A proof of Waring's Expression for $\Sigma\alpha^r$ in terms of the Coefficients of an Equation.

By R. F. MUIRHEAD, M.A., D.Sc.

While Newton's Theorem on the Sums of Powers of the Roots of an equation furnishes a set of lineo-linear equations connecting the quantities $s_1, s_2, s_3, \ldots$ and the quantities $p_1, p_2, p_3, \ldots$ Waring gives the solution of these equations by which the s's are expressed in terms of the p's.

The general formula for s_r given by Waring, both in his *Meditationes Algebraicae* and in his *Miscellanea Analytica* is sometimes named after Albert Girard, who a century earlier, in his *Invention Nouvelle en l'Algèbre* gave the formulas for the sums of the squares, cubes and fourth powers; but as this mathematician gave no hint as to the form of the general formula, and perhaps even did not suspect the possibility of a general formula, it seems to me that if any name is to be associated with the formula, that name ought to be Waring's.

Waring gives a succinct proof by Mathematical Induction. This, though quite complete, has of course the disadvantage of requiring a knowledge of the formula to start with. A variety of other proofs have been given, of which the simplest are those which, like that indicated in Burnside and Panton's *Theory of Equations*, § 133, Ex. 8, use expansions by the Multinomial Theorem, and equate coefficients of like powers.

The proof here given is of that character, but it is perhaps unique in being *elementary* in the sense that it does not use infinite series.

Let $\alpha, \beta, \gamma, \ldots \alpha_n$ be the roots of the equation

$$x^n - p_1x^{n-1} + p_2x^{n-2} \ldots \pm p_n = 0\,;$$

and let s_r denote the sum $\alpha^r + \beta^r + \gamma^r + \ldots + \alpha_n{}^r$.

Then we have the identity

$$(1+\alpha x)(1+\beta x)(1+\gamma x)\ldots\ldots\ldots = 1 + p_1x + p_2x^2 + \ldots + p_nx^n. \quad (1)$$

Hence

$$(1+\alpha x)^m(1+\beta x)^m(1+\gamma x)^m\ldots\ldots = (1 + p_1x + p_2x^2 + \ldots + p_nx^n)^m. \quad (2)$$

But $(1+\alpha x)^m = 1 + \binom{m}{1}\alpha x + \binom{m}{2}\alpha^2x^2 + \ldots + \binom{m}{r}\alpha^r x^r + \ldots + \binom{m}{m}\alpha^m x^m.$

https://doi.org/10.1017/S0013091500033083 Published online by Cambridge University Press

Expanding both sides of (2) we find the general term on the right to be

$$\frac{m!\,p_1^{r_1}p_2^{r_2}p_3^{r_3}\ldots p_n^{r_n}\,.\,x^{r_1+2r_2+\ldots+nr_n}}{r_0!\,r_1!\,r_2!\ldots r_n!}$$

where $r_0, r_1, .. r_n$ are integers or zeros such that $r_0+r_1+r_2+..+r_n=m$.

The general term on the left is

$$\binom{m}{a}\binom{m}{b}\ldots\ldots\binom{m}{a_n}a^a\beta^b\gamma^c\ldots a_n^{a_n}\,.\,x^{a+b+c\ldots+a_n}$$

where $a, b, c, \ldots$ may have any values from 0 up to m inclusive.

Thus on the right, the coefficient of x^r is

$$\Sigma\frac{m!\,p_1^{r_1}p_2^{r_2}\ldots p_n^{r_n}}{r_0!\,r_1!\ldots r_n!}$$

where the summation extends over all values of $r_0, r_1, \ldots$ for which

$$r_0+r_1+\ldots+r_n=m \quad\text{and}\quad r_1+2r_2+3r_3+\ldots+nr_n=r.$$

On the left, the coefficient of x^r is

$$\Sigma\left\{\binom{m}{a}\binom{m}{b}\ldots\ldots\binom{m}{a_n}a^a\beta^b\ldots a_n^{a_n}\right\}$$

where the summation extends over all possible sets of positive integral or zero values of $a, b, c, \ldots a_n$ for which $a+b+c\ldots+a_n=r$.

Of these sets, some will differ only as to the *order* in which the indices a, b and c occur. In order to group together such sets, let us denote by $(a, b, c, \ldots a_n)$ the symmetrical function $\Sigma(a^a\beta^b\ldots a_n^{a_n})$ where the summation extends over all *different* products which can be got by permuting the *indices* while $a, \beta, \ldots$ are kept in one definite order.

The coefficient of x^r on the left can then be written

$$\Sigma\left\{\binom{m}{a}\binom{m}{b}\ldots\ldots\binom{m}{a_n}(a, b, c, \ldots a_n)\right\}.$$

Equating coefficients of x^r on the right and on the left, we get

$$(3)\qquad \Sigma\left\{\binom{m}{a}\binom{m}{b}\ldots\ldots\binom{m}{a_n}(a, b, c, \ldots a_n)\right\}$$
$$=\Sigma\frac{m(m-1)(m-2)\ldots(m+1-r_1-r_2-\ldots-r_n)}{r_1!\,r_2!\ldots\ldots r_n!}p_1^{r_1}p_2^{r_2}\ldots p_n^{r_n}$$

where $a+b+\ldots+a_n=r$, and $r_1+2r_2+3r_3+\ldots+nr_n=r$, and on the left the order of $a, b, c, \ldots a_n$ is indifferent.

With respect to m each side of this identity is a rational integral algebraic function of degree r, the form of which is independent of m provided m is large enough. Hence we may equate coefficients of powers of m.

The coefficient of m on the left arises from such terms as have only *one* of the quantities $a, b, c, \ldots$ different from zero, that one being $= r$ and it is therefore

$$=\frac{(-1)(-2)\ldots(-r+1)}{1.2\ldots\ldots\ldots\ldots r}(r, 0, 0, 0, \ldots),$$

which may be written $\qquad (-1)^{r-1}\frac{1}{r}\Sigma a^r.$

The coefficient of m on the right is

$$\Sigma\frac{(-1)(-2)\ldots(-r_1-r_2-r\ldots-r_n+1)}{r_1!\,r_2!\ldots r_n!}p_1^{r_1}p_2^{r_2}\ldots p_n^{r_n}.$$

Thus we have Waring's Formula

$$(4) \quad \cdot \quad \Sigma a^r \quad = r\Sigma\frac{(r_1+r_2+\ldots+r_n-1)!\,p_1^{r_1}p_2^{r_2}\ldots p_n^{r_n}(-1)^{\Sigma r_1 - r}}{r_1!\,r_2!\,r_3!\ldots r_n!}$$

$$\text{or} \quad \Sigma(-a^r) = r\Sigma\frac{(r_1+r_2+\ldots+r_n-1)!\,(-p_1)^{r_1}(-p_2)^{r_2}\ldots(-p_n)^{r_n}}{r_1!\,r_2!\ldots r_n!}$$

where $r_1, r_2, \ldots r_n$ have all possible positive integer or zero values making

$$r_1+2r_2+3r_3\ldots nr_n=r.$$

It may be of interest to note the more complicated formula which arises when we equate the coefficients of m^k on the right and left of (3), k being a positive integer not greater than r. It may be written

$$(-1)^{r-k}\Sigma\frac{[a, b, c, \ldots a_n; k](a, b, c, \ldots a_a)}{a!\,b!\,c!\ldots\ldots\ldots\ldots a_n!}$$

$$=\Sigma\frac{(-1)^{p-k}[p; k]p_1^{r_1}p_2^{r_2}\ldots p_n^{r_n}}{r_1!\,r_2!\ldots\ldots r_n!} \quad - \quad - \quad (5)$$

where $[a, b, c, \ldots a_n; k]$ denotes the sum of all products of the numbers $1, 2, \ldots\ldots a-1$; $1, 2, \ldots\ldots b-1$; $\ldots\ldots$; $1, 2, \ldots\ldots a_n-1$, taken $a+b+\ldots+a_n-k$ at a time, the value of $[a, b, c, \ldots a_n; k]$ being

https://doi.org/10.1017/S0013091500033083 Published online by Cambridge University Press

reckoned $=1$ if $a+b+\ ..+a_n-k$ is zero, and $=0$ if the latter is negative; and p denotes the sum $r_1+r_2+r_3+\dots+r_n$; and as before a, b, etc., have any positive integral or zero values making $a+b+c+\dots+a_n=r$, and r_1, r_2, etc., have any positive integral or zero values making $r_1+2r_2+3r_3+\dots+nr_n=r$.

It may be useful to collect the formulae by which the elementary symmetrical functions p_1, p_2, etc., the sums of powers s_1, s_2, etc., and the sums of homogeneous products H_1, H_2, etc., are expressed in terms of one another. In addition to Waring's Formula, we have five others, as follows:—

$$(6)\qquad s_r = r\sum\frac{(r_1+r_2+\dots+r_m-1)!(-1)^{r_1+r_2+\dots+r_m-1}H_1^{r_1}H_2^{r_2}\dots H_m^{r_m}}{r_1!\,r_2!\,r_3!\,\dots\dots\dots\,r_m!}$$

$$(7)\qquad p_r=(-1)^r\sum\frac{(-s_1)^{r_1}(-s_2)^{r_2}\dots(-s_m)^{r_m}}{(r_1!\,r_2!\dots r_m!)(2^{r_2}3^{r_3}\dots m^{r_m})}$$

$$(8)\qquad H_r=\sum\frac{s_1^{r_1}s_2^{r_2}\dots s_m^{r_m}}{(r_1!\,r_2!\dots r_m!)(2^{r_2}3^{r_3}\dots m^{r_m})}$$

$$(9)\qquad H_r=(-1)^r\sum\frac{(r_1+r_2+\dots+r_n)!\,(-p_1)^{r_1}(-p_2)^{r_2}\dots(-p_n)^{r_n}}{r_1!\,r_3!\,\dots\dots\,r_n!}$$

$$(10)\qquad p_r=(-1)^r\sum\frac{(r_1+r_2+\dots+r_m)!\,(-H_1)^{r_1}(-H_2)^{r_2}\dots(-H_m)^{r_m}}{r_1!\,r_2!\,r_3!\,\dots\dots\,r_m!}$$

where in each case r_1, r_2, ... are to have all possible positive integral or zero values for which $r_1+2r_2+3r_3+\dots=r$.

It is to be noted that the series of p's ends with p_n, while the series of s's and of H's do not end. Note also that many writers use $(-1)^r p_r$ to denote what is here denoted by p_r.
