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On Uniqueness of Meromorphic Functions with Shared Values in Some Angular Domains

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Abstract. In this paper we investigate the uniqueness of transcendental meromorphic function dealing with the shared values in some angular domains instead of the whole complex plane.

1 Introduction and Main Results

Let $f\colon \mathbf{C} \to \overline{\mathbf{C}}$ be a transcendental meromorphic function, where $\mathbf{C}$ is the complex plane and $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. We assume that the readers are familiar with the Nevanlinna theory of meromorphic functions and the standard notations such as Nevanlinna deficiency $\delta(a, f)$ of $f(z)$ with respect to $a \in \overline{\mathbf{C}}$ and Nevanlinna characteristic $T(r, f)$ of $f(z)$. And the lower order μ and the order λ are in turn defined as follows:

$$\mu = \mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

$$\lambda = \lambda(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

For the references please see [9]. An $a \in \overline{C}$ is called an IM (ignoring multiplicities) shared value in $X \subseteq \mathbf{C}$ of two meromorphic functions $f(z)$ and $g(z)$ if in X, $f(z) = a$ if and only if $g(z) = a$. R. Nevanlinna [12] proved that if two meromorphic functions $f(z)$ and $g(z)$ have five distinct IM shared values in $X = \mathbf{C}$, then $f(z) \equiv g(z)$. After his very fundamental work, the uniqueness of meromorphic functions with shared values in the whole complex plane attracted many investigations. In this paper, we consider the uniqueness dealing with shared values in a proper subset of $\mathbf{C}$. It is obvious that Nevanlinna's result is true if $X = \mathbf{C}$ is replaced by X being the remaining part removing a bounded set from $\mathbf{C}$. We establish the following results.

We consider q pair of real numbers $\{\alpha_j, \beta_j\}$ such that

$$(1) \qquad -\pi \le \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \le \cdots \le \alpha_q < \beta_q \le \pi$$

and define

$$\omega = \max\left\{ \frac{\pi}{\beta_1 - \alpha_1}, \dots, \frac{\pi}{\beta_q - \alpha_q} \right\}.$$

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Theorem 1 *Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions and let $f(z)$ be of the finite lower order μ and such that for some $a \in \overline{\mathbf{C}}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. For q pair of real numbers $\{\alpha_j, \beta_j\}$ satisfying (1) and*

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma} \arcsin \sqrt{\frac{\delta}{2}}, \tag{2}$$

where $\sigma = \max\{\omega, \mu\}$, assume that $f(z)$ and $g(z)$ have five distinct IM shared values in $X = \bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$. If $\omega < \lambda(f)$, then $f(z) \equiv g(z)$.

If we remove the condition "$\mu(f) < \infty$" in Theorem 1, then we have the following.

Theorem 2 *Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions and such that for some $a \in \overline{\mathbf{C}}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. Assume that for q radii $\arg z = \alpha_j$, $(1 \leq j \leq q)$, satisfying*

$$-\pi \leq \alpha_1 < \alpha_2 < \cdots < \alpha_q < \pi, \quad \alpha_{q+1} = \alpha_1 + 2\pi,$$

$f(z)$ and $g(z)$ have five distinct IM shared values in $X = \mathbf{C} \setminus \bigcup_{j=1}^{q}\{z : \arg z = \alpha_j\}$. If

$$\max\left\{\frac{\pi}{\alpha_{j+1} - \alpha_j} : 1 \leq j \leq q\right\} < \lambda(f), \tag{3}$$

then $f(z) \equiv g(z)$.

Remark A. Observe the functions

$$f(z) = z + 1 + \sum_{n=1}^{\infty}\left(\frac{1}{a_n - z} - \frac{1}{a_n}\right)$$

where $a_n = n^{2/3}$, and $g(z)$ is defined by replacing "a_n" with "$-a_n$" in the form of $f(z)$. It is easy to see $\lambda(f) = \lambda(g) = 3/2$. From a result of Cebotarev [5] (see also Levin [11], or [3]), there exist five distinct real numbers b_j $(1 \leq j \leq 5)$ such that all the equations $f(z) = b_j$ and $g(z) = b_j$ have only real roots, and then all the b_j are IM shared values of $f(z)$ and $g(z)$ in $\mathbf{C} \setminus \mathbf{R}$. But $f(z) \not\equiv g(z)$. This implies that the condition "$\delta(a, f^{(p)}) > 0$" cannot be removed in Theorem 1 and Theorem 2.

B. It is easy to see that for each real number a, $0 \leq a \leq 1$, $\sin z$ and $\cos z$ can take value a only on the real axis, and then they have five distinct IM shared values in $\mathbf{C} \setminus \mathbf{R}$. Obviously $\lambda(\sin z) = \lambda(\cos z) = 1$ and $\delta(\infty, \sin z) = \delta(\infty, \cos z) = 1$. This shows that the condition "$\omega < \lambda(f)$" or (3) cannot be removed in Theorem 1 and Theorem 2, respectively.

C. We shall give an example to show that "$\mu(f) < \infty$" in Theorem 1 cannot be removed by using the theory of complex dynamics. For the basic knowledge of complex dynamics, please see [4]. We take into account the following function

$$g(z) = z - (1 + a) + \frac{1}{2\pi i}\int_L \frac{e^{e^t}}{t - z}\, dt$$

where L is the boundary of the region $\{\mathrm{Re}\, z > 0, -\pi < \mathrm{Im}\, z < \pi\}$ described in a clockwise direction. Then $f(z)$ is an entire function with infinite lower order. From the proof of Theorem 2 in Baker [2], the Julia set $J(g)$ of $g(z)$ lies in the region $\{\mathrm{Re}\, z > -a, -h < \mathrm{Im}\, z < h\}$ for suitable a and h. Since $J(g)$ does not contain any isolated Jordan arcs, there exists a horizontal straight line which intersects $J(f)$ at at least five points. By a translation, we conjugate $g(z)$ to an entire function $f(z)$ such that the Julia set $J(f)$ of $f(z)$ contains at least five real points c_j $(1 \leq j \leq 5)$. Then all the roots of $f(z) = c_j$ $(1 \leq j \leq 5)$ lie in $G = \{\mathrm{Re}\, z > -a, -2h < \mathrm{Im}\, z < 2h\}$. It is well-known that $\tan z = c_j$ $(1 \leq j \leq 5)$ have only real roots. Thus $f(z)$ and $\tan z$ have five distinct IM shared values in $\mathbf{C} \setminus (G \cup \mathbf{R})$ and obviously $\delta(\infty, f) = 1$, but $f(z) \not\equiv \tan z$. This shows that "$\mu(f) < \infty$" in Theorem 1 is necessary.

The method in this paper was used to investigate the growth of transcendental meromorphic functions with radially distributed values in Zheng [16].

2 Proofs of Theorems 1 and 2

First we need some auxiliary results for the proofs of the theorems. The following result was proved in [13] (also see [7]).

Lemma 1 *Let $f(z)$ be transcendental and meromorphic in $\mathbf{C}$ with the lower order $0 \leq \mu < \infty$ and the order $0 < \lambda \leq \infty$. Then for arbitrary positive number σ satisfying $\mu \leq \sigma \leq \lambda$ and a set E with finite linear measure, there exist a sequence of positive numbers $\{r_n\}$ such that*

(1) $r_n \notin E$, $\lim_{n\to\infty} \frac{r_n}{n} = \infty$;
(2) $\liminf_{n\to\infty} \frac{\log T(r_n, f)}{\log r_n} \geq \sigma$;
(3) $T(t, f) < \big(1 + o(1)\big)\,(\frac{t}{r_n})^{\sigma} T(r_n, f)$, $t \in [r_n/n, nr_n]$.

A sequence $\{r_n\}$ satisfying (1), (2) and (3) in Lemma 1 is called a Polya peak of order σ outside E in this paper. For $r > 0$ and $a \in \mathbf{C}$ define

$$D(r, a) := \left\{ \theta \in [-\pi, \pi) : \log^+ \frac{1}{|f(re^{i\theta}) - a|} > \frac{1}{\log r} T(r, f) \right\} \tag{4}$$

and

$$D(r, \infty) := \left\{ \theta \in [-\pi, \pi) : \log^+ |f(re^{i\theta})| > \frac{1}{\log r} T(r, f) \right\}.$$

The following result is a special version of the main result of Baernstein [1].

Lemma 2 *Let $f(z)$ be transcendental and meromorphic in $\mathbf{C}$ with the finite lower order μ and the order $0 < \lambda \leq \infty$ and for some $a \in \overline{\mathbf{C}}$, $\delta = \delta(a, f) > 0$. Then for arbitrary Polya peak $\{r_n\}$ of order $\sigma > 0$, $\mu \leq \sigma \leq \lambda$, we have*

$$\liminf_{n\to\infty} \operatorname{mes} D(r_n, a) \geq \min\left\{ 2\pi, \frac{4}{\sigma} \arcsin \sqrt{\frac{\delta}{2}} \right\}.$$

Although Lemma 2 was proved in [1] for the Polya peak of order μ, the same argument of Baernstein [1] can derive Lemma 2 for the Polya peak of order σ, $\mu \leq \sigma \leq \lambda$.

In order to prove our theorems, we need Nevanlinna theory on an angular domain. Let $f(z)$ be a meromorphic function on the angular domain $\overline{\Omega}(\alpha,\beta)=\{z : \alpha\le\arg z\le\beta\}$, where $0<\beta-\alpha\le 2\pi$. Following Nevanlinna (see [10]) define

$$
\begin{aligned}
A_{\alpha,\beta}(r,f)&=\frac{\omega}{\pi}\int_1^r\Big(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}}\Big)\{\log^+|f(te^{i\alpha})|+\log^+|f(te^{i\beta})|\}\,\frac{dt}{t},\\
B_{\alpha,\beta}(r,f)&=\frac{2\omega}{\pi r^\omega}\int_\alpha^\beta\log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)\,d\theta,\\
C_{\alpha,\beta}(r,f)&=2\sum_{1<|b_n|<r}\Big(\frac{1}{|b_n|^\omega}-\frac{|b_n|^\omega}{r^{2\omega}}\Big)\sin\omega(\theta_n-\alpha),
\end{aligned}
\tag{5}
$$

where $\omega=\frac{\pi}{\beta-\alpha}$ and $b_n=|b_n|e^{i\theta_n}$ are the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ appearing according to their multiplicities. $C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\overline{\Omega}(\alpha,\beta)$ and Nevanlinna's angular characteristic is defined as follows:

$$S_{\alpha,\beta}(r,f)=A_{\alpha,\beta}(r,f)+B_{\alpha,\beta}(r,f)+C_{\alpha,\beta}(r,f).$$

Throughout, we denote by $R_{\alpha,\beta}(r,*)$ a quantity satisfying

$$R_{\alpha,\beta}(r,*)=O\big\{\log\big(rS_{\alpha,\beta}(r,*)\big)\big\},\quad r\notin E,$$

where E denotes a set of positive real numbers with finite linear measure. It is not necessarily the same for every occurrence in the context [14].

Lemma 3 *Let $f(z)$ be meromorphic on $\overline{\Omega}(\alpha,\beta)$. Then for arbitrary complex number a, we have*

$$S_{\alpha,\beta}\Big(r,\frac{1}{f-a}\Big)=S_{\alpha,\beta}(r,f)+O(1)\tag{6}$$

and for an integer $p\ge 0$,

$$
\begin{aligned}
S_{\alpha,\beta}(r,f^{(p)})&\le 2^pS_{\alpha,\beta}(r,f)+R_{\alpha,\beta}(r,f),\\
A_{\alpha,\beta}\Big(r,\frac{f^{(p)}}{f}\Big)+B_{\alpha,\beta}\Big(r,\frac{f^{(p)}}{f}\Big)&=R_{\alpha,\beta}(r,f)
\end{aligned}
$$

and $R_{\alpha,\beta}(r,f^{(p)})=R_{\alpha,\beta}(r,f)$.

Lemma 3 can be proved by induction and noting

$$
\begin{aligned}
S(r,f')&\le C(r,f')+(A+B)(r,f)+(A+B)\Big(r,\frac{f'}{f}\Big)\\
&\le S(r,f)+\overline{C}(r,f)+R(r,f).
\end{aligned}
$$

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But in general, we do not know if $R_{\alpha,\beta}(r, f) = R_{\alpha,\beta}(r, f^{(p)})$.

Lemma 4 *Let $f(z)$ be meromorphic on $\overline{\Omega}(\alpha,\beta)$. Then for arbitrary q distinct $a_j \in \overline{\mathbf{C}}$ $(1 \le j \le q)$, we have*

$$(q-2)S_{\alpha,\beta}(r, f) \le \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\Big(r, \frac{1}{f-a_j}\Big) + R(r, f),$$

where the term $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j})$ will be replaced by $\overline{C}_{\alpha,\beta}(r, f)$ when some $a_j = \infty$.

Lemma 4 can be proved by the same argument as in the proof of Nevanlinna's second fundamental theorem.

Proof of Theorem 1 Suppose $f(z) \not\equiv g(z)$. Let $a_j \in \overline{C}$ $(1 \le j \le 5)$ be the five distinct IM shared values in X of $f(z)$ and $g(z)$. For convenience, below we omit the subscript of all the notations, such as $S(r,*)$ and $C(r,*)$. By applying Lemma 4 to g and (6), we have

$$\begin{aligned} 3S(r,g) &\le \sum_{j=1}^{5} \overline{C}\Big(r, \frac{1}{g-a_j}\Big) + R(r,g) \\ &\le C\Big(r, \frac{1}{f-g}\Big) + R(r,g) \\ &\le S(r, f-g) + R(r,g) \\ &\le S(r, f) + S(r,g) + R(r,g), \end{aligned}$$

so that

$$2S(r,g) - R(r,g) \le S(r, f). \tag{7}$$

This implies that $R(r,g) = R(r, f)$. We have also (7) for alternation of f and g, and combining (7) gives

$$2S(r, f) - R(r, f) \le S(r,g) \le S(r, f) + R(r, f).$$

Thus

$$S(r, f) = O(\log r), \quad r \notin E. \tag{8}$$

We assume that $a \in \mathbf{C}$. By the same argument we can show Theorem 1 for the case when $a = \infty$. By applying Lemma 3 and (8), we estimate

(9)

$$\begin{aligned} B\Big(r, \frac{1}{f^{(p)}-a}\Big) &\le S(r, f^{(p)}) + O(1) \\ &= (A+B)\Big(r, \frac{f^{(p)}}{f}\Big) + (A+B)(r, f) + p\overline{C}(r, f) + C(r, f) + O(1) \\ &\le (p+1)S(r, f) + R(r, f) \\ &= O(\log r), \quad r \notin E. \end{aligned}$$

The following method comes from [16]. Note that $\lambda(f) > \omega$. We need to treat two cases.

(I) $\lambda(f) > \mu$ Then $\lambda(f^{(p)}) = \lambda(f) > \sigma \geq \mu = \mu(f^{(p)})$. And by the inequality (2), we can take a real number $\epsilon > 0$ such that

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j + 2\epsilon) + 2\epsilon < \frac{4}{\sigma + 2\epsilon} \arcsin\sqrt{\frac{\delta}{2}}, \tag{10}$$

where $\alpha_{q+1} = 2\pi + \alpha_1$, and

$$\lambda(f^{(p)}) > \sigma + 2\epsilon > \mu.$$

Applying Lemma 1 to $f^{(p)}(z)$ gives the existence of the Polya peak $\{r_n\}$ of order $\sigma + 2\epsilon$ of $f^{(p)}$ such that $r_n \notin E$, and then from Lemma 2 for sufficiently large n we have

$$\operatorname{mes} D(r_n, a) > \frac{4}{\sigma + 2\epsilon} \arcsin\sqrt{\frac{\delta}{2}} - \epsilon, \tag{11}$$

since $\sigma + 2\epsilon > 1/2$. We can assume for all the n, (11) holds. Set

$$K := \operatorname{mes}\Big(D(r_n, a) \cap \bigcup_{j=1}^{q}(\alpha_j + \epsilon, \beta_j - \epsilon)\Big).$$

Then from (10) and (11) it follows that

$$\begin{aligned}
K &\geq \operatorname{mes}\big(D(r_n, a)\big) - \operatorname{mes}\Big([0, 2\pi) \setminus \bigcup_{j=1}^{q}(\alpha_j + \epsilon, \beta_j - \epsilon)\Big) \\
&= \operatorname{mes}\big(D(r_n, a)\big) - \operatorname{mes}\Big(\bigcup_{j=1}^{q}(\beta_j - \epsilon, \alpha_{j+1} + \epsilon)\Big) \\
&= \operatorname{mes}\big(D(r_n, a)\big) - \sum_{j=1}^{q}(\alpha_{j+1} - \beta_j + 2\epsilon) > \epsilon > 0.
\end{aligned}$$

It is easy to see that there exists a j_0 such that for infinitely many n, we have

$$\operatorname{mes}\big(D(r_n, a) \cap (\alpha_{j_0} + \epsilon, \beta_{j_0} - \epsilon)\big) > \frac{K}{q}. \tag{12}$$

We can assume for all the n, (12) holds. Set $E_n = D(r_n, a) \cap (\alpha_{j_0} + \epsilon, \beta_{j_0} - \epsilon)$. Thus from the definition (4) of $D(r, a)$ it follows that

$$\begin{aligned}
\int_{\alpha_{j_0}+\epsilon}^{\beta_{j_0}-\epsilon} \log^+ \frac{1}{|f^{(p)}(r_n e^{i\theta}) - a|}\, d\theta &\geq \int_{E_n} \log^+ \frac{1}{|f^{(p)}(r_n e^{i\theta}) - a|}\, d\theta \\
&\geq \operatorname{mes}(E_n) \frac{T(r_n, f^{(p)})}{\log r_n} \\
&> \frac{K}{q} \frac{T(r_n, f^{(p)})}{\log r_n}.
\end{aligned} \tag{13}$$

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On the other hand, by the definition (5) of $B_{\alpha,\beta}(r,*)$ and (9), we have

$$\int_{\alpha_{j_0}+\epsilon}^{\beta_{j_0}-\epsilon} \log^+ \frac{1}{|f^{(p)}(re^{i\theta})-a|}\,d\theta \le \frac{\pi}{2\omega_{j_0}\sin(\epsilon\omega_{j_0})} r^{\omega_{j_0}} B_{\alpha_{j_0},\beta_{j_0}}\Big(r, \frac{1}{f^{(p)}-a}\Big) < \tilde{K_{j_0}} r^{\omega_{j_0}} \log r, \quad r \notin E. \tag{14}$$

Combining (13) with (14) gives

$$T(r_n, f^{(p)}) \le \frac{q\tilde{K_{j_0}}}{K} r_n^{\omega_{j_0}} \log^2 r_n.$$

Thus from (2) in Lemma 1 for $\sigma + 2\epsilon$, we have

$$\sigma + 2\epsilon \le \limsup_{n\to\infty} \frac{\log T(r_n, f^{(p)})}{\log r_n} \le \omega_{j_0} \le \sigma + \epsilon.$$

This is impossible.

(II) $\lambda(f) = \mu$ Then $\sigma = \mu = \lambda(f) = \lambda(f^{(p)}) = \mu(f^{(p)})$. By the same argument as in (I) with all the $\sigma + 2\epsilon$ replaced by $\sigma = \mu$, we can derive

$$\max\{\omega, \mu\} = \sigma \le \omega < \lambda(f).$$

This is impossible.

Theorem 1 follows.

In order to prove Theorem 2, we need a result of Edrei [6].

Lemma 5 *Let $f(z)$ be a meromorphic function with $\delta = \delta(\infty, f) > 0$. Then given $\varepsilon > 0$, we have*

$$\text{mes}\, E(r,f) > \frac{1}{T^{\varepsilon}(r,f)[\log r]^{1+\varepsilon}}, \quad r \notin F,$$

where

$$E(r,f) = \Big\{\theta \in [-\pi,\pi) : \log^+|f(re^{i\theta})| > \frac{\delta}{4} T(r,f)\Big\}$$

and F is a set of positive real numbers with finite logarithmic measure depending on ε.

Proof of Theorem 2 As in the proof of Theorem 1, we have for each j,

$$B_{\alpha_j,\alpha_{j+1}}\Big(r, \frac{1}{f^{(p)}-a}\Big) = O(\log r), \quad r \notin E. \tag{15}$$

Applying Lemma 5 to $f^{(p)}(z)$ implies the existence of a sequence $\{r_n\}$ of positive numbers such that $r_n \to \infty$ $(n \to \infty)$ and $r_n \notin E$ and

$$\text{mes}\, E\Big(r_n, \frac{1}{f^{(p)}-a}\Big) \ge \frac{1}{T^{\varepsilon}(r_n, f^{(p)})[\log r_n]^{1+\varepsilon}}. \tag{16}$$

Set

$$\varepsilon_n = \frac{1}{2q+1}\frac{1}{T^{\varepsilon}(r_n, f)[\log r_n]^{1+\varepsilon}}.$$

Then from (16) it follows that

$$\begin{aligned}
&\operatorname{mes}\Bigl(E\Bigl(r_n, \frac{1}{f^{(p)}-a}\Bigr) \cap \bigcup_{j=1}^{q}(\alpha_j+\varepsilon_n, \alpha_{j+1}-\varepsilon_n)\Bigr)\\
&\qquad \geq \operatorname{mes} E\Bigl(r_n, \frac{1}{f^{(p)}-a}\Bigr) - \operatorname{mes}\Bigl(\bigcup_{j=1}^{q}(\alpha_j-\varepsilon_n, \alpha_j+\varepsilon_n)\Bigr)\\
&\qquad > (2q+1)\varepsilon_n - 2q\varepsilon_n = \varepsilon_n > 0,
\end{aligned}$$

so that there exists a j such that for infinitely many n, we have

$$\operatorname{mes} E_n > \frac{\varepsilon_n}{q}, \tag{17}$$

where $E_n = E(r_n, \frac{1}{f^{(p)}-a}) \cap (\alpha_j+\varepsilon_n, \alpha_{j+1}-\varepsilon_n)$. We can assume that (17) holds for all the n. Thus from the definition of $E(r, f)$ it follows that

$$\begin{aligned}
\int_{\alpha_j+\varepsilon_n}^{\alpha_{j+1}-\varepsilon_n} \log^+ \frac{1}{|f^{(p)}(r_ne^{i\theta})-a|}\,d\theta &\geq \int_{E_n} \log^+ \frac{1}{|f^{(p)}(r_ne^{i\theta})-a|}\,d\theta\\
&\geq \operatorname{mes}(E_n)\frac{\delta}{4}T(r_n, f^{(p)})\\
&> \frac{\delta\varepsilon_n}{4q}T(r_n, f^{(p)}).
\end{aligned} \tag{18}$$

On the other hand, by the definition of $B_{\alpha,\beta}(r,*)$ and (15), we have

$$\begin{aligned}
\int_{\alpha_j+\varepsilon_n}^{\alpha_{j+1}-\varepsilon_n} \log^+ \frac{1}{|f^{(p)}(re^{i\theta})-a|}\,d\theta &\leq \frac{\pi}{2\omega_j\sin(\varepsilon_n\omega_j)}r^{\omega_j}B_{\alpha_j,\alpha_{j+1}}\Bigl(r, \frac{1}{f^{(p)}-a}\Bigr)\\
&< \frac{\pi^2}{4\omega_j^2\varepsilon_n}O(r^{\omega_j}\log r), \quad r \notin E,
\end{aligned} \tag{19}$$

where $\omega_j = \frac{\pi}{\alpha_{j+1}-\alpha_j}$. Combining (18) with (19) gives

$$\varepsilon_n^2 T(r_n, f^{(p)}) \leq O(r_n^{\omega_j}\log r_n),$$

so that

$$T^{1-2\varepsilon}(r_n, f^{(p)}) \leq O(r_n^{\omega_j}[\log r_n]^{3+2\varepsilon}).$$

Thus $\mu(f) = \mu(f^{(p)}) \leq \omega/(1-2\varepsilon) < \infty$.

Theorem 2 follows from Theorem 1.

3 Conclusion

For many uniqueness theorems established for meromorphic functions with shared values in the complex plane, we can establish their counterparts for meromorphic functions with shared values in angular domains. For example, by the arguments of Frank and Schwick [8] and this paper, and noting the fact that the zeros of non-zero analytic functions are isolated, we can prove the following

Theorem 3 *Let $f(z)$ be a transcendental meromorphic function of finite lower order μ and such that for some $a \in \overline{\mathbf{C}}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. For q pairs of real numbers $\{\alpha_j, \beta_j\}$ satisfying (1) and (2), assume that $f(z)$ and $f^{(k)}(z)$ have three distinct IM shared values in $X = \bigcup_{j=1}^{q}\{z : \alpha_j \leq \arg z \leq \beta_j\}$. If $\omega < \lambda(f)$, then $f(z) \equiv f^{(k)}(z)$.*

We can also establish the result similar to Theorem 3 corresponding to Theorem 2. Finally, what the author emphasizes is that the topic on the uniqueness of meromorphic functions dealing with shared values in an unbounded proper subset of the whole complex plane is interesting and deserves to be investigated.

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