# CONTINUOUS SOLUTIONS OF THE FUNCTIONAL EQUATION $f^n(x) = f(x)$

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In this note the authors find all continuous real functions defined on the real axis and such that for an integer $n \geqslant 2$, and for each $x$,

$$f^n(x) = f(x). \tag{1}$$

The symbol $f^n$ denotes $f$ iterated $n$ times.

The following two classes of functions occur as solutions.

CLASS I.

($a_1$) *The function $f(x)$ is continuous for all real $x$,*

($b_1$) $f(x) \equiv x$ *on a connected subset $S$ of the $x$-axis, and*

($c_1$) $g \leqslant f(x) \leqslant G$, *in which $g$ and $G$ denote, respectively, the infinum and supremum of $f(x)$ on $S$.*

The set $S$ must be a point, a closed interval, a closed ray, or the entire $x$–axis. Thus Class I includes all constants and the function $x$. If $S$ is a closed interval $[a, b]$ then $f(x)$ is arbitrary outside of $[a, b]$ except for continuity and the condition $a \leqslant f(x) \leqslant b$. If $S$ is a ray, $f(x)$ is similarly described.

CLASS II.

($a_2$) *The function $f(x)$ is continuous for all real $x$ and either,*

($b_2$) $f^2(x) \equiv x$, *or*

($c_2$) $f^2(x) \equiv x$ *on a non-degenerate closed interval* $[a, b]$, $f(a) = b$, $f(b) = a$, *and* $a \leqslant f(x) \leqslant b$.

A function satisfies ($b_2$) if and only if $y = f(x)$ implies $x = f(y)$. Its graph is, accordingly, symmetric with respect to the line $y = x$. If $f(x)$ is a solution of ($b_2$), then the inverse of the transformation $x \rightarrow f(x)$ is clearly single valued and continuous. Hence the transformation $x \rightarrow f(x)$ defines a homeomorphism of the $x$-axis onto itself.

One can easily see that Classes I and II have only the function $x$ in common.

LEMMA 1. *$f(x)$ is a continuous solution of $f^2(x) = f(x)$ if, and only if, $f(x)$ is of Class* I.

*Proof.* That every Class I function is a solution is easily verified. Conversely, if $f(x)$ is a continuous solution then $x = f(r)$ satisfies $f(x) = x$ for every real $r$. In case $f(x) = x$ has only one solution, $f(x)$ is constant and hence of Class I. If $f(x) = x$ has two solutions $a$ and $b$, $a < b$, then $f(a) = a$ and $f(b) = b$; and given $c$ between $a$ and $b$ there exists, from the continuity of $f(x)$,

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a number $m$, $a < m < b$, such that $f(m) = c$. It follows that $f(c) = ff(m) = f(m) = c$, and hence if $f(x) = x$ at the ends of an interval the relation holds identically on the interval. The maximal set $S$ on which $f(x) = x$ is thus connected. The continuity of $f(x)$ implies that $S$ is closed. Finally $f(x)$ has property ($c_1$), for if there exists an $r$ not in $S$ such that $f(r)$ does not satisfy ($c_1$), the relation $f(x) = ff(x) = f^2(x)$ with $x = r$, contradicts the fact that $S$ is maximal.

LEMMA 2. *If $f^n(x) = x$ on a non-degenerate closed connected subset $S$ of the real axis and if $f(x)$ maps $S$ continuously into $S$, then*

(i) *$f(x)$ is a homeomorphism of $S$ onto itself,*

(ii) *if $S$ is an interval $[a, b]$, $f(x) \equiv x$ on $S$ or $f^2(x) \equiv x$ on $S$ and $f(x)$ is equivalent to a reflection of $[a, b]$ about the single fixed point $p$,*

(iii) *if $S$ is a ray, $f(x) \equiv x$ on $S$, and*

(iv) *if $S$ is the entire axis, $f(x) \equiv x$ on $S$ or $f^2(x) \equiv x$ on $S$ and $f(x)$ is equivalent to a reflection of $S$ about the single fixed point $p$.*

*Proof.* Conclusions (i) for the case of an interval and (ii) are special cases of results in Whyburn [**2**, pp. 240, 264].

If $S$ is a ray, the mapping [**2**, p. 240] is (1–1) and onto. Thus $f(x)$ is monotone on the ray and the end point is fixed under $f(x)$. If there were an interior point $b$ of the ray such that $f(b) \neq b$, the monotonicity of $f(x)$ would imply that $f^n(b) \neq b$. Hence (iii), which implies (i) for the ray.

If $S$ is the real axis, the mapping is again (1–1) and onto and $f(x)$ is monotone. If $f(x)$ increases with $x$, we see that $f(x) \equiv x$ by the argument employed for the ray. If $f(x)$ is monotone decreasing its graph cuts $y = x$ in exactly one point and $f(x)$ is topologically equivalent to a reflection of the $x$-axis about the abscissa of this point.

COROLLARY 1. *If $n$ is odd, the functional equation $f^n(x) = x$ has only the function $x$ as a continuous solution. If $n$ is even, the continuous solutions of $f^n(x) = x$ are those of $f^2(x) = x$.*

*Proof.* By conclusion (iv) of Lemma 2, there are two possibilities. If $n = 2m + 1$ and if $f^2(x) \equiv x$ then $f^{2m}(x) \equiv f^2f^2 \dots f^2(x) \equiv x$, and hence $f^{2m+1}(x) \equiv f(x) \equiv x$. If $n$ is even, the stated result is immediate from Lemma 2 since $f(x) \equiv x$ is a solution of $f^2(x) = x$.

THEOREM 1. *The continuous real solutions of $f^n(x) = f(x)$, $n \geqslant 2$, are the functions of Class* I *if $n$ is even and the functions of Classes* I *and* II *if $n$ is odd.*

*Proof.* If $f(x)$ is of Class I then $f^2(x) = f(x)$. Whence

$$f^3(x) = f^2(x) = f(x), \dots .$$

If $f(x)$ is of Class II we verify that $f^3(x) = f(x)$. Then

$$f^5(x) = f^3(x) = f(x), \quad f^7(x) = f^5(x) = f(x), \dots .$$

Conversely, let $f(x)$ be a continuous solution of $f^n(x) = f(x)$. Then

$$f^{n-1} f^{n-1}(x) = f^{n-2} f^n(x) = f^{n-2} f(x) = f^{n-1}(x)$$

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so that $f^{n-1}(x)$ if of Class I by Lemma 1. Let $S$ be the maximal subset of the $x$-axis on which $f^{n-1}(x) \equiv x$.

If $S$ is a point, then $f^{n-1}(x) \equiv c$ and $f(x) \equiv ff^{n-1}(x) \equiv f(c)$, so that $f(x)$ is of Class I.

If $S$ is the closed interval $[a, b]$ then $f^{n-1}(x) \equiv x$ on $[a, b]$ and $a \leqslant f^{n-1}(x) \leqslant b$ by ($b_1$) and ($c_1$). Moreover $a \leqslant f(x) \leqslant b$, as a consequence of the relations

$$f(x) = f^n(x) = f^{n-1}f(x). \tag{2}$$

Thus $f(x)$ maps the real axis into $[a, b]$. In particular $[a, b]$ goes into $[a, b]$ and Lemma 2 is applicable. If $f(x) \equiv x$ on $[a, b]$, $f(x)$ is of Class I. The other possibility is that $f^2(x) \equiv x$ on $[a, b]$, and that $f(a) = b$, $f(b) = a$, in which event $f(x)$ is of Class II.

If $f^{n-1}(x) \equiv x$ on a ray, we see that $f(x)$ maps the reals into $[a, \infty]$ or $[-\infty, b]$. Hence $f(x) \equiv x$ on the ray by Lemma 2 and is of Class I.

Finally, if $S$ is the $x$-axis, the desired conclusion is given by Corollary **1**.

The functional equation

$$f^n(x) = f^m(x), \tag{3}$$

$m$ and $n$ integers, $1 < m < n$, has among its continuous solutions the functions of Classes I and II if $m + n$ is even and those of Class I if $m + n$ is odd. In either case, if $f(x)$ is a solution of (3) there exists an integer $k$ such that $f^k(x)$ is of Class I. However, each equation (3) has continuous solutions in neither of our classes. For example $f^3(x) = f^2(x)$ has the solution

$$f(x) = \begin{cases} \sin \pi x & x > 1 \\ 0 & |x| \leqslant 1 \\ -x-1 & -2 \leqslant x < -1 \\ 1 & x < -2. \end{cases}$$

Let $y = f(x)$ denote a transformation of a topological space $X$ into itself. A necessary and sufficient condition that $f^2(x) = f(x)$ have $f(x)$ as a continuous solution is that the set $S$ of fixed points under $f(x)$ be non-vacuous and that $f(x)$ map $X$ continuously into $S$. That $f(x)$ be a continuous solution of $f^2(x) = f(x)$ is a restriction on both $S$ and $f(x)$. If $X$ is the real axis these restrictions are given by our Lemma 1. In general the solutions of $f^2(x) = f(x)$ are the "retractions" of the space $X$. These have been studied by Borsuk [**1**] but results for higher-dimensional or abstract cases comparable to those of Lemma 1 do not seem to be available and appear difficult to achieve.

## References

**1.** K. Borsuk, *Sur les retractes*, Fund. Math., *17* (1931), 152–170.

**2.** G. T. Whyburn, *Analytic topology* (Amer. Math. Soc. Colloquium Publications, vol. 28, 1942).

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