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# INTEGRAL DOMAINS WHICH HAVE FINITE CHARACTER LOCALLY

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ABSTRACT. In recent papers Brewer and Mott have studied integral domains which have finite character globally. This paper concentrates on domains which have finite character locally. Examples include global finite character domains plus Prufer, almost Dedekind, and almost Krull domains. General properties are given, including a valuation-theoretic characterization. The effect of requiring essential and/or rank one valuations is also studied.

**General properties.** An integral domain $D$ with quotient field $K$ has *finite character* if there exists a family $F$ of valuations of $K$ having the following properties:

(1) *$F$ defines $D$*, i.e., $D=\bigcap_{v\in F} R_v$, where $R_v$ denotes the valuation ring of $v$, and

(2) $F$ has *finite character*, i.e., each nonzero element of $K$ is a nonunit in at most finitely many of the $R_v$ for $v\in F$.

Examples include Dedekind, Krull, and generalized Krull domains, plus domains of finite real character and domains of Krull type. A domain $D$ has *finite character locally* if for each maximal ideal $M$ of $D$, the quotient ring $D_M$ has finite character. In [5] Gilmer studied domains which are locally Dedekind, refering to them as *almost Dedekind*. Pirtle in [16] introduced *almost Krull* domains, i.e., domains which are locally Krull. Following Gilmer and Pirtle, we call a domain which has finite character locally a *domain of almost finite character* (AFC-*domain*). Our first result shows that the seeming dependence on maximal ideals is only superficial.

THEOREM 1. *$D$ is an* AFC-*domain if and only if $D_P$ is a domain of finite character for every proper prime ideal $P$ of $D$.*

**Proof.** $D_P=(D_M)_{PD_M}$ for some maximal ideal $M$, and domains of finite character are stable under localization [9, p. 721].

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We next characterize AFC-domains in terms of families of valuations. The proof is immediate from the definitions and the fact that any integral domain $D$ is the intersection of it's quotient rings $D_M$, $M$ ranging over the maximal ideals of $D$[18, p. 94].

THEOREM 2. *$D$ is an* AFC-*domain if and only if there exists a family $F$ of valuations of the quotient field having the following properties*:

(1) $D=\bigcap_{v\in F} R_v$, *and for each maximal ideal $M$ of $D$, there exists a subfamily $F_M$ of $F$ such that* $D_M=\bigcap_{v\in F_M} R_v$.

(2) *Each subfamily $F_M$ has finite character.*

COROLLARY 3. AFC-*domains are integrally closed.*

The family $F$ of Theorem 2 is called a *defining family* for $D$. It need not be unique. However, later in this paper we shall mention conditions under which a unique defining family does exist.

We look next at stability properties for AFC-domains. The proofs are straightforward and will be omitted. They involve manipulations with quotient rings and the corresponding stability properties for domains of finite character, properties that were proven by Griffin in [9]. It is also convenient to use Griffin's concept of coarseness, where a family of valuations $F'$ is said to be *coarser* than a family $F$ if for each $v'\in F'$, there exists $v\in F$ with $R_v\subseteq R_{v'}$.

THEOREM 4. *Let $D$ be an* AFC-*domain with defining family $F$ and quotient field $K$.*

(a) *If $S$ is a multiplicatively closed set in $D$* ($0\notin S$), *then the quotient ring $D_S$ is an* AFC-*domain with defining family coarser than $F$.*

(b) *Let $K'$ be a finite, algebraic extension of $K$, let $D'$ denote the integral closure of $D$ in $K'$, and let $F'$ denote the family of all extensions of valuations in $F$ to $K'$. Then $D'$ is an* AFC-*domain with defining family coarser than $F'$.*

(c) *Let $\{X_i\}$ be an arbitrary set of indeterminates, let $F'$ denote the family of canonical extensions of valuations in $F$ to $K(X_i)$, and let $G$ denote the family of all valuations of $K(X_i)$ defined by the irreducible polynomials in $K[X_i]$. Then $D[X_i]$ is an* AFC-*domain with defining family coarser than $F'\cup G$.*

The next result provides a sufficient condition for an AFC-domain to be a domain of finite character. The proof is similar to [16, p. 445] and will be omitted.

THEOREM 5. *Let $D$ be an* AFC-*domain with defining family $F$. If every proper ideal of $D$ is contained in only a finite number of maximal ideals, then $D$ is a domain of finite character with $F$ as a defining family. In particular, the result holds if $D$ has only a finite number of maximal ideals.*

We close this section by generalizing results of Brewer in [2] on the ideal transform. In [14] Nagata defined the transform $T(I)$ of an ideal $I$ in a commutative ring $R$ with identity and having total quotient ring $S$ as follows: $T(I)=\bigcup_{n=1}^{\infty} J_n$, where $J_n=R:I^n=\{x\in S: xI^n\subseteq R\}$.

THEOREM 6. *Let $D$ be an integral domain which is not quasi-local. Then $D$ is an* AFC-*domain if and only if $T((x))$ is an* AFC-*domain for each nonunit $x \in D$.*

**Proof.** For the "if" part, let $M$ be a maximal ideal of $D$. There exists a nonunit $x \in D$ such that $x \notin M$, and by [2, p. 302] there exists a maximal ideal $M'$ of $T((x))$ such that $M = M' \cap D$ and $D_M = T((x))_M$. The converse follows from Theorem 4 since $T((x)) = D_N$, where $N = \{x^i : i = 1, 2, \ldots\}$ [2, p. 303].

**Essential valuations.** A valuation $v$ is *essential* for a domain $D$ if $R_v = D_P$ for some prime ideal $P$ of $D$. In [9] Griffin calls a domain $D$ which has a defining family consisting of essential valuations an *essential domain*, while a domain of finite character with a defining family of essential valuations is a *domain of Krull type*. It is easy to see that a domain which is locally essential is also globally essential, but a domain which has Krull type locally leads to a new concept, which we call a *domain of almost Krull type* (*AKT-domain*). Krull type implies *AKT* which in turn implies both *AFC* and essential. Almost Dedekind, almost Krull, and Prufer domains have *AKT* but not Krull type, while [10, Ex. 2, p. 84] provides an example of an AFC-domain that does not have *AKT*. We have no example of an essential domain not having *AKT*.

Theorems 1, 4, 5, and 6 remain true if we replace *AFC* by *AKT* and finite character by Krull type, while for Theorem 2 we need to add the following condition:

(3) Each $v \in F$ is essential for $D$.

Corollary 3 can be strengthened using the concept of regular integral closure [1, p. 89].

COROLLARY 3′. *AKT-domains are regularly integrally closed.*

**Proof.** *AKT*-domains are essential and essential domains are regularly integrally closed [3, Remark 1, p. 7].

In [10, Prop. 12, p. 83] Griffin proved that if $D$ is a domain of finite character with defining family $F$, then any essential valuation of $D$ is coarser than some valuation in $F$. Moreover, it is easy to see that if $F'$ is any family of valuations coarser than $F$, then $F'$ also has finite character. Using these two facts gives us the following result.

THEOREM 7. *Let $D$ be an AKT-domain with defining family $F$. Then $D$ is a domain of Krull type with $F$ as a defining family if and only if $D$ has finite character for some defining family $G$. In this case, $F$ is a coarser family than $G$.*

The next result provides sufficient conditions for an *AKT*-domain to have Krull type and shows the existence of large classes of *AKT*-domains.

THEOREM 8. *Let $D$ be an AKT-domain with defining family $F$ and quotient field $K$. Then $D$ is a domain of Krull type with $F$ as a defining family in either of the following* *two cases*:

(a) $D[X_i]$ *has finite character, where* $\{X_i\}$ *is any set of indeterminates, or*

(b) $D'$ *has finite character, where* $D'$ *denotes the integral closure of* $D$ *in a finite, algebraic extension* $K'$ *of* $K$.

*In particular, if* $D$ *is an AKT-domain that does not have Krull type, then the same is true for both* $D[X_i]$ *and* $D'$.

**Proof.** We indicate a proof for part (b). By [9, p. 718] the family $F'$ of all extensions of valuations in $F$ to $K'$ consists of valuations essential for $D'$ and Theorem 7 then shows that $F'$ has finite character. But this implies that $F$ has finite character since for $0\neq x \in K$ and $v \in F$, $v(x)=v'(x)$ for any extension $v'$ of $v$.

**Rank one valuations.** A *domain of finite real character* is a domain of finite character which has a defining family consisting of rank one valuations, while a *generalized Krull domain* is a domain of Krull type which has a defining family consisting of rank one valuations. Ribenboim has studied these domains in [17]. By a *domain of almost finite real character* (AFRC-*domain*) we will mean a domain which has finite real character locally, and similarly, by an *almost generalized Krull domain* (AGK-*domain*) a domain that is locally generalized Krull. AGK implies both AFRC and AKT, AFRC implies AFC, generalized Krull implies AGK, and finite real character implies AFRC. Almost Dedekind, almost Krull, and one-dimensional Prufer domains are AGK but not generalized Krull, while Prufer domains having (Krull) dimension greater than one are not AFRC and hence not AGK. Thus the ring of entire functions $E$, known to be an infinite-dimensional Bezout domain ([11, p. 351] and [12, p. 717]), is AKT but not AFRC. Moreover, $E$ does not have Krull type by Theorem 7 and [1, p. 88]. A two-dimensional AKT-domain which has neither Krull type nor AFRC appears in [7, Ex. 1, p. 299]. The ring of all algebraic integers, a one-dimensional Bezout domain [13, p. 86], is AGK but by [6, p. 597] neither generalized Krull nor almost Krull. Finally, [10, Ex. 1, p. 84] and [15, p. 330] provide examples of AFRC-domains which are not AGK.

Theorems 1, 4, 5, and 6 remain true if we replace AFC and finite character by either AGK and generalized Krull or AFRC and finite real character. Substituting AFRC for AFC in Theorem 2 will give a valid result if and only if we include the following condition:

(4) Each $v \in F$ has rank one.

Theorem 2 remains true with AGK in place of AFC if and only if we include both conditions (3) and (4). Corollaries 3 and 3′ can be strengthened using the concept of complete integral closure [18, p. 250], and it will follow from this corollary that if $D$ is an AFRC-domain, then the monoid of divisors $\mathscr{D}(D)$ is a group [1, p. 5].

COROLLARY 3″. AFRC-*domains are completely integrally closed.*

Theorems 7 and 8 remain true with AGK and generalized Krull in place of AKT and Krull type, and for AFRC-domains we have the following result.

THEOREM 9. *Let $D$ be an AFRC-domain with defining family $F$ and quotient field $K$.*

(a) *If $P$ is a minimal prime ideal of $D$, then $D_P$ is a valuation ring, necessarily essential and of rank one.*

(b) *If $w$ is a nontrivial valuation of $K$ essential for $D$, then $w \in F$. Thus $w$ has rank one and $R_w = D_P$ for some minimal prime ideal $P$ of $D$.*

**Proof.** For part (a) we note that $D_P$ is a quasi-local, one-dimensional domain of finite real character and hence a valuation ring by [4, Th. 5, p. 36]. For part (b), suppose $R_w = D_P$ for some proper prime ideal $P$ of $D$. Then $R_w$ is a domain of finite real character with defining family $F_w \subseteq F$. But $F$ consists of rank one valuations and $R_w$ is already a valuation ring, so $F_w = \{w\}$ and hence $w \in F$.

An immediate corollary of Theorem 9 is that an AGK-domain has a unique defining family.

COROLLARY 10. *If $D$ is an AGK-domain with defining family $F$, then $\{R_v : v \in F\} = \{D_P : P$ is a minimal prime ideal of $D\}$.*

Without further restrictions neither AFRC-, AKT-, nor AFC-domains have unique defining families. However, using the concept of $S$-representation introduced by Gilmer and Heinzer in [8] and recent results of Brewer and Mott [4, Th. 14, p. 38] and of Brewer [3, Th. 1.1, p. 8], it is easy to see that a unique defining family $F$ does exist for AFRC- and AKT-domains $D$ provided we make the additional requirements that $F = \cup F_M$, $M$ ranging over all maximal ideals of $D$, and that each $F_M$ provide an $S$-representation for the quotient ring $D_M$.

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