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Noncircular algebraic curves of constant width: an answer to Rabinowitz

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Abstract. In response to an open problem raised by S. Rabinowitz, we prove that

$$\begin{aligned}&\left(\left(x^2+y^2\right)^2+8y\left(y^2-3x^2\right)\right)^2+432y\left(y^2-3x^2\right)\left(351-10\left(x^2+y^2\right)\right)\\&=567^3+28\left(x^2+y^2\right)^3+486\left(x^2+y^2\right)\left(67\left(x^2+y^2\right)-567\times 18\right)\end{aligned}$$

is the equation of a plane convex curve of constant width.

1 Introduction

A disk has the property that it can be rotated between two fixed parallel lines without losing contact with either line. It has been known for a long time that there are many other plane convex bodies with the same property. Such plane convex bodies are called plane convex bodies "of constant width" or "orbiforms." Their boundaries are of course called "plane convex curves of constant width." A classical noncircular example is the Reuleaux triangle [6], which is made of three circular arcs. But a noncircular plane convex curve of constant width can be smooth, and not having any circular arc in its boundary. The notion of a convex body of constant width can of course be extended to higher dimensions. For a recent book on the topic, we refer the reader to [4].

In this paper, we are essentially interested in noncircular algebraic curves of constant width. Rabinowitz [5] found that the zero set of the following polynomial $P \in \mathbb{R}[X,Y]$ forms a noncircular algebraic curve of constant width in $\mathbb{R}^2$:

$$\begin{aligned}P(x,y) \quad := &\left(x^2+y^2\right)^4-45\left(x^2+y^2\right)^3-41283\left(x^2+y^2\right)^2\\&+7950960\left(x^2+y^2\right)+16\left(x^2-3y^2\right)^3+48\left(x^2+y^2\right)\left(x^2-3y^2\right)^2\\&+x\left(x^2-3y^2\right)\left(16\left(x^2+y^2\right)^2-5544\left(x^2+y^2\right)+266382\right)-720^3.\end{aligned}$$

Then, he raised the following open questions: "*The polynomial curve found is pretty complicated. Can it be put in simpler form? Our polynomial is of degree 8. Is there one with lower degree? What is the lowest degree polynomial whose graph is a noncircular curve of constant width?*" A couple of years ago, Bardet and Bayen [1, Corollary 2.1] proved that the degree of P, that is 8, is the minimum possible degree for a noncircular plane *convex* curve of constant width. Here, we emphasize the convexity assumption, because it is implicit in the statement of Corollary 2.1 in [1]. In this short note, we

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provide additional answers to Rabinowitz's open questions. First, we recall the notion of a plane hedgehog curve of constant width, and we notice that in this setting, we can find algebraic curves of constant width much simpler. Second, we give an example of a noncircular algebraic curve of constant width whose equation is simpler than the one of Rabinowitz. Finally, we notice that we can deduce from it (relatively) simple examples in higher dimensions.

2 Plane algebraic hedgehogs of constant width

Here, we will follow more or less [2].

Definition For any smooth function $h : \mathbb{S}^1 = \mathbb{R} \setminus 2\pi\mathbb{Z} \to \mathbb{R}, \theta \mapsto h(\theta)$, we let $\mathcal{H}_h$ denote the envelope of the family of lines given by

$$x \cos\theta + y \sin\theta = h(\theta), \tag{1}$$

where (x, y) *are the coordinates in the canonical basis of the Euclidean vector plane* $\mathbb{R}^2$. We say that $\mathcal{H}_h$ is the plane hedgehog with support function h, and that $\mathcal{H}_h$ is projective if $h(\theta + \pi) = -h(\theta)$ for all $\theta \in \mathbb{S}^1$.

Partial differentiation of 1 yields

$$-x \sin\theta + y \cos\theta = h'(\theta). \tag{2}$$

From 1 and 2, the parametric equations for $\mathcal{H}_h$ are

$$\begin{cases} x = h(\theta)\cos\theta - h'(\theta)\sin\theta, \\ y = h(\theta)\sin\theta + h'(\theta)\cos\theta. \end{cases}$$

The family of lines $(D(\theta))_{\theta\in\mathbb{S}^1}$ of which $\mathcal{H}_h$ is the envelope is the family of "support lines" of $\mathcal{H}_h$. Suppose that $\mathcal{H}_h$ has a well-defined tangent line at the point (x, y), say T. Then, T is the support line with equation 1: the unit vector $u(\theta) = (\cos\theta, \sin\theta)$ is normal to T and $h(\theta)$ may be interpreted as the signed distance from the origin to T.

A plane hedgehog is thus simply a plane envelope that has exactly one oriented support line in each direction. A singularity-free plane hedgehog is simply a convex curve. A plane hedgehog is projective if it has exactly one nonoriented support line in each direction.

Now, we can define the width, say $w_h(\theta)$, of such a plane hedgehog $\mathcal{H}_h$ in the direction $u(\theta)$ to be the signed distance between the two support lines of $\mathcal{H}_h$ that are orthogonal to $u(\theta)$, that is, by

$$w_h(\theta) = h(\theta) + h(\theta + \pi).$$

Thus, plane projective hedgehogs are hedgehogs of constant width 0, and the condition that a plane hedgehog $\mathcal{H}_h$ is of constant width $2r$ is simply that its support function h has the form $f + r$, where f is the support function of a projective hedgehog. Here are three examples of plane hedgehogs: (a) a convex hedgehog of constant width; (b) a hedgehog with four cusps; and (c) a plane projective hedgehog which is a hypocycloid with three cusps (Figure 1).

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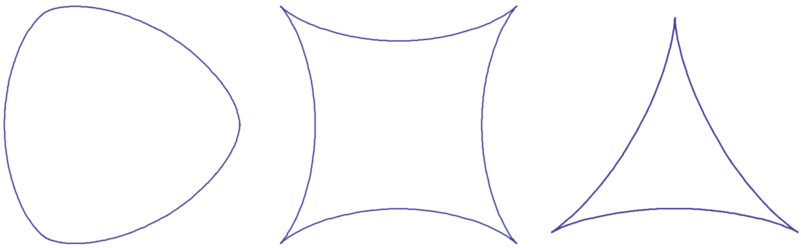

(a) $h(\theta) = 10 + \cos(3\theta)$ (b) $h(\theta) = \cos(2\theta)$ (c) $h(\theta) = \sin 3\theta$

Figure 1: Three examples of plane hedgehogs.

Theorem *The projective hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ with support function $h : \mathbb{S}^1 \to \mathbb{R}$, $\theta \mapsto \sin(3\theta)$ is a noncircular algebraic curve of constant width* 0 *with equation*

$$\left(x^2 + y^2\right)^2 + 18\left(x^2 + y^2\right) - 8y\left(y^2 - 3x^2\right) = 27.$$

Proof We already know that $\mathcal{H}_h$ is a noncircular curve of constant width 0. From the parametric equations for $\mathcal{H}_h$, we deduce that

$$x^2 + y^2 = h(\theta)^2 + h'(\theta)^2 = \sin^2(3\theta) + 9\cos^2(3\theta) = 5 + 4\cos(6\theta).$$

Now, $h : \mathbb{S}^1 \to \mathbb{R}$, $\theta \mapsto \sin(3\theta)$ is the restriction of the polynomial $-y\left(y^2 - 3x^2\right)$ to the unit circle $\mathbb{S}^1$, and the linearization of $-y\left(y^2 - 3x^2\right)$ as a trigonometric function of θ gives

$$-y\left(y^2 - 3x^2\right) = -12 - 14\cos(6\theta) - \cos(12\theta) = -11 - 14\cos(6\theta) - 2\cos^2(6\theta).$$

From the above two equations, we deduce easily that

$$\left(x^2 + y^2\right)^2 + 18\left(x^2 + y^2\right) - 8y\left(y^2 - 3x^2\right) = 27.$$

■

3 A noncircular algebraic curve of constant width whose equation is not too complicated

Any hedgehog whose support function $h : \mathbb{S}^1 \to \mathbb{R}$ is of the form $h(\theta) = r - \sin(3\theta)$, for some constant r, is a hedgehog of constant width $2r$. Such a function $h : \mathbb{S}^1 \to \mathbb{R}$ is the support function of a convex body if and only if $(h + h'')(\theta) = r - 8\sin(3\theta) \geq 0$, for all $\theta \in \mathbb{S}^1$, that is, if and only if $r \geq 8$. We choose $r = 8$ in order to be "as closed as possible" to the previous example.

Theorem *The plane hedgehog $\mathcal{H}_h$ with support function $h : \mathbb{S}^1 \to \mathbb{R}$, $\theta \mapsto 8 - \sin(3\theta) = 4\sin^3\theta - 3\sin\theta + 8$ is a noncircular convex algebraic curve of constant*

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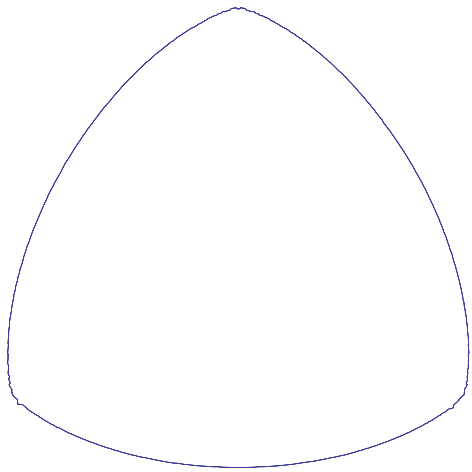

Figure 2: The noncircular convex curve of constant width 16 with equation (3).

width 16 *with equation* (Figure 2)

$$\left(\left(x^2+y^2\right)^2+8y\left(y^2-3x^2\right)\right)^2+432y\left(y^2-3x^2\right)\left(351-10\left(x^2+y^2\right)\right) = 567^3+28\left(x^2+y^2\right)^3+486\left(x^2+y^2\right)\left(67\left(x^2+y^2\right)-567\times 18\right). \tag{3}$$

Proof The parametric equations for $\mathcal{H}_h$ are equivalent to:

$$\begin{cases} x=-8\left(\sin^3(\theta)-1\right)\cos(\theta), \\ y=-2\cos(2\theta)-\cos(4\theta)+8\sin(\theta). \end{cases}$$

Expanding x and y in terms of $c=\cos\theta$ and $s=\sin\theta$, we obtain after simplification:

$$\begin{cases} x=-8\left(s^3-1\right)c, \\ y=-3+4s\left(2+3s-2s^3\right). \end{cases}$$

Squaring the first equation and substituting in $c^2=1-s^2$ gives us the following system of equations in the three unknowns x, y, and s:

$$\begin{cases} 64\left(1-s^2\right)\left(s^3-1\right)^2-x^2=0, \\ -3+4s\left(2+3s-2s^3\right)-y=0. \end{cases}$$

We then eliminate s by computing the resultant of the polynomials $A(s)=64\left(1-s^2\right)\left(s^3-1\right)^2-x^2$ and $B(s)=-3+4s\left(2+3s-2s^3\right)-y$ with Mathematica, and find after simplification that:

$$\left(\left(x^2+y^2\right)^2+8y\left(y^2-3x^2\right)\right)^2+432y\left(y^2-3x^2\right)\left(351-10\left(x^2+y^2\right)\right) = 567^3+28\left(x^2+y^2\right)^3+486\left(x^2+y^2\right)\left(67\left(x^2+y^2\right)-567\times 18\right).$$

■

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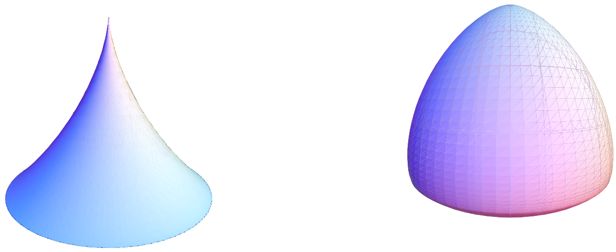

Figure 3: Our two algebraic surfaces of constant width.

4 Higher dimension

The notion of a hedgehog of constant width can of course be extended to higher dimensions (see, e.g., [3]). Each of the above two examples of algebraic curves of constant width admits an axis of symmetry in $\mathbb{R}^2$. By rotating it around such an axis, we deduce immediately an example of algebraic surface of revolution that is of constant width in $\mathbb{R}^3$. More precisely, the algebraic surface with equation.

$$\left(x^2+y^2+z^2\right)^2+18\left(x^2+y^2+z^2\right)-8z\left(z^2-3\left(x^2+y^2\right)\right)=27$$

is a "projective hedgehog" of revolution and a surface of constant width 0 in $\mathbb{R}^3$ (see Figure 3, left), and the algebraic surface with equation

$$\begin{gathered}\left(\left(x^2+y^2+z^2\right)^2+8z\left(z^2-3\left(x^2+y^2\right)\right)\right)^2\\+432z\left(z^2-3\left(x^2+y^2\right)\right)\left(351-10\left(x^2+y^2+z^2\right)\right)\\=567^3+28\left(x^2+y^2+z^2\right)^3\\+486\left(x^2+y^2+z^2\right)\left(67\left(x^2+y^2+z^2\right)-567\times 18\right)\end{gathered}$$

is a convex surface of constant width 16 in $\mathbb{R}^3$ (see Figure 3, right).

There are several methods to explicitly find algebraic constant width bodies, even without being of revolution (see, e.g., [4, Section 8.5]).

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