline{C}_{\overline{\text{conv}} F}$, the closures taken in $C(T, X)$.*

PROOF: Let $y \in C_{\overline{\text{conv}} F}$. Then by definition $y(t) = \int_0^t f(s)ds$, $t \in T$, $f \in S^1_{\overline{\text{conv}} F}$. Invoking Theorem 2 of Chuong [4], we can find $g \in S_F^1$ such that

$$\sup_{t,t' \in T} \left\| \int_t^{t'} (g(s) - f(s))ds \right\| < \varepsilon$$
$$\Longrightarrow \left\| \int_0^t (g(s) - f(s))ds \right\| < \varepsilon \text{ for all } t \in T.$$

Set $x(t) = \int_0^t g(s)ds$, $t \in T$. Clearly $x(\cdot) \in C_F$. We have:

$$\|x - y\|_\infty < \varepsilon$$
$$\Longrightarrow \overline{C}_F = \overline{C}_{\overline{\text{conv}} F}.$$

∎

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REMARK. If $F(\cdot)$ is $P_{wk}(X)$-valued and integrably bounded, then from the Krein–Smulian theorem, $\overline{\text{conv}}F(t) \in P_{wkc}(X)$ and $t \to \overline{\text{conv}}F(t)$ is integrably bounded. So from Remark (a) following Theorem 3.1, we get that $C_{\overline{\text{conv}}F}$ is closed in $C(T, X)$. Our result extends Theorem 3.3 of Bridgland [3]. Even when $X = \mathbb{R}^n$ our result is stronger than that of Bridgland since we do not assume $F(\cdot)$ to be integrably bounded.

Theorem 4.1 above, suggests that there is a close relation between the convexity of the values of $F(\cdot)$ and the closedness of C_F in $C(T, X)$. This relation is revealed in the next theorem.

THEOREM 4.2. *If $F: T \to P_f(X)$ is a measurable multifunction such that $S_F^1 \neq \emptyset$ and C_F is closed in $C(T, X)$, then $F(t)$ is a.e. convex.*

PROOF: Let $|\cdot|_w$ denote the weak norm on $L^1(X)$ that is $|f|_w = \sup_{t',t\in T} \left\| \int_t^{t'} f(s)ds \right\|$ (see Chuong [4]). We claim that S_F^1 is $|\cdot|_w$-closed. So let $\{f_n\}_{n\geqslant 1} \subseteq S_F^1$ and assume that $f_n \xrightarrow{|\cdot|_w} f$. This means that $y_n \xrightarrow{\|\cdot\|_\infty} y$, where $y_n(t) = \int_0^t f_n(s)ds$ and $y(t) = \int_0^t f(s)ds$. But C_F is closed in $C(T, X)$. So $y \in C_F \Longrightarrow f \in S_F^1$. Invoking Theorem 2 of Chuong [4], we get that $S_F^1 = S_{\overline{\text{conv}}F}^1 \Longrightarrow F(t) = \overline{\text{conv}}F(t)$ a.e. $\square$

5. DIFFERENTIATION OF MULTIFUNCTIONS

In this Section we examine some "differentiation" properties of the multifunction $K(t) = \int_0^t F(s)ds$, $t \in T$.

Our first result extends Theorem 4.3 of Artstein [1].

THEOREM 5.1. *If X^* is separable and $F: T \to P_{wkc}(X)$ is integrably bounded, then $1/h \int_t^{t+h} F(s)ds \xrightarrow{K-M} F(t)$ a.e. as $h \to 0$.*

PROOF: Let $R_h(t) = 1/h \int_t^{t+h} F(s)ds$. From the Corollary to Proposition 3.1 in [11] we have that $R_h(t) \in P_{wkc}(X)$ for all $t \in T$.

Let $x^* \in X^*$. We have:

$$\sigma_{R_h(t)}(x^*) = \frac{1}{h} \int_t^{t+h} \sigma_{F(s)}(x^*)ds.$$

From the properties of the Lebesgue integral, we know that

$$\frac{1}{h} \int_t^{t+h} \sigma_{F(s)}(x^*)ds \to \sigma_{F(t)}(x^*)$$

for all $t \in T \setminus N(x^*)$, $\lambda(N(x^*)) = 0$, where $\lambda(\cdot)$ is the Lebesgue measure on T. Let $\{x_m^*\}_{m\geqslant 1}$ be dense in X^*. Set $N_1 = \bigcup_{m\geqslant 1} N(x_m^*)$, $\lambda(N_1) = 0$. Also let $N_2 \subseteq T$

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be such that $\lambda(N_2) = 0$ and for $t \in T \setminus N_2$, we have $1/h \int_t^{t+h} |F(s)|\,ds \to |F(t)|$. Set $N_3 = N_1 \cup N_2$. Then $\lambda(N_3) = 0$. We claim that for every $t \in T \setminus N_3$ and for every $x^* \in X^*$, we have $\sigma_{R_h(t)}(x^*) \to \sigma_{F(t)}(x^*)$ as $h \to 0$. To this end let $\{x_k^*\}_{k \geqslant 1} \subseteq \{x_m^*\}_{m \geqslant 1}$ be such that $x_k \xrightarrow{s} x^*$. Also let $h_n \to 0$ and set $R_n(t) = R_{h_n}(t)$. We have for $t \in T \setminus N_3$ and $k \geqslant 1$

$$\sigma_{R_n(t)}(x_k^*) \to \sigma_{F(t)}(x_k^*) \text{ as } n \to \infty.$$

Furthermore because of the strong continuity of the support function, we have for $t \in T \setminus N_3$

$$\sigma_{F(t)}(x_k^*) \to \sigma_{F(t)}(x^*).$$

By diagonalisation, we get

$$\sigma_{R_n(t)}\left(x_{k(n)}^*\right) \to \sigma_{F(t)}(x^*).$$

Hence for $t \in T \setminus N_3$, we have:

$$\begin{aligned} \left|\sigma_{R_n(t)}(x^*) - \sigma_{F(t)}(x^*)\right| &\leqslant \left|\sigma_{R_n(t)}(x^*) - \sigma_{R_n(t)}\left(x_{k(n)}^*\right)\right| + \left|\sigma_{R_n(t)}\left(x_{k(n)}^*\right) - \sigma_{F(t)}(x^*)\right| \\ &\leqslant |R_n(t)| \cdot \left\|x^* - x_{k(n)}^*\right\| + \left|\sigma_{R_n(t)}\left(x_{k(n)}^*\right) - \sigma_{F(t)}(x^*)\right|. \end{aligned}$$

But $|R_n(t)| \leqslant 1/h_n \int_t^{t+h_n} |F(s)|\,ds$ and $\{1/h \int_t^{t+h_n} |F(s)|\,ds\}_{n \geqslant 1}$ is convergent for $t \in T \setminus N_3$. So there exists $M(t) > 0$ such that $|R_n(t)| \leqslant M(t)$ for all $n \geqslant 1$. Hence we have

$$\left|\sigma_{R_n(t)}(x^*) - \sigma_{F(t)}(x^*)\right| \leqslant M(t)\left\|x^* - x_{k(n)}^*\right\| + \left|\sigma_{R_n(t)}\left(x_{k(n)}^*\right) - \sigma_{F(t)}(x^*)\right| \to 0\, n \to \infty.$$

Therefore for all $x^* \in X^*$ and all $t \in T \setminus N_3$ we have:

$$\lim \sigma_{R_n(t)}(x^*) = \sigma_{F(t)}(x^*).$$

Invoking Proposition 4.1 of [**12**], we get

$$w - \overline{\lim}\, R_h(t) \subseteq F(t) \text{ a.e.}. \tag{1}$$

Next let $z \in X$ and let $f_k \colon T \to X$ be measurable functions such that $F(t) = cl\{f_k(t)\}_{k \geqslant 1}$. We have

$$d(z, R_n(t)) \leqslant \left\| z - \frac{1}{h_n} \int_t^{t+h_n} f_k(s)ds \right\| k \geqslant 1.$$

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We know that there exists a Lebesgue null set $\widehat{N}_k$ where $k \geqslant 1$ such that for $t \in T \setminus \widehat{N}_k$ we have

$$\frac{1}{h_n} \int_t^{t+h_n} f_k(s)ds \to f_k(t) \text{ as } n \to \infty.$$

Set $N_4 = \bigcup_{k \geqslant 1} \widehat{N}_k$. Then $\lambda(N_4) = 0$ and for $t \in T \setminus N_4$ we have

$$\left\| z - \frac{1}{h_n} \int_t^{t+h_n} f_k(s)ds \right\| \to \|z - f_k(t)\|.$$

Hence we have:

$$\begin{aligned} &\overline{\lim}\, d(z, R_n(t)) \leqslant \|z - f_k(t)\| \text{ a.e. for all } k \geqslant 1 \\ \Longrightarrow &\overline{\lim}\, d(z, R_n(t)) \leqslant d(z, F(t)) \text{ a.e. .} \end{aligned}$$

From Theorem 2.1 of Tsukada [**15**], we get that for all $t \in T \setminus N_4$, we have

$$F(t) \subseteq s - \underline{\lim}\, R_n(t) \text{ a.e. .} \tag{2}$$

Combining (1) and (2) above we conclude that

$$\frac{1}{h_n} \int_t^{t+h} F(s)ds \xrightarrow{K-M} F(t) \text{ a.e. .}$$

$\square$

Remark. When $\dim X < \infty$, from Corollary 3A of Salinetti and Wets [**14**], we deduce that

$$\frac{1}{h_n} \int_t^{t+h} F(s)ds \xrightarrow{h} F(t) \text{ a.e. .}$$

In [**6**] DeBlasi introduced a derivative for multifunctions, which is useful in the perturbation theory of ordinary differential equations. By this definition [**6**, Definition 2.5] a multifunction $F: T \to 2^X \setminus \{\emptyset\}$ with bounded values is *differentiable at* $t_0 \in T$, if there exists a map $D_0: T \to P_{fc}(X)$ which is upper semicontinuous and homogeneous and a $\delta > 0$ such that

$$h(F(t+h), F(t) + D_0(h)) = O(h), \text{ when } \|h\| < \delta.$$

An interesting consequence of Theorem 5.1 is the following result:

Theorem 5.2. *If* $\dim X < \infty$ *and* $F\colon T \to P_f(X)$ *is integrably bounded then* $K(\cdot)$ *is DeBlasi differentiable at almost all* $t \in T$.

Proof: We have:

$$\begin{aligned}
&\frac{1}{h} h(K(t+h),\, K(t) + h\,\overline{\operatorname{conv}}\, F(t)) \\
=&\frac{1}{h} h\left(K(t) + \int_t^{t+h} F(s)ds,\, K(t) + h\,\overline{\operatorname{conv}}\, F(t)\right) \\
\leqslant& h\left(\frac{1}{h}\int_t^{t+h} F(s)ds,\, \overline{\operatorname{conv}}\, F(t)\right) \to 0 \text{ a.e. (see the remark following Theorem 5.1).}
\end{aligned}$$

So $K(\cdot)$ is almost everywhere DeBlasi differentiable. ∎

We will conclude this paper, with a theorem that tells us when a multifunction can be expressed as the indefinite set-valued integral of an integrable one. Our result extends Theorem 4.1 of Artstein [**1**].

Theorem 5.3. *If* X *and* X^* *both have the Radon–Nikodym property and* $K\colon T \to T_{fc}(X)$ *is a measurable multifunction such that*

(a) $K(t) \subseteq W \in P_{wkc}(X)$ *for all* $t \in T$,
(b) $|\sigma(x^*) - \sigma(x^*)| \leqslant \|x^*\|\,(\phi(t') - \phi(t))0 \leqslant t \leqslant t' \leqslant b$ *and* $\phi\colon T \to \mathbb{R}$ *is an increasing absolutely continuous function, and*
(c) *for every* t', $t \in T$, $t < t'$, $x^* \to \sigma_{K(t')}(x^*) - \sigma_{K(t)}(x^*)$ *is sublinear, then there exists* $F\colon T \to P_{wkc}(X)$ *integrably bounded such that* $K(t) = \int_0^t F(s)ds$, $t \in T$.

Proof: Let $m(x^*)(\cdot)$ and $\mu(\cdot)$ be the bounded variation measures on $\left(T, \widehat{B(T)}\right)$ ($\widehat{B(T)}$ being the Lebesgue σ-field of T), corresponding to functions $\sigma_{K(\cdot)}(x^*)$ and $\phi(\cdot)$ respectively. By considering instead $\widehat{W} = \overline{\operatorname{conv}}(W \cup (-W)) \in P_{wkc}(X)$, we may assume without any loss of generality that W is absolutely convex. Because of hypothesis (a) we have $|m(x^*)(A)| \leqslant 2\sigma_W(x^*)$ for all $A \in \widehat{B(T)}$. Since, by hypothesis (c), $x^* \to m(x^*)(A)$ is sublinear, the above inequaltiy tells us that $x^* \to m(x^*)(A)$ is $m(X^*, X)$-continuous (here $m(X^*, X)$ denotes the Mackey topology for the dual pair (X^*, X)). So there exists $M(A) \in P_{wkc}(X)$ such that $m(x^*)(A) = \sigma_{M(A)}(x^*) \Longrightarrow M(\cdot)$ is a multimeasure with values in $P_{wkc}(X)$. Furthermore, note that because of hypothesis (b), we have

$$\left|\sigma_{M(A)}(x^*)\right| \leqslant \|x^*\| \cdot \mu(A)$$

and since by hypothesis $\phi(\cdot)$ is absolutely continuous, $\mu << \lambda$ and so $M << \lambda$. Thus we can apply Theorem 3 of Costé [**5**], to get $F\colon \Omega \to P_{wkc}(X)$ integrably bounded such that $M(A) = \int_A F(s)ds$ for all $A \in B(T)$. Since $\sigma_{K(t)}(x^*) = \sigma_{M(A)}(x^*)$ for $A = [0, t]$, we conclude that $K(t) = \int_0^t F(s)ds$, $t \in T$. $\square$

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