VOL. XXXIII. (SESSION 1914-15)

PART 2.

https://doi.org/10.1017/S0013091500002376 Published online by Cambridge University Press

Two remarkable Cubics associated with a Triangle.

By F. GLANVILLE TAYLOR.

(*Received 1st December 1912. Read 13th December 1912.*)

I.

1. *Introductory.*

Let O_1 and H_1 be two points, in the plane of any triangle of reference ABC, so related that if O_1P, O_1Q, O_1R be the perpendiculars drawn to the sides of ABC, then AP, BQ, CR meet in H_1. We shall find that O_1 and H_1 describe respectively two cubics which are related to each other in a remarkable manner. We shall show, for instance, that points in each curve may be derived from each other by two sets of three alternative rational quadric transformations, and that the join of correspondents passes through a fixed point as in plane projection. We shall then discuss the homographic relation between corresponding pencils formed by rays through pairs of related points—not direct correspondents—and investigate the relation between these latter points.

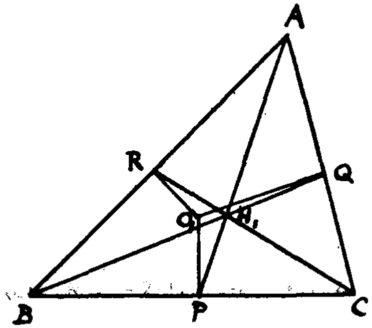

Fig. 2.

2. *Equations of the loci.*

To find the loci, let $(\alpha_1 \beta_1 \gamma_1)$ be the trilinear coordinates of O_1 (see Fig. 2), and $(\alpha\beta\gamma)$ those of H_1. Then, by considering perpendiculars from P to the sides b and c, we have

$$\left.\begin{aligned} &\frac{\beta}{\gamma}=\frac{\beta_1+\alpha_1\cos C}{\gamma_1+\alpha_1\cos B}=\frac{n'}{m}, \text{ say,} \\ &\text{and similarly} \\ &\frac{\gamma}{\alpha}=\frac{\gamma_1+\beta_1\cos A}{\alpha_1+\beta_1\cos C}=\frac{l'}{n}, \quad \frac{\alpha}{\beta}=\frac{\alpha_1+\gamma_1\cos B}{\beta_1+\gamma_1\cos A}=\frac{m'}{l} \end{aligned}\right\} \text{(A)}$$

Eliminating α, β, γ, and suppressing suffixes, the locus of O_1 is

$$\left.\begin{aligned} &(\beta+\gamma\cos A)(\gamma+\alpha\cos B)(\alpha+\beta\cos C) \\ &\qquad =(\gamma+\beta\cos A)(\alpha+\gamma\cos B)(\beta+\alpha\cos C) \\ &\text{or } lmn=l'm'n' \ \ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots\ldots\ (1)$$

which reduces to

$$\alpha(\beta^2-\gamma^2)(\cos A-\cos B\cos C)+\beta(\gamma^2-\alpha^2)(\cos B-\cos C\cos A) \\ +\gamma(\alpha^2-\beta^2)(\cos C-\cos A\cos B)=0 \ \ldots\ldots\ldots\ldots\ (2)$$

Again, to find the locus of H_1 we eliminate α_1, β_1, γ_1 from equations (A), obtaining the determinant

$$\begin{vmatrix} \beta\cos B-\gamma\cos C & -\gamma & \beta \\ \gamma & \gamma\cos C-\alpha\cos A & -\alpha \\ -\beta & \alpha & \alpha\cos A-\beta\cos B \end{vmatrix}=0,$$

or $$\Sigma(\sin^2 B\cos C\,\beta^2\gamma-\sin^2 C\cos B\,\beta\gamma^2)=0,$$

or $$a^2\alpha^2(\beta\cos B-\gamma\cos C)+b^2\beta^2(\gamma\cos C-\alpha\cos A) \\ +c^2\gamma^2(\alpha\cos A-\beta\cos B)=0 \ \ldots\ldots\ldots\ldots\ (3)$$

the equation required.

These two loci are therefore cubics through A, B, C.

It can be easily shown that the O-locus passes through the ortho-, circum-, in-, and ex- centres (H, O, I, I_1, etc.). Other points will be named later.

3. *The isogonal conjugates of all points on the O-locus lie on the same locus*, as is seen by changing $(\alpha\,\beta\,\gamma)$ to $\left(\frac{1}{\alpha}\,\frac{1}{\beta}\,\frac{1}{\gamma}\right)$ in (1) or (2). ~~Examples—(O, H), (I, I), etc.~~

4. Points which are symmetrical about the circumcentre O will be called opposite points, or *opposites*.

Let O_1, O_1' be a pair of opposites of which O_1 lies on the O-locus, and let P_1, P_1' be the feet of $\perp^{rs}$ on BC, etc. Then since $BP_1=P_1'C$, and similarly for the points Q, R; since also

AP_1, BQ_1, CR_1 meet in H_1, it follows that AP_1', BQ_1', CR_1' meet at the isotomic conjugate H_1'.* Hence O_1' must lie on the O-locus, and we have the following :—

(1) *The O-locus is symmetrical about O the circumcentre ;*

(2) *The isotomic conjugates of all points on the H-locus lie on the same locus ;* and

(3) *All such pairs of isotomic conjugates correspond to pairs of opposites on the O-locus.*

5. *Asymptotes of O-locus.*

If lines be drawn through the vertices of ABC, $\perp^r$ to the adjacent sides, we obtain two triangles LMN, $L'M'N'$ (Fig. 3). The sides l, m, n, l', m', n' have equations

$$[l]\ \beta+\gamma\cos A=0\ ; \quad [l']\ \gamma+\beta\cos A=0\ ; \quad \text{etc.}\dagger$$

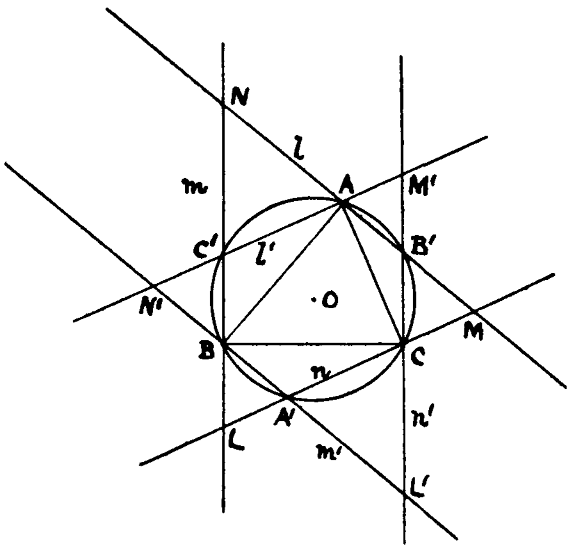

Fig. 3.

* This is, of course, the definition of the isotomic conjugate of a point. Also the isotomic conjugate of $(\alpha\beta\gamma)$ is $\left(\frac{1}{a^2\alpha}, \frac{1}{b^2\beta}, \frac{1}{c^2\gamma}\right)$.

† By Ceva's Theorem we may easily derive, for a point on the O-locus, that $lmn=l'm'n'$, which is equation (1).

https://doi.org/10.1017/S0013091500002376 Published online by Cambridge University Press

Equation (1) shows that *the O-locus passes through* the points $A'B'C'$ (as already seen), and also through *the three points at infinity in directions $\perp^r$ to the sides of ABC.* We shall call these A_1^∞, B_1^∞, C_1^∞.

Since pairs of opposites lie on the curve it follows that there are two points at infinity in each direction whose join therefore passes through O. *Hence there are three asymptotes, namely, the lines through O $\perp^r$ to the sides of ABC.*

This can be easily verified by analysis.

6. *The point H'. Notation.*

If H' be the opposite of H, the coordinates of H' are seen to be $(\cos A - \cos B\cos C, \quad \cos B - \cos C\cos A, \quad \cos C - \cos A\cos B)$. This is a very important point.

Hereafter the opposite of a point P on the O-locus will be denoted by P', and similarly for other letters. Also, with the exceptions given below and previously, the corresponding points on the H-locus will be represented by corresponding small letters p and p'. Thus p and p' are isotomic conjugates on the H-locus. And, as far as practicable, isogonal conjugates will be denoted thus, P and P_1.

The point H' lies on the H-locus (*as well as on the O-locus*), as may be seen by substituting its coordinates in (3).

7. We can easily prove that *all lines through H' meet the O-locus in pairs of isogonal conjugates.*

Denoting the coordinates of H' by (LMN), the condition that $(\alpha\beta\gamma)$, $\left(\frac{1}{\alpha}\,\frac{1}{\beta}\,\frac{1}{\gamma}\right)$, (LMN) are collinear is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \\ L & M & N \end{vmatrix} = 0$$

which gives the equation to the O-locus, whence the statement follows.

8. We can similarly show that *all lines through h', the isotomic conjugate of h, meet the H-locus in pairs of isotomic conjugates.*

https://doi.org/10.1017/S0013091500002376 Published online by Cambridge University Press

For, since the coordinates of h' are $\cos A/a^2$, etc., the condition that $(\alpha\beta\gamma)$, $(1/a^2\alpha,\ 1/b^2\beta,\ 1/c^2\gamma)$, $(\cos A/a^2,\ \cos B/b^2,\ \cos C/c^2)$ are collinear is

$$\begin{vmatrix} \alpha & \beta & \gamma \\ b^2c^2\beta\gamma & c^2a^2\gamma\alpha & a^2b^2\alpha\beta \\ b^2c^2\cos A & c^2a^2\cos B & a^2b^2\cos C \end{vmatrix} = 0$$

giving equation (3) of Art. 2.

9. Finally, *all lines through H' meet the two cubics in pairs of correspondents.*

For, from the second and third equations of (A), Art. 2, the coordinates of H_1 are $(m'n,\ nl,\ l'm')$, and the condition that $O_1 H' H_1$ are collinear is

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ L & M & N \\ m'n & nl & l'm' \end{vmatrix} = 0.$$

Also $\beta_1 N - \gamma_1 M = l\cos C - l'\cos B$, etc., whence we easily obtain $lmn = l'm'n'$, which is equation (1).

Thus any line through H' meets the cubics in P, P_1, p, p_1, where P, p; P_1, p_1 are correspondents, and P, P_1 are isogonal conjugates.

10. Since I, I_1, I_2, I_3 are the isogonal self-conjugates, it follows that $H'I$, $H'I_1$, etc., are tangents to the O-locus. Hence also $H'i$, $H'i_1$, etc., are tangents to the H-locus.

Or, *the common tangents of the two cubics meet at H'.*

Other points on the loci.

11. Putting $\alpha = 0$ in (2), we get

$$\frac{\beta}{\gamma} = \frac{\cos B - \cos C\cos A}{\cos C - \cos A\cos B} \text{ for the point } K \text{ in the figure,}$$

which evidently lies on AH'. (See Fig. 1 at end of paper.)

Hence $H'A$, $H'B$, $H'C$ meet BC, CA, AB in points K, L, M on the O-locus.

If the triangle $a_1b_1c_1$ be formed by drawing parallels to the opposite sides through A, B, C, then $a_1\,b_1\,c_1\,G$ are the four isotomic self-conjugates, and as they clearly correspond to the points $A_1^\infty B_1^\infty C_1^\infty O$, which are the four self-opposites, it follows that the former lie on the H-locus. Also, by Art. 8, h' is the tangential of $a_1\,b_1\,c_1\,G$.

12. If X, Y, Z be the points (BI_3', CI_2'), etc., then will XYZ lie on the O-locus. For we can show that AI_2', AI_3' are isogonal lines since ABC is the nine-point circle of $I_1'\,I_2'\,I_3'$, and D' the mid-point of $I_2'\,I_3'$ lies on the circle; similarly BI_3', BI_1' and CI_1', CI_2' are isogonal lines. Hence X is the isogonal conjugate of I_1', and therefore lies on the curve; so for Y and Z.

It follows that $H'XI_1'$, etc., are collinear triads; also $AI_2'\,Z$, $AI_3'\,Y$, etc., are collinear.

We can similarly show that AI', AI_1'; BI', BI_2'; CI', CI_3' are pairs of isogonal lines; hence if V is the isogonal conjugate of I', V lies on the O-locus, and $AI_1'\,V$, $BI_2'\,V$, $CI_3'\,V$, $H'I'\,V$ are collinear.

Thus if $XYZV$ be the isogonal conjugates of $I_1'\,I_2'\,I_3'\,I'$ respectively, then

$A\,I'\,X$	$A\,I_1'\,V$	$A\,I_2'\,Z$	$A\,I_3'\,Y$
$B\,I'\,Y$	$B\,I_1'\,Z$	$B\,I_2'\,V$	$B\,I_3'\,X$
$C\,I'\,Z$	$C\,I_1'\,Y$	$C\,I_2'\,X$	$C\,I_3'\,V$
$H'I'\,V$	$H'I_1'\,X$	$H'I_2'\,Y$	$H'I_3'\,Z$

are collinear triads.

II.

13. *The quadric transformations.*

The equations (A) are

$$\frac{\beta}{\gamma}=\frac{n'}{m},\quad \frac{\gamma}{\alpha}=\frac{l'}{n},\quad \frac{\alpha}{\beta}=\frac{m'}{l}.$$

From the second and third we have $\frac{\alpha}{m'n}=\frac{\beta}{nl}=\frac{\gamma}{l'm'}$ giving a quadric transformation from the H- to the O-plane; similarly, two other transformations are obtained, namely, from the third and first, and from the first and second.

Since the conics $m'n=0$, $nl=0$, $l'm'=0$, corresponding to $\alpha=0$, $\beta=0$, $\gamma=0$ pass through the points A', B_1^{∞}, C_1^{∞} (see Fig. 3), the transformation is a rational one, for a pair of lines in the H-plane is transformed into a pair of conics, in the O-plane, through these fixed points and a fourth point which corresponds uniquely to the meet of the lines. Similarly for the other two transformations.

Hence to a given point in the H-plane corresponds one in the O-plane for each transformation respectively. When the given point is on the H-locus, the three latter points, of course, coincide on the O-locus.

It follows that, to any given line in the H-plane cutting the H-cubic in p, q, r, correspond three hyperbolas in the O-plane which contain in common the three correspondents P. Q, R on the O-cubic, besides passing through the fixed points $A' B_1^\infty C_1^\infty$; $A_1^\infty B' C_1^\infty$; $A_1^\infty B_1^\infty C'$ respectively.

14. Similarly from equations (A), solving for α_1, β_1, γ_1, we have three alternative quadric transformations from the O- to the H-plane. We have, in fact, for the first transformation,

$$\frac{\alpha_1}{\alpha^2 \sin^2 A - \beta\gamma \cos B \cos C + \gamma\alpha \cos C \cos A + \alpha\beta \cos A \cos B}$$
$$= \frac{\beta_1}{\beta\gamma \cos B - \gamma\alpha \cos A + \alpha\beta} = \frac{\gamma_1}{\beta\gamma \cos C + \gamma\alpha - \alpha\beta \cos A},$$

and similarly for the other two.

It can easily be verified that the denominators equated to zero give three conics, each passing through the points a_1, b, c; and similarly for the others. Thus the transformation is again rational.

15. *Directions of the asymptotes of the H-cubic.*

Let $A_3 B_3 C_3$ be the O-correspondents of the points at infinity on the H-cubic. The line at infinity in the H-plane will be transformed into three alternative hyperbolas

$$a\, m'n + b\, nl + c\, l'm' = 0, \text{ etc.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

or, in full,

$$\left.\begin{aligned} &a\,\alpha^2 + b\cos C \,.\, \beta^2 + c\cos B \,.\, \gamma^2 + a\sin B\sin C \,.\, \beta\gamma \\ &\qquad\qquad\qquad\qquad + 2c\,\gamma\alpha + 2b\,\alpha\beta = 0 \\ &a\cos C \,.\, \alpha^2 + b\,\beta^2 + c\cos A \,.\, \gamma^2 + 2c\,\beta\gamma + b\sin C\sin A \,.\, \gamma\alpha \\ &\qquad\qquad\qquad\qquad + 2a\,\alpha\beta = 0 \\ &a\cos B \,.\, \alpha^2 + b\cos A \,.\, \beta^2 + c\,\gamma^2 + 2b\,\beta\gamma + 2\,a\,\gamma\alpha \\ &\qquad\qquad\qquad\qquad + c\sin A\sin B \,.\, \alpha\beta = 0 \end{aligned}\right\} \quad (B)$$

The first hyperbola will be seen to have its centre at A and its asymptotes $\perp^r$ to AB and AC; also it passes through A'; similarly for the others. The three all pass through A_3, B_3, C_3, the fourth point for each pair being A_1^∞, B_1^∞, C_1^∞ respectively.

Hence $H'A_3$, etc., are the directions of the H-asymptotes.

Multiplying equations (B) by a, b, c and adding, we obtain

$$2\,\Sigma\, a^2\alpha^2 + \Sigma\, bc\,(4+\sin^2 A)\,\beta\gamma = 0$$

$$\text{or}\quad a\,\beta\gamma + b\,\gamma\alpha + c\,\alpha\beta + \frac{2R}{\Delta}\,(a\,\alpha + b\,\beta + c\,\gamma)^2 = 0,$$

which is the circle $A_3 B_3 C_3$.

This circle is obviously concentric with the circumcircle ABC, and its radius is easily found to be $3R$. It also cuts the O-cubic in the three opposites A_3', B_3', C_3'.

16. *To prove that H' is the orthocentre of $A_3 B_3 C_3$.*

The circle, radius $3R$, meets each of the hyperbolas (B) in four points. Taking the first hyperbola only, three of these points are $A_3 B_3 C_3$; and, noting that A is the centre and that A', given by $n=0$, $m'=0$, lies on the curve [see equation (4) and Fig. 3], the fourth point, A''' say, is on OA produced.

The equation to the hyperbola, centre A, and asymptotes AB', AC', referred to rectangular axes through O is

$$(x\sin\gamma - y\cos\gamma - R\sin C)\,(-x\sin\beta + y\cos\beta - R\sin B) = \delta^2 \text{ say,}$$

and since this passes through A''' $(3R\cos\overline{\beta+B},\ 3R\sin\overline{\beta+B})$, we find that $\delta^2 = 4R^2\sin B\sin C$.

Now, let $x = 3R\cos\chi$, $y = 3R\sin\chi$, and let $\cos\chi \equiv c$, $\sin\chi \equiv s$; then

$$(3\,c\sin\gamma - 3\,s\cos\gamma - \sin C)\,(3\,c\sin\beta - 3\,s\cos\beta + \sin B)$$
$$+\,4\sin B\sin C = 0.$$

Solving for c, remembering that one root is $\cos\overline{\beta+B}$, we have

$$9\,c^2\sin\beta\sin\gamma + 9\,(1-c^2)\cos\beta\cos\gamma + 3\,c\,(\sin\gamma\sin B - \sin\beta\sin C)$$
$$+\,3\sin B\sin C = s\,[9\,c\sin\overline{\beta+\gamma} + 3\,(\cos\gamma\sin B - \cos\beta\sin C)]$$

$$\therefore\quad [-3\,c^2\cos\overline{\beta+\gamma} + c\,(\sin\gamma\sin B - \sin\beta\sin C) + \ldots]^2$$
$$= (1-c^2)\,[3\,c\sin\overline{\beta+\gamma} + \cos\gamma\sin B - \cos\beta\sin C]^2.$$

The coefficient of c^4 is 9.

The coefficient of c^3

$$= -6\,[\cos\overline{\beta+\gamma}\,(\sin\gamma\sin B - \sin\beta\sin C)$$
$$-\sin\overline{\beta+\gamma}\,(\cos\gamma\sin B - \cos\beta\sin C)]$$
$$= -6\,(\sin\gamma\sin C - \sin\beta\sin B)$$
$$= -6\,[\sin\gamma\sin\overline{\beta-\alpha} - \sin\beta\sin\overline{\alpha-\gamma}]$$
$$= -3\,[\Sigma\cos\overline{\beta+\gamma-\alpha} - 3\cos\overline{\beta+\gamma-\alpha}]$$
$$= 3\,\Sigma\cos\overline{\alpha+A} - 9\cos\overline{\beta+B}.$$

$$\therefore\qquad c_1 + c_2 + c_3 + c_4 = \cos\overline{\beta+B} - \tfrac{1}{3}\,\Sigma\cos\overline{\alpha+A}\,;$$

and, taking $c_4 = \cos\overline{\beta+B}$, $\quad c_1 + c_2 + c_3 = -\tfrac{1}{3}\,\Sigma\cos\overline{\alpha+A}$.

Now, the abscissae of A_3, B_3, C_3 are $3Rc_1$, $3Rc_2$, $3Rc_3$; and those of A', B', C' are $-R\cos\overline{\beta+B}$, $-R\cos\overline{\gamma+C}$, $-R\cos\overline{\alpha+A}$. Hence the centroids of $A_3B_3C_3$ and $A'B'C'$ coincide, since Ox is any line through O. But the circumcentres likewise coincide, and therefore so do the orthocentres. And H' is the orthocentre of $A'B'C'$. Q.E.D.

17. *The mid-points of the sides of $A_3B_3C_3$ lie on the O-locus.*

This can be shown by elementary geometry as follows:—

Let $D_3E_3F_3$ denote these points, and let $P_1Q_1R_1$, $P_2Q_2R_2$, $P_3Q_3R_3$ be the feet of perpendiculars from $A_3B_3C_3$ to the sides of ABC. Then, since $A_3B_3C_3$ are the O-correspondents of the points at infinity on the H-locus (see Fig. 4),

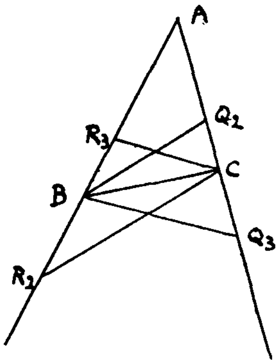

Fig. 4.

$$BQ_2 \parallel CR_2; \quad \therefore \quad \frac{BA}{R_2A} = \frac{Q_2A}{CA};$$

and similarly
$$\frac{BA}{R_3A} = \frac{Q_3A}{CA};$$

$$\therefore \quad \frac{R_2A}{R_3A} = \frac{Q_3A}{Q_2A}.$$

If P, Q, R are the mid-points of P_2P_3, Q_2Q_3, R_2R_3, then

$$\frac{R_2R_3}{RA} = \frac{Q_3Q_2}{QA}, \text{ and similarly } \frac{P_2P_3}{PB} = \frac{R_3R_2}{RB}, \quad \frac{Q_2Q_3}{QC} = \frac{P_3P_2}{PC};$$

thus $\dfrac{BP}{PC} \cdot \dfrac{CQ}{QA} \cdot \dfrac{AR}{RB} = 1$, or AP, BQ, CR are concurrent, at d_3 say.

https://doi.org/10.1017/S0013091500002376 Published online by Cambridge University Press

Since $P\,Q\,R$ are the feet of perpendiculars from D_3, it follows that D_3 and d_3 are two correspondents on the O- and H-locus respectively. Similarly for E_3 and F_3.

Note that $H'\,D_3\,A_3'$ are collinear; hence D_3, E_3, F_3 are the isogonal conjugates of A_3', B_3', C_3' respectively.

III.

18. *Chords through a fixed point.*

Let any line in the O-plane cut the O-locus in P, Q, R. Then, as we have seen, using the first transformation of Art. 14, the line $P\,Q\,R$ is transformed into a conic through a_1, b, c cutting the H-locus in p, q, r.

If, now, we keep P, and therefore p, fixed while we vary $Q\,R$, then the variable conic passes through four fixed points. Hence (Salmon's *Higher Plane Curves*, 3rd Ed., p. 134) qr passes through a fixed point s on the H-locus.

To find s, let $Q=H'$, $\therefore$ $R=P_1$, the isogonal conjugate of P; also $q=h'$, and $r=p_1$, say.

But $h'\,p_1\,s$ are collinear, whence s is the isotomic conjugate of p_1. Thus P and s are connected by the relation that the *isogonal conjugate of* P (viz. P_1) *corresponds to the isotomic conjugate of* s (viz. p_1).

We shall in this paper call such points *cross-correspondents* (*c.c.*), and denote them, *when practicable*, by $(P,\,p_1')$ or $(P',\,p_1)$.

Examples (see Fig. 1):—

Points on O-cubic	A	O	I	K	X	A_1^∞	H'	H	T
Isogonal conjugates	K	H	I	A	I_1'	A'	T	O	H'
H-correspondents	k	h	i	a	i_1'	a'	t	G	h'
Isotomic conjugates (c.c. of first line)	k'	h'	i'	a'	i_1	a	t'	G	h

Observe that H' $(=t')$ is its own c.c.

19. An important corollary is that if P is the tangential of a tetrad of four points on the O-cubic, its c.c. p_1' is that of the tetrad of H-correspondents: *e.g.* (1) $O\,(A_1^\infty B_1^\infty C_1^\infty O)$, $h'(a_1\,b_1\,c_1\,G)$; (2) $H'(II_1I_2I_3)$, $t'(ii_1i_2i_3)$.

Hence, also, *the (constant) cross-ratio of the pencil formed by four tangents from any point on either curve to that curve is the same for both curves.*

This cross-ratio is given by $\lambda = a^2(c^2 - b^2) / c^2(a^2 - b^2)$, where $\lambda = O(OA_1^\infty B_1^\infty C_1^\infty)$.

20. *The two pencils of corresponding chords through a pair of c.c. are homographic.*

For, to any line through a point P on the O-cubic, cutting the cubic again in Q and R correspond two lines through any given point s on the H-cubic, namely, those joining s to the corresponding points q and r. But if s is the c.c. of P, the points q and r are collinear with s (by definition of c.c.), and hence there is only one line through p_1' corresponding to the line through P, and *vice-versâ.* Hence the pencils are homographic, and their product is, of course, a conic through P and p_1'.

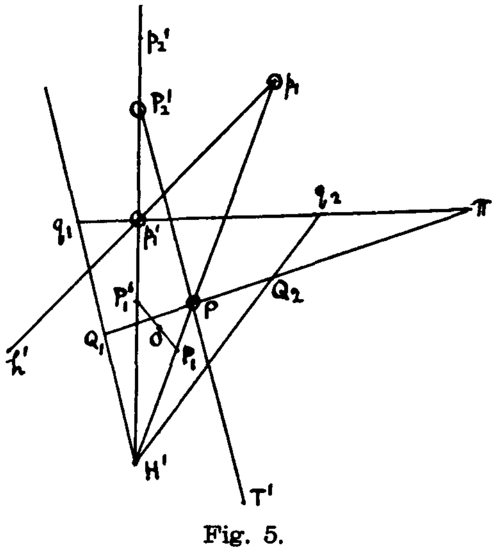

Fig. 5.

21. Since A, B, C, H are self-correspondents, they lie on the conic which is thus a *rectangular hyperbola.*

Let Q_1PQ_2, $q_1p_1'q_2$ (Fig. 5) be corresponding rays meeting at

https://doi.org/10.1017/S0013091500002376 Published online by Cambridge University Press

π on the hyperbola. If $Q_1 = H'$, then $q_1 = h'$, therefore $q_2 = p_1$, the isotomic conjugate of p_1'; Q_2 becoming P_1. Hence p_1 lies on the hyperbola.

Again, let $q_1 = H'$; then $Q_1 =$ the O-correspondent of H', viz. T' in the figure; $q_2 = p_2'$, say; $Q_2 = P_2'$.*

But $Q_1 P Q_2$, $q_1 p_1' q_2$, $H' Q_2 q_2$ are collinear triads: the same therefore is true of $T' P P_2'$, $H' p_1' p_2'$, and $H' P_2' p_2'$.

Hence the corresponding rays through H', T' meet at P_2' which is thus on the hyperbola.

Since the isogonal conjugate of P_2' corresponds to p_1', the isotomic conjugate of p_1, therefore p_1 is the c.c. of P_2'.

And the four points P, p_1', P_2', p_1 lie on the hyperbola $ABCH$.

Hence the following statements:—

(1) *Every member of the system of rectangular hyperbolas through $ABCH$ cuts the two cubics in two pairs of c.c.*

(2) *Every member cuts the H-cubic in a pair of isotomic conjugates.*

(3) *Every member cuts the O-cubic in the isogonal conjugates of a pair of opposites.* [Art. 4.]

(4) If $PQpq$ are the points, then Pp, Qq pass through H'; PQ through the O-correspondent of H' (viz. T'); and pq through the H-correspondent of H' (viz. h').

(5) *The joins of the isogonal conjugates of opposites on the O-cubic pass through a fixed point* (T').

(6) The common chords of all members of the system of rectangular hyperbolas $ABCH$ and the two cubics respectively pass through the two fixed points h', T'.

22. *If the join of PQ, two points on the O-cubic, meets the curve in a fixed point R, the join of their c.c. $p_1' q_1'$ will meet the H-cubic in a fixed point s_2, and conversely.*

---

* P_2' is the isogonal conjugate of P_1', and, according to our notation, should be written P'. Unfortunately, this clashes with the notation for opposites, since P_2' and P are *not* opposites.

To prove this :—

(1) Using Maclaurin's properties of three lines and the cubic, let T be the tangential of H' for the O cubic, and let the join of the isogonal conjugates of P and Q, *i.e.* P_1 and Q_1, meet the cubic in S. Then we have the scheme

$$\begin{array}{ccc} P_1 & Q_1 & S \\ P & Q & R \\ H' & H' & T \end{array}$$

whence RST are collinear, or *the join of the isogonal conjugates of* PQ *passes through a fixed point* S*;* since R and T are fixed.

(2) Again, we have $P_1 \; Q_1 \; S$; whence

$p_1 \; q_1 \; s_1'$ (the c.c. of S, and $\therefore$ fixed).

Therefore we have

$$\begin{array}{ccc} p_1' & q_1' & s_2 \\ p_1 & q_1 & s_1' \\ h' & h' & h \end{array}$$

whence $hs_1's_2$ are collinear, or s_2 is fixed.

The converse easily follows. Q.E.D.

Also, *the join of the isotomic conjugates of any two points* $p_1 q_1$ *passes through a fixed point, if that of* $p_1 q_1$ *does so.*

By putting $Q = O$ we may show that *the opposite of* R *and the isotomic conjugate of* s_2 *are c.c.*; or, *the opposite of* R *and the* O*-correspondent of* s_2 *are isogonal conjugates.* Also it may be shown that the rays RPQ, $s_2p_1'q_1'$ are homographic.

Note that T is the opposite of T', the O-correspondent of H', since the isogonal conjugates of the opposites H' and H are T' and O respectively, and by Art. 21 (5) TO passes through T'.

23. From the preceding we have the important corollary :— *If four points on either cubic form a tetrad with a common tangential, so also will their c.c.*

The same property holds good for *four points and their isogonal conjugates*, the join of the tangentials passing through T.

It also holds good for *four points and their isotomic conjugates,* the join of the tangentials passing through h.

The last two cases are, of course, only particular cases of the more general property of a cubic, namely that if $PQRS$ be a tetrad and PP', QQ', etc., pass through *any* fixed point E on the curve, then $P'Q'R'S'$ form a tetrad, the join of the tangentials passing through the tangential of E.

24. Examples of the above may be of interest:—

(1) If $R = O$, then $s_2 = h$. Hence the joins of c.c.'s of opposites pass through h. The c.c.'s of self-opposites are a, b, c, h'. Hence the tetrad $h\,(abch')$.

(2) If $R = H'$, then $s_2 = G$. Hence the joins of c.c.'s of isogonal conjugates pass through G, leading to the tetrad $G\,(i'i_1'i_2'i_3')$,

(3) From the tetrad $O\,(A_1^\infty B_1^\infty C_1^\infty O)$ we deduce for the isogonal conjugates $T'\,(A'B'C'H)$, since $T'O$ passes through T. Hence also, from the symmetry of the curve, $T\,(ABCH')$. This gives, again, for the isogonal conjugates the tetrad $(K\,L\,M\,T)$, and, therefore, $(K\,L'M'\,T')$.

(4) From the tetrad $T(ABCH')$ and the scheme of Art. 22 (1) it follows that if $R = A$, then $S = A$. Hence, if APQ are collinear, so also are AP_1Q_1. See also Art. 12.

(5) It may be proved by elementary geometry that $A_1^\infty II_1'$, $A_1^\infty I'I_1$, etc., are collinear. Hence by Art. 18 we have aii_1', $ai'i_1$, etc. And by Art. 22 we deduce $K'VI_1$, $K'XI$, etc. In each case there are 16 triads. Similar groups are given by $A_1^\infty L'M$, $A_1^\infty KT'$, etc., whence $al'm$, $ak't$, etc. Tables may be drawn up, as in Art. 12, with the help of the tetrads $XYZV$, $KLMT$, etc.

(6) *The joins of opposites are parallel to the joins of their c.c.* For, from (1), corresponding pencils OPQ, $hp_1'q_1'$ are homographic. Three pairs of corresponding rays are OA_1^∞, ha; OB_1^∞, hb; OC_1^∞, hc. But these are parallel; hence all corresponding rays are parallel. Thus, for example, AA' is parallel to $k'a_1$.

If $P = Q = O$, $p_1' = q_1' = h'$. Hence *the tangents at O and h' to the two curves are parallel, the common direction being $h'h$.*

25. If (see Fig. 5) $P_2' = P$, T' is the tangential, and $P = A'$, B', C', or H. Similarly, $p_1 = p_1' = a_1$, b_1, c_1, or G, h' being the tangential. Hence there are four, and only four, members of the system of

rectangular hyperbolas $ABCH$ which simultaneously touch the cubics in $A'a_1$, $B'b_1$, $C'c_1$, HG, respectively; these being pairs of c.c.

26. We may note finally that *the asymptotes of the H-cubic are not concurrent in general.* For if they were, then $A_3B_3C_3$ would form part of a tetrad. Hence (Salmon, *Higher Plane Curves*, p. 132) the fourth point would be the meet of A_3D_3, B_3E_3, C_3F_3, *i.e.* the centroid of $A_3B_3C_3$. But this point lies between O and H' (see Art. 16) and cannot therefore lie on the O-cubic.

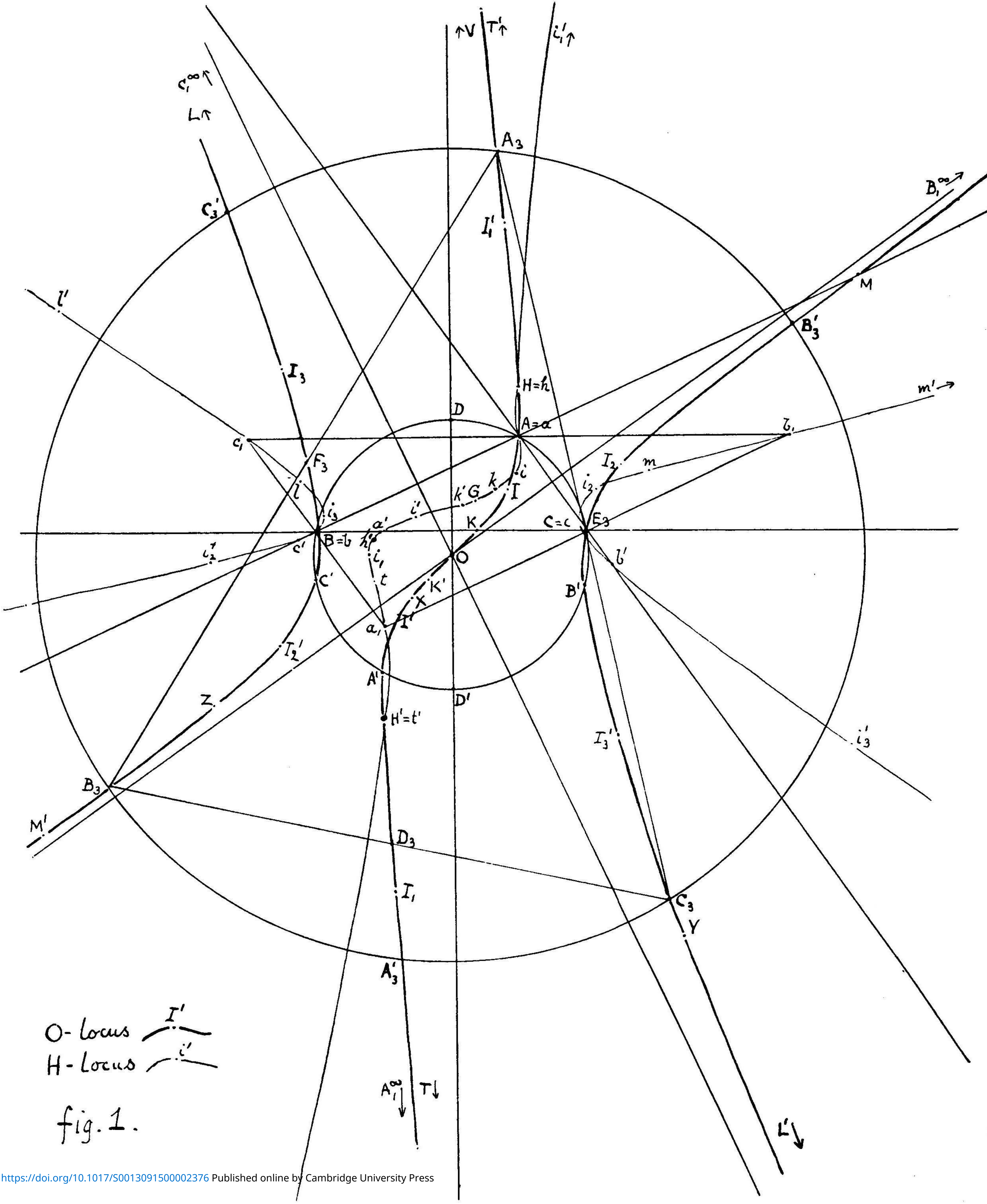

fig. 1.

https://doi.org/10.1017/S0013091500002376 Published online by Cambridge University Press