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PRIME RINGS FOR WHICH THE SET OF NONZERO IDEALS IS A SPECIAL CLASS

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Abstract

We show that the rings described in the title are precisely the indecomposable injectives for the category whose objects are the associative rings and whose morphisms are the ring homomorphisms with accessible images. These rings are more or less completely known. Those of cardinality greater than that of the continuum are subdirectly irreducible but there are some nontrivial principal ideal domains in the class.

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Throughout this note, all rings are associative but are not required to have an identity.

Since a class $\mathscr{M}$ of prime rings is special if and only if it is hereditary with respect to nonzero ideals and closed under essential extensions (see [3] for this characterization, which differs slightly from the original definition given by Andrunakievich [1]) it is clear that every nonempty intersection of special classes is special, and in particular every prime ring A generates a special class (which, following [6] and [2], we shall call π_A). This π_A is the intersection of all special classes containing A.

Ryabukhin [6] has obtained the following characterization of π_A (see also [2, page 239]).

For a prime ring A, π_A contains a prime ring B if and only if B has a nonzero ideal which is isomorphic to an accessible subring of A.

Useful though it is, this description of π_A is not overly explicit. For instance, one sees easily that if A is simple, then π_A consists of all subdirectly irreducible rings with hearts isomorphic to A (though this can easily be deduced by other means) but this does not really tell us much about the "size" of π_A. On the other hand, for any prime ring A, if $0 \neq I \triangleleft A$, then $|A| \leq 2^{|I|}$ (see [6] or [2, page 240]) so we can say something about the size of *members of* π_A.

Our concern in this note is with those π_A which are in a sense as small as possible: with those A for which π_A is (up to isomorphisms) the set of nonzero ideals of A. Since π_A is closed under essential extensions and every prime ring has an essential extension (which is a prime ring) with identity, it is clear that A must have an identity. What we seek, therefore, is a characterization of the prime rings A with identity such that whenever B is prime with a nonzero ideal isomorphic to an accessible subring of A, B itself is isomorphic to an ideal of A. (There is a slightly better-*looking* condition obtained by the substitution of "accessible subring" for "ideal" in a couple of places, but this turns out to be equivalent to the one we have enunciated.) Now what we have is starting to look vaguely like an injective-type concept. Our main result will make this much more precise, and the relevant injectivity is the following.

A ring R is acc-*injective* [4] if it is an injective object in the category whose objects are rings (not necessarily with identity) and whose morphisms are ring homomorphisms with accessible images. The acc-injectives have been fairly completely described in [4] and [5]. In particular, they have identities.

THEOREM. *The following conditions are equivalent for a prime ring* A:

(i) π_A *is (up to isomorphisms) the set of nonzero ideals of* A;

(ii) π_A *is (up to isomorphisms) the set of nonzero accessible subrings of* A;

(iii) A *is* acc-*injective.*

PROOF. (i) $\Rightarrow$ (ii). This is clear since in any case π_A contains all nonzero accessible subrings of A.

(ii) $\Rightarrow$ (iii). Let A be a prime ring satisfying (ii). As noted above, A has an identity. If $0 \neq I \triangleleft A$ then by factoring out of the standard unital extension of I an ideal maximal with respect to having zero intersection with I, we obtain a ring $\bar{I}$ with identity in which I is an essential ideal. This $\bar{I}$ is prime and so, as $\bar{I} \in \pi_A$, $\bar{I}$ is (isomorphic to) an accessible subring of A. But $\bar{I}^2 = \bar{I}$ so $\bar{I}$ is isomorphic to an ideal, hence a direct summand, of A, whence as A is prime, $\bar{I} \cong A$. It follows that A is generated, as a ring, by

I and 1; we denote this as follows:

(1) $$A = \langle I , 1\rangle \quad \textit{whenever } 0 \neq I \triangleleft A.$$

If $0 \neq K \triangleleft I \triangleleft A$, then $AK + KA = \langle I , 1\rangle K + K\langle I , 1\rangle \subseteq K$. Thus

(2) *all accessible subrings of A are ideals.*

Now consider two nonzero, isomorphic, ideals I and J of A. Since $I \cap J \neq 0$, we have

$$\langle I , 1\rangle = \langle J , 1\rangle = \langle I \cap J , 1\rangle = A.$$

Suppose $I \neq J$. Then we may assume that $J \not\subseteq I$. For every $j \in J$ (as $A = \langle I \cap J , 1\rangle$) there exist $x \in I \cap J$, $k \in \mathbb{Z}$ such that $j = x + k \cdot 1$. Then $k \cdot 1 = j - x \in J$, so as $J \neq I \cap J$, at least one such k is nonzero. Clearly $\{n \in \mathbb{Z}: n \cdot 1 \in J\} \triangleleft \mathbb{Z}$, so taking n_0 as the positive generator for this ideal we see that (by (1)) every element of A has a unique representation

$$j + n \cdot 1, \qquad j \in J, \qquad n \in \mathbb{Z}, \qquad 0 \leq n < n_0.$$

Now I and J are isomorphic rings, so let $f: J \to I$ be a ring isomorphism. Then $f(n_0 \cdot 1) = i_0$ for some $i_0 \in I$, and then for every element $f(j)$ of I we have

$$i_0 f(j) = f(n_0 \cdot 1)f(j) = f(n_0 j) = n_0 f(j) = (n_0 \cdot 1)f(j).$$

But then $(i_0 - n_0 \cdot 1)I = 0$, so, A being a prime ring, $i_0 - n_0 \cdot 1 = 0$. Hence $\{n \in \mathbb{Z}: n \cdot 1 \in I\}$ is an ideal of $\mathbb{Z}$ containing n_0. Let m_0 be the positive generator of this ideal. Then $m_0 \leq n_0$ and every element of A has a unique representation

$$i + n \cdot 1, \qquad i \in I, \qquad n \in \mathbb{Z}, \qquad 0 \leq n < m_0.$$

But by interchanging the roles of I and J in the isomorphism argument used above, we can deduce that $n_0 \leq m_0$. Thus we get

(3) $$A = \{i + n \cdot 1: i \in I , 0 \leq n < n_0\} = \{j + k \cdot 1: j \in J , 0 \leq k < n_0\},$$

where i and n, j and k are unique. The first part of the argument just given also shows that

(4) $$f(n_0 \cdot 1) = n_0 \cdot 1.$$

We extend our isomorphism f to a map $\hat{f}: A \to A$ by the rule

$$\hat{f}(j + n \cdot 1) = f(j) + n \cdot 1 \ (j \in J , \ n \in \mathbb{Z}, \quad 0 \leq n < n_0).$$

By (3), $\hat{f}$ is well defined. Let $n , n' \in \mathbb{Z}$ be such that $0 \leq n < n_0$, $0 \leq n' < n_0$, and let $n + n' = kn_0 + r$, where $0 \leq r < n_0$. Then (for $j , j' \in J$) we

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have

$$\begin{aligned}\hat{f}(j+n\cdot 1)+\hat{f}(j'+n'\cdot 1) &= f(j)+n\cdot 1+f(j')+n'\cdot 1\\ &= f(j)+f(j')+(n+n')\cdot 1\\ &= f(j)+f(j')+kn_0\cdot 1+r\cdot 1\\ &= f(j)+f(j')+f(kn_0\cdot 1)+r\cdot 1 \text{ (by (4))}\\ &= f(j+j'+kn_0\cdot 1)+r\cdot 1\\ &= \hat{f}(j+j'+kn_0\cdot 1+r\cdot 1)\\ &= \hat{f}(j+j'+(n+n')\cdot 1)\\ &= f(j+n\cdot 1+j'+n'\cdot 1)\end{aligned}$$

and similarly one shows that $\hat{f}$ preserves multiplication. Thus $\hat{f}$ is a homomorphism. From (3) and the fact that f is an isomorphism, it is clear that $\hat{f}$ is in fact an isomorphism, so

(5) *every isomorphism between ideals of A extends to an automorphism of A.*

Now let R be a prime ring having a nonzero ideal L isomorphic to an ideal I of A, and let $g\colon L \to I$ be an isomorphism. Then R is in π_A so by assumption (that is, (ii)) and (2), there is an injective homomorphism $h\colon R \to A$ such that $h(R) \lhd A$ and then $h(L) \lhd h(R) \lhd A$ so $h(L) \lhd A$. But $h(L) \cong I$ *via* the correspondence

$$x \mapsto gh^{-1}(x).$$

Hence by (5) there is an automorphism $F\colon A \to A$ such that $F(x) = gh^{-1}(x)$ for all $x \in h(L)$. Now Fh is an injective homomorphism and $Fh(R) \lhd A$. Since for each $z \in L$ we have $Fh(z) = gh^{-1}h(z) = g(z)$ we have the following commutative diagram

$$\begin{array}{ccc} L & \lhd & R \\ {\scriptstyle g}\downarrow & & \swarrow {\scriptstyle Fh} \\ I & & \\ \Delta & & \\ A & & \end{array}$$

(with $Fh(R) \lhd A$). This shows that A satisfies a "special case" of acc-injectivity. We now consider the general case.

Let T be any ring, $0 \neq V \lhd T$ and let $G\colon V \to A$ be a homomorphism with $G(V)$ an accessible subring and therefore an ideal of A (by (6)). Then $G(V)$ is a prime ring so $\operatorname{Ker}(G)$ is an ideal of T. Let us re-name $V/\operatorname{Ker}(G)$ and $T/\operatorname{Ker}(G)$ as V_1, T_1 respectively. Now $V_1 \lhd T_1$. Let M be an ideal of T_1 which has zero intersection with V_1 and is maximal for this. Then

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T_1/M is an essential extension of $(V_1+M)/M \cong V_1$, so T_1/M is a prime ring. Since $(V_1+M)/M \cong V_1 = V/\mathrm{Ker}(G) \cong G(V) \triangleleft A$ there is an injective homomorphism $H\colon T_1/M \to A$ with $H(T_1/M) \triangleleft A$ such that H extends the map of $(V_1+M)/M$ into A (by the "special case" above). In other words we have a commutative diagram

$$\begin{array}{ccc} V & \triangleleft & T \\ \downarrow & & \downarrow \\ V_1 = V/\mathrm{Ker}(G) & \triangleleft & T/\mathrm{Ker}(G) = T_1 \\ \downarrow & & \downarrow \\ (V_1+M)/M & \triangleleft & T_1/M \\ \downarrow & & \swarrow \\ V/\mathrm{Ker}(g) & \swarrow & \\ \downarrow\cong & & \\ G(V) & \swarrow & \\ \triangle & \swarrow & \\ A. & & \end{array}$$

By induction we can replace the condition " $V \triangleleft T$ " with " V is an accessible subring of T ". This proves that A is acc-injective.

(iii) $\Rightarrow$ (i). Let A be acc-injective, $B \in \pi_A$. Then B has a nonzero ideal I which is isomorphic to an accessible subring of A. Thus there is an injective homomorphism $g\colon I \to A$ with $g(I)$ an accessible subring of A. Then there is a homomorphism $h\colon B \to A$ with accessible image such that $h(i) = g(i)$ for all $i \in I$. This latter condition requires that $I \cap \mathrm{Ker}(h) = 0$, whence, B being a prime ring, $\mathrm{Ker}(h) = 0$ and B is isomorphic to an accessible subring of A and hence, by (2), an ideal of A.

From [4] and [5] we know that the prime (or, equivalently, indecomposable) acc-injectives are

(i) the unital simple rings,

(ii) the standard characteristic p unital extensions of certain simple rings of characteristic p which do not admit algebra structures over fields other than $\mathbb{Z}_p$ and

(iii) rings (they are in fact principal ideal domains) all of whose proper homomorphic images are isomorphic to rings $\mathbb{Z}_n$ (various n).

A ring A of type (iii) is embeddable in the ring of p-adic integers whenever $pA \neq A$ and thus $|A| \leq 2^{\aleph_0}$. Hence we have

COROLLARY. *Every prime ring which satisfies the conditions of the Theorem and has greater cardinality than the continuum is subdirectly irreducible.*

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