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REAL ZEROS OF ALGEBRAIC POLYNOMIALS WITH STABLE RANDOM COEFFICIENTS

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Abstract

We consider a random algebraic polynomial of the form $P_{n,\theta,\alpha}(t) = \theta_0\xi_0 + \theta_1\xi_1 t + \cdots + \theta_n\xi_n t^n$, where ξ_k, $k = 0, 1, 2, \ldots, n$ have identical symmetric stable distribution with index α, $0 < \alpha \leq 2$. First, for a general form of $\theta_{k,\alpha} \equiv \theta_k$ we derive the expected number of real zeros of $P_{n,\theta,\alpha}(t)$. We then show that our results can be used for special choices of θ_k. In particular, we obtain the above expected number of zeros when $\theta_k = \binom{n}{k}^{1/2}$. The latter generate a polynomial with binomial elements which has recently been of significant interest and has previously been studied only for Gaussian distributed coefficients. We see the effect of α on increasing the expected number of zeros compared with the special case of Gaussian coefficients.

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1. Introduction

Let $\{\xi_k\}_{k=0}^n$ be a sequence of independent identically distributed random variables. The classical random algebraic polynomial defined as $Q_n(t) = \sum_{k=0}^n \xi_k t^k$ has been well studied. However, most of the known results are for the case of Gaussian distributed coefficients ξ_k. Assuming symmetric distribution for the ξ_k and large n, Kac [5] found an asymptotic estimate for the expected number of real zeros of $Q_n(t)$ in the entire real line as $EN_{Q,n}(-\infty, \infty) \sim (1/2\pi)\log n$. This is significantly fewer than that of random trigonometric polynomial $T_n(t) = \sum_{k=0}^n \xi_k \cos kt$ which has $EN_{T,n}(0, 2\pi) \sim n/\sqrt{3}$ zeros. The error terms occurring in the above asymptotic formula are reduced to $O(1)$ in the two interesting papers by Wilkins in [9] for the algebraic case and in [10] for the trigonometric polynomials. There are reviews on the earlier results for the above polynomials and related topics in [1] and more recent works in [3].

Until recently the above classes of polynomials were most commonly studied, and their various characteristics, such as their level crossings and maxima (minima), were considered. Motivated by some physical applications, stated in Ramponi [8], Edelman and Kostlan [2] introduced polynomials of the form

$$P_n(t) = \sum_{k=0}^{n} \xi_k \binom{n}{k}^{1/2} t^k. \tag{1}$$

They found that in this case the expected number of real zeros that $P_n(t)$ possesses is significantly more than $Q_n(t)$ but fewer than $T_n(t)$. For the symmetric Gaussian coefficients they found $EN_{P,n}(-\infty, \infty) \sim \sqrt{n}$. Interestingly, unlike classical algebraic polynomials $Q_n(t)$, this asymptotic value persists for the case of level crossings or when the number of maxima (minima) is considered. The variance of the kth term of $P_n(t)$, which is $\binom{n}{k}$, increases to its maximum at the middle term. This initiated another type of random algebraic polynomials with this property in [4]. However, so far the only class of polynomials with $EN(-\infty, \infty) \sim \sqrt{n}$ is in the form of $P_n(t)$ given in (1).

As for distributions other than Gaussian, very little is known. This is not surprising as the analysis involved for any other distribution becomes complicated. One can see the latter in the works of Logan and Shepp [6, 7] where, for distributions not belonging to the domain of attraction of normal law, it is shown that there is a slight increase in the expected number of real zeros for $Q_n(t)$. However, the order of $\log n$ obtained previously in the Gaussian case persists. Therefore it is of interest, and natural to ask, whether this increase in the expected number of real zeros will also remain the same for the polynomials with binomial elements $P_n(t)$ given in (1). We will see, as a consequence of our results here, that this is in fact the case. A very general case considered in Theorem 1 and a formula for the expected number of real zeros is given. Then for a special case, which is of interest as mentioned above, Theorem 2 gives an asymptotic formula for EN_n. We prove the following result.

THEOREM 1. *Let $P_{n,\theta,\alpha}(t) = \sum_{k=0}^{n} \theta_k \xi_k t^k$ where $\theta_{n,k} \equiv \theta_k$ is a variable independent of t and ξ_k have identical symmetric stable distribution with index α, $0 < \alpha \le 2$. Then the expected number of real zeros of $P_{n,\theta,\alpha}(t)$ is*

$$EN_{n,\theta,\alpha}(-\infty, \infty) = \frac{2}{\pi^2 \alpha} \int_0^\infty \frac{dt}{t} \int_{-\infty}^{\infty} \log \left\{ \frac{\sum_{k=0}^{n} \pi_k |u-k|^\alpha}{|\sum_{k=0}^{n} \pi_k (u-k)|^\alpha} \right\} du$$

where

$$\Pi_{k,\alpha} = \pi_k = \frac{t^{\alpha k} \theta_k^\alpha}{\sum_{j=0}^{n} t^{\alpha j} \theta_j^\alpha}. \tag{2}$$

Now, in Theorem 1, we assume $\theta_k^\alpha = \binom{n}{k}$, which leads to a natural definition for

$$P_{n,\alpha}(t) = \sum_{k=0}^{n} \binom{n}{k}^{1/\alpha} \xi_k t^k. \tag{3}$$

This enables us to evaluate the expected number of real zeros of $P_{n,\alpha}(t)$ obtained in Theorem 1 explicitly as the following result.

THEOREM 2. *With the same assumptions as Theorem 1 and for $P_n(t)$ defined as in (3),*

$$EN_{n,\alpha}(-\infty, \infty) \sim C(\alpha)\sqrt{n}$$

where

$$C(\alpha) = \frac{4}{\pi\alpha^2}\int_0^\infty dx \log\left\{\int_{-\infty}^{\infty}\left|1-\frac{y}{x}\right|^\alpha \exp\left(-\frac{y^2}{2}\right)\frac{dy}{\sqrt{2\pi}}\right\}.$$

2. Proof of Theorem 1

Let $p_{t,\theta,\alpha}(x, y) \equiv p_t(x, y)$ be the joint probability density function of $P_{n,\theta}(t)$ and its derivative with respect to t, $P'_{n,\theta}(t)$. Then, from [5] or [3, p. 12], we know that the expected number of real zeros of $P_{n,\theta}(t)$ in the interval (a, b) is given by

$$EN_{n,\theta,\alpha}(a, b) = \int_a^b f_n(t)\, dt$$

where

$$f_{n,\theta,\alpha}(t) \equiv f_n(t) = \int_{-\infty}^{\infty} p_t(0, y)|y|\, dy.$$

Let $\varphi(z) = E\{\exp(iz\xi_k)\} = \exp(-|z|^\alpha)$ be the common characteristic function of ξ_k, $k = 0, 1, 2, \ldots, n$, $a_k = t^k$ and $b_k = kt^k$; then the inversion formula yields

$$f_n(t) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} e^{-iy\omega} \prod_{k=0}^{n} \varphi\{\theta_k(a_k z + b_k\omega/t)\}\, d\omega.$$

Now with the change of variable $z = u\omega$, similar to [7], we obtain

$$\begin{aligned} f_n(t) &= \lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty} |y| \exp(-\epsilon|y|) p_t(0, y)\, dy \\ &= \lim_{\epsilon\downarrow 0}\left(\frac{1}{\pi^2}\right)\int_{-\infty}^{\infty} du \int_0^\infty \mathrm{Re}\frac{\omega}{(\epsilon - i\omega)^2}\prod_{k=0}^{n}\varphi(t^k|u-k|\theta_k\omega)\, d\omega. \end{aligned} \tag{4}$$

As established in [7], we use the following identity valid for nonzero constants A and B:

$$\mathrm{Re}\left(\frac{1}{\pi^2}\right)\int_0^\infty d\omega \int_{-\infty}^{\infty} \frac{1}{(\epsilon + i\omega)^2} \exp(-|Az + B\omega|^\alpha)\, dz = 0. \tag{5}$$

We subtract (5) from (4) and let $\omega = -tv$ to derive

$$f_n(t) = \left(\frac{1}{\pi^2 t}\right)\int_{-\infty}^{\infty} du \int_0^\infty \frac{\varphi^{n+1}\{(Au - B)v\} - \prod_{k=0}^{n}\varphi(t^k|u-k|\theta_k v)}{v}\, dv. \tag{6}$$

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In order for $f_n(t)$ in (6) to be convergent as $\epsilon \downarrow 0$ we need to chose constants A and B such that

$$\frac{(n+1)|Au-B|^\alpha - \sum_{k=0}^{n} |u-k|^\alpha t^{k\alpha}\theta_k^\alpha}{u^\alpha} = O\left(\frac{1}{u^2}\right) \quad \text{as } u \to \infty.$$

This would be the case for

$$A = \left(\frac{1}{n+1}\sum_{k=0}^{n} t^{\alpha k}\theta_k^\alpha\right)^{1/\alpha}$$

and

$$B = \frac{A^{1-\alpha}}{n+1}\sum_{k=0}^{n} kt^{k\alpha}\theta_k^\alpha.$$

We can now evaluate $f_n(t)$ in (6) further. We first modify the identity given in [7, p. 310] to

$$\int_0^\infty \frac{\exp(-a^\alpha v^\alpha) - \exp(-b^\alpha v^\alpha)}{v}\,dv = \log\left(\frac{b}{a}\right).$$

This, together with (6), yields

$$\begin{aligned}
f_n(t) &= \left(\frac{1}{\pi^2 t}\right)\int_{-\infty}^{\infty} du \\
&\quad \times \int_0^\infty \left[\frac{\exp\{-(n+1)^{1-\alpha}|(\sum_{k=0}^{n} t^{\alpha k}\theta_k^\alpha)^{1/\alpha}u - A^{1-\alpha}\sum_{k=0}^{n} kt^{\alpha k}\theta_k^\alpha|^\alpha v^\alpha\}}{v}\right. \\
&\quad \left. - \frac{-\exp\{-\sum_{k=0}^{n} t^{k\alpha}|u-k|^\alpha\theta_k^\alpha\}}{v}\right] dv \\
&= \left(\frac{1}{\pi^2 t}\right) \\
&\quad \times \int_{-\infty}^{\infty} \log\left\{\frac{(\sum_{k=0}^{n} t^{\alpha k}|u-k|^\alpha\theta_k^\alpha)^{1/\alpha}}{|(\sum_{k=0}^{n} t^{\alpha k}\theta_k^\alpha)^{1/\alpha}u - A^{1-\alpha}\sum_{k=0}^{n} kt^{\alpha k}\theta_k^\alpha|/(n+1)^{1-1/\alpha}|}\right\} du \\
&= \left(\frac{1}{\pi^2 t}\right) \\
&\quad \times \int_{-\infty}^{\infty} \log\left\{\frac{(\sum_{k=0}^{n} t^{\alpha k}|u-k|^\alpha\theta_k^\alpha)^{1/\alpha}}{|(\sum_{k=0}^{n} t^{\alpha k}\theta_k^\alpha)^{1/\alpha}u - (\sum_{k=0}^{n} t^{\alpha k}\theta_k^\alpha)^{1/\alpha-1}\sum_{k=0}^{n} kt^{\alpha k}\theta_k^\alpha|}\right\} du \\
&= \left(\frac{1}{\pi^2 t\alpha}\right) \\
&\quad \times \int_{-\infty}^{\infty} \log\left\{\frac{\sum_{k=0}^{n} t^{\alpha k}|u-k|^\alpha\theta_k^\alpha}{(\sum_{k=0}^{n} t^{\alpha k}\theta_k^\alpha)|u - \sum_{k=0}^{n} kt^{\alpha k}\theta_k^\alpha / \sum_{k=0}^{n} t^{\alpha k}\theta_k^\alpha)|^\alpha}\right\} du. \qquad (7)
\end{aligned}$$

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Now from (7) we have a formula for the expected number of zeros of $P_n(t)$:

$$EN_{n,\theta,\alpha}(-\infty,\infty)=\left(\frac{2}{\pi^2\alpha}\right)\int_0^\infty\frac{dt}{t}\int_{-\infty}^{\infty}\log\left\{\frac{\sum_{k=0}^n\pi_k|u-k|^\alpha}{|\sum_{k=0}^n\pi_k(u-k)|^\alpha}\right\}du, \tag{8}$$

where π_k is given in (2). This completes the proof of Theorem 1. If we assume $\theta_k\ge 0$ for all k, since t and α are nonnegative, π_k is also nonnegative for all k. Therefore,

$$\sum_{k=0}^n\pi_k|u-k|^\alpha\ge\left|\sum_{k=0}^n\pi_k(u-k)\right|^\alpha,$$

and we note that the term inside $\{\cdot\}$ in (8) is greater than or equal to one and therefore the integrand is positive.

3. Random polynomials with binomial elements

Now we assume that in Theorem 1, $\theta_k^\alpha\equiv\binom{n}{k}$. This, therefore, will be the case for Theorem 2. We first evaluate the terms on the right-hand side of (8). To this end, we note that if we let $t^\alpha=p/(1-p)$ then the value of π_k given in (2) becomes

$$\pi_k=\frac{t^{\alpha k}\binom{n}{k}}{(1+t^\alpha)^n}=\binom{n}{k}p^k(1-p)^{n-k}$$

which is binomial $B(n,p)$. Therefore, from (8),

$$EN_{n,\alpha}(-\infty,\infty)=\left(\frac{2}{\pi^2\alpha^2}\right)\int_0^1\frac{dp}{p(1-p)}\int_{-\infty}^{\infty}\log\left\{\frac{E|u-B(n,p)|^\alpha}{|u-np|^\alpha}\right\}du. \tag{9}$$

Now we let $B(n,p)=np+\sqrt{np(1-p)}\eta$, $u=np+v$ and $v=y\sqrt{np(1-p)}$. Then from (9) we obtain

$$\begin{aligned}EN_{n,\alpha}(-\infty,\infty)&=\left(\frac{2}{\pi^2\alpha^2}\right)\int_0^1\frac{dp}{p(1-p)}\int_{-\infty}^{\infty}\log E\left|\frac{v-\sqrt{np(1-p)}\eta}{v}\right|^\alpha dv\\&=\left(\frac{2\sqrt{n}}{\pi^2\alpha^2}\right)\int_0^1\frac{dp}{\sqrt{p(1-p)}}\int_{-\infty}^{\infty}\log E\left|\frac{y-\eta}{y}\right|^\alpha dy\\&=\left(\frac{2\sqrt{n}}{\pi^2\alpha^2}\right)\int_0^1\frac{dp}{\sqrt{p(1-p)}}\\&\quad\times\int_{-\infty}^{\infty}\log\left\{\int_{-\infty}^{\infty}\left|1-\frac{x}{y}\right|^\alpha\frac{\exp(-x^2/2)}{\sqrt{2\pi}}dx\right\}dy.\end{aligned} \tag{10}$$

Now, since we can evaluate $\int_0^1 dp/\sqrt{p(1-p)}$ as $\Gamma^2(1/2)/\Gamma(1)=\pi$, from (10)

$$EN_{n,\alpha}(-\infty,\infty)=\left(\frac{4\sqrt{n}}{\pi\alpha^2}\right)\int_0^\infty\log\left\{\int_{-\infty}^{\infty}\left|1-\frac{x}{y}\right|^\alpha\frac{\exp(-x^2/2)}{\sqrt{2\pi}}\,dx\right\}dy.$$

This proves Theorem 2.

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In order to show that our general result in Theorem 2 corresponds with the known results for $\alpha = 2$ we make this substitution in the result of Theorem 2. We obtain

$$\begin{aligned} EN_{n,2}(-\infty, \infty) &= \frac{\sqrt{n}}{\pi} \int_0^\infty \log \left\{ \int_{-\infty}^\infty \left(1 - \frac{2x}{y} + \frac{x^2}{y^2}\right) \frac{\exp(-x^2/2)}{\sqrt{2\pi}}\, dx \right\} dy \\ &= \left(\frac{\sqrt{n}}{\pi}\right) \int_0^\infty \log\left(1 + \frac{1}{y^2}\right) dy = \left(\frac{\sqrt{n}}{\pi}\right) \int_0^\infty \frac{dy}{1+y^2}\, dy \\ &= \sqrt{n}. \end{aligned}$$

Also numerical calculation shows that $C(\alpha)$ decreases from one to zero as α decreases from 2 to 0.

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