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SOME PROPERTIES OF PERIODIC MODULES

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Abstract

In this paper periodic modules over group rings and algebras are considered. A new lower bound for the p-part of the rank of a periodic module with abelian vertex is given, and results on periodic modules with odd/even and small periods are obtained. In particular, it is shown that characters afforded by periodic lattices of odd period satisfy strong properties and that irreducible periodic lattices are always of even period.

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1. Introduction

Let G be a finite group, R a complete discrete valuation ring with quotient field K of characteristic 0, and residue field F of characteristic $p > 0$, $A \in \{R, F\}$. In this article we will consider periodic AG-modules, where the term "AG-modules" denotes modules which are finitely generated and free over A; for $A = R$ we also use the term "RG-lattice".

For periodic AG-modules with abelian vertices we shall improve the lower bound on the p-part of their A-rank, given by Green's theorem [4, 59.6] or by the improved version of this theorem involving complexity [1, 3.1]. This is the content of Theorem 2.1; an application of a theorem of Jon Carlson [2] is crucial in proving 2.1. In the rest of this section we shall derive consequences of the existence of an irreducible periodic RG-lattice with abelian vertex.

The topic of the next section is the period of periodic modules. Theorem 3.2 deals with periodic RG-lattices of odd periods. They satisfy very strong properties. In particular, their rank is always divisible by the highest power of p dividing the order of G. As a corollary of this, irreducible periodic lattices are always of even period, if K is sufficiently large. Moreover, results on RG-lattices of period 1 and 2 are obtained. We also consider periodic FG-modules, but although some results are parallel to those on lattices, the implications of odd period are less strong. In particular, when $p = 2$ the simplest example shows that the full power of p dividing the order of G does in general not divide the dimension of a periodic module of odd period, and that a simple FG-module need not have even period.

For standard terminology we refer to the books of Dornhoff [4] and Feit [6].

2. Periodic modules with abelian vertices

In the following, for a natural number n, n_p denotes the highest power of p dividing n.

2.1. THEOREM. *Let U be an indecomposable periodic AG-module with abelian vertex D. Then $|G|_p \exp(D)^{-1} | \operatorname{rank}_A U$, where $\exp(D)$ denotes the exponent of D.*

PROOF. (i) Let $A = F$.

Without loss of generality we can assume that F is algebraically closed. Take a Sylow p-subgroup P of G. Then $U_P = \bigoplus_{i=1}^{n} U_i$ with indecomposable FP-modules U_i, and each U_i is $(D^{x_i} \cap P)$-projective, for some $x_i \in G$. Set $D_i = D^{x_i} \cap P$. By Green's theorem [4, 52.6] we have $U_i \simeq V_i^P$, with indecomposable FD_i-modules V_i, for $1 \leqslant i \leqslant n$. As U is periodic, so is each U_i and hence each V_i. Now since the D_i are abelian, by [2, 5.6] we have $|D_i| \exp(D_i)^{-1} | \dim_F V_i$, for all i. This implies $|P| \exp(D_i)^{-1} | \dim_F U_i$, for all i. But obviously $\exp(D_i) \leqslant \exp(D)$ for all i, so $|P| \exp(D)^{-1} | \dim_F U_i$, for all i, and thus $|P| \exp(D)^{-1} | \dim_F U$.

(ii) Now suppose $A = R$. Then $U \otimes_R F \simeq \bigoplus_{j=1}^{m} V_j$, with indecomposable periodic FG-modules V_j, for all j. As $vx\, V_j \leqslant_G D$, $vx\, V_j$ is abelian and by (i)

$$|G|_p \exp(D)^{-1} \big| \dim_F V_j, \quad \text{for all } j.$$

This gives

$$|G|_p \exp(D)^{-1} \big| \dim_F \big(U \otimes_R F\big) = \operatorname{rank}_R U.$$

2.2. REMARKS. (1) This lower bound for the p-part of $\operatorname{rank}_A U$ improves the bound $|G : D|_p p^{r(D)-1}$, where $r(D)$ is the rank of D, which we get from [1, 3.1].

(2) If D is not abelian, the lower bound given in 2.1 does not hold, as already the example $G = Q_8$ (the quaternion group of order 8) and $U = A_G$ (the trivial AG-module) shows.

2.3. COROLLARY. *If K is a splitting field for G and its subgroups and U an irreducible periodic RG-lattice with elementary abelian vertex D, then D is cyclic.*

PROOF. By Theorem 2.1 $|G|_p p^{-1} | \operatorname{rank}_R U$. Now by [9, Theorem 4.5B] this implies that U belongs to a block of defect 0 or 1. Hence D must be cyclic.

2.4. REMARK. The above corollary was also obtained in [1] as Corollary 3.5.

In the following, we will combine Theorem 2.1 with well-known theorems on upper bounds of character degrees. First, let us state these results for convenience. For proofs see [8, 5.11], [7, Chapter V, 17.9] or [3, 53.17]. Observe that in (b) we can take K to be a splitting field instead of algebraically closed.

2.5. THEOREM. *Let M be a simple KG-module.*

(a) *If* $\operatorname{End}_{KG}(M) \simeq K$ *and H is an abelian normal subgroup of G, then* $\dim_K M \,|\, |G : H|$.

(b) *Suppose that K is a splitting field for G and its subgroups, and that H is a subnormal subgroup of G. Then* $\dim_K M \,|\, |G : H| d$, *where* $d = \dim_K V, V$ *any simple KH-module with* $V | M_H$.

Before we turn to periodic modules, we derive a corollary to Theorem 2.5. First we have a definition.

2.6. DEFINITION. Let U be an indecomposable AG-module with vertex D. Then U is said to be of vertex height 0, if $|G : D|_p = (\operatorname{rank}_A U)_p$.

2.7. COROLLARY. *Let U be an irreducible RG-lattice with vertex D.*

(a) *If* $\operatorname{End}_{RG}(U) \simeq R$ *and D is normal and abelian, then U is of vertex height* 0.

(b) *If K is a splitting field for G and its subgroups, D is subnormal and there exists an irreducible RD-lattice V of rank not divisible by p, such that* $\operatorname{Hom}_{RG}(U, V^G) \neq 0$ *(e.g., if D is abelian), then U is of vertex height* 0.

PROOF. (a) is clear by 2.5(a).

(b) By 2.5(b) $\operatorname{rank}_R U \,|\, |G : D| \operatorname{rank}_R V$, as $KV | KU_D$. Thus, as $p \nmid \operatorname{rank}_R V$, $(\operatorname{rank}_R U)_p \,|\, |G : D|_p$. By Green's theorem, $|G : D|_p | \operatorname{rank}_R U$, so U is of vertex height 0.

For the rest of this section we assume that R is sufficiently large, so that K is a splitting field for G and its subgroups.

2.8. PROPOSITION. *Let U be an irreducible periodic RG-lattice with abelian vertex D. Let H be a subnormal subgroup of G, and suppose* $\operatorname{Hom}_{RG}(U, V^G) \neq 0$ *for some irreducible RH-lattice V of rank not divisible by p. (In particular, this is satisfied if H is abelian.) Then* $|H|_p \mid \exp(D)$.

PROOF. By 2.1 we have $|G|_p \exp(D)^{-1} | \operatorname{rank}_R U$. On the other hand, by 2.5(b) $(\operatorname{rank}_R U)_p \,|\, |G : H|_p$. Hence $|H|_p | \exp(D)$.

From this we get immediately

2.9. COROLLARY. *Let U be an irreducible periodic RG-lattice with subnormal abelian vertex D. Then D is cyclic.*

2.10. PROPOSITION. *Let U be an irreducible periodic RG-lattice with abelian vertex D, and let H be a subnormal p-subgroup of G, $|H| = p^n$, $n \geqslant 1$. Then* $p^{[n/2]+1} | \exp(D)$, *where $[n/2]$ is the integer part of $n/2$.*

PROOF. By 2.5(b) $\operatorname{rank}_R U \,|\, |G : H| \dim_K M$, where M is a simple KH-module. As $|H| = p^n$, $\dim_K M = p^a$ and

$$0 \leqslant a \leqslant \begin{cases} \dfrac{n}{2} - 1, & \text{if } n \text{ is even,} \\ \dfrac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Now by 2.1 $|G|_p \exp(D)^{-1} | \operatorname{rank}_R U$, so $|H| p^{1-n/2} | \exp(D)$, if n is even, $|H| p^{(1-n)/2} | \exp(D)$, if n is odd. Since

$$p^{[n/2]+1} = \begin{cases} p^{n/2+1}, & \text{if } n \text{ is even,} \\ p^{(n+1)/2}, & \text{if } n \text{ is odd,} \end{cases}$$

we get the result.

3. Periods of periodic modules

In this paragraph we will derive properties of periodic modules from the fact that they are of odd/even or small period. For an RG-lattice U, the character afforded by U will be denoted by χ_U.

3.1. PROPOSITION. *Let U be a periodic RG-lattice, $0 \to U \to P_n \to \cdots \to P_1 \to U \to 0$ an exact sequence with projective RG-lattices P_i, $1 \leqslant i \leqslant n$.*

(a) *If n is odd, then* $2 \cdot \chi_U = \sum_{i=1}^{n} (-1)^{i-1} \chi_{P_i}$.

(b) *If n is even, then* $\bigoplus_{i=1}^{n/2} P_{2i-1} \simeq \bigoplus_{i=1}^{n/2} P_{2i}$.

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PROOF. From the exact sequence we get

$$(*) \qquad \chi_U - \sum_{i=1}^{n} (-1)^i \chi_{P_i} + (-1)^{n+1} \chi_U = 0.$$

Now if n is odd, $(*)$ immediately implies (a). If n is even, $\sum_{i=1}^{n}(-1)^i \chi_{P_i} = 0$. But as projective RG-lattices with the same character are isomorphic [3, 77.14], this gives (b).

This proposition has strong implications for periodic RG-lattices of odd period.

3.2. THEOREM. *Let U be a periodic RG-lattice of odd period. Then*

(a) $\chi_U(x) = 0$, *for all p-singular $x \in G$ (i.e. p divides the order of x),*

(b) *there exist projective RG-lattices Q_i and $n_i \in \mathbf{Z}$, $1 \leqslant i \leqslant m$, such that $\chi_U = \sum_{i=1}^{m} n_i \chi_{Q_i}$ (or equivalently, χ_U is in the image of the map e in the cde-triangle),*

(c) $|G|_p \mid \operatorname{rank}_R U$.

PROOF. (a) Without loss of generality we can assume that K is a splitting field for G and its subgroups. As U is of odd period, we have an exact sequence as in 3.1 with odd n. Hence (with the notation as in 3.1) $2 \cdot \chi_U(x) = \sum_{i=1}^{n}(-1)^{i-1}\chi_{P_i}(x) = 0$, for all p-singular $x \in G$, because $\chi_{P_i}(x) = 0$ as P_i is projective (see [4, 59.7] or [6, 2.5]). Thus $\chi_U(x) = 0$ for all p-singular $x \in G$.

(b) By [10, Theorem 36], (a) implies (b).

(c) As $|G|_p \mid \operatorname{rank}_R Q_i$, for all i, because the Q_i are projective, $|G|_p \mid \sum_{i=1}^{m} n_i \operatorname{rank}_R Q_i = \sum_{i=1}^{m} n_i \chi_{Q_i}(1) = \chi_U(1) = \operatorname{rank}_R U$.

For irreducible RG-lattices we now have

3.3. COROLLARY. *Let K be a splitting field for G and its subgroups. Then every irreducible periodic non-projective RG-lattice is of even period.*

PROOF. Suppose U is an irreducible periodic RG-lattice of odd period. By 3.2(c), $|G|_p \mid \operatorname{rank}_R U$. Now [9, Theorem 4.5B] implies that U belongs to a block of defect 0, so U is projective.

3.4. COROLLARY. *Let U be a periodic RG-lattice of period 1. Then χ_U is also afforded by a projective RG-lattice Q, and $Q \oplus Q$ is the projective cover of U.*

PROOF. If P is the projective cover of U, we obtain from 3.1 $2 \cdot \chi_U = \chi_P$. Now [10, Proposition 44] implies that χ_U is also afforded by a (unique) projective RG-lattice Q, so $\chi_U = \chi_Q$. Thus $\chi_P = 2 \cdot \chi_U = 2 \cdot \chi_Q = \chi_{Q \oplus Q}$, so the projective

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RG-lattices P and $Q \oplus Q$ afford the same character, and hence by [3, 77.14] $P \simeq Q \oplus Q$.

The next corollary shows how the Q_i in 3.2(b) may be chosen.

3.5. COROLLARY. *Let U be a periodic RG-lattice of odd period n. Then $\chi_U = \chi_P - \sum_{i=1}^{n/2} \chi_{P_{2i-1}}$, where P_{2i-1} is the projective cover of $\Omega^{2i-1}U$, and $P \oplus P$ is the projective cover of $\oplus_{i=0}^{n-1} \Omega^i U$ (where Ω denotes Heller's operator).*

PROOF. $V = \oplus_{i=0}^{n-1} \Omega^i U$ is a periodic RG-lattice of period 1. By 3.4 $\chi_V = \chi_P$ with a projective RG-lattice P, such that $P \oplus P$ is the projective cover of V. Moreover, $\chi_{\Omega^i U} + \chi_{\Omega^{i+1} U} = \chi_{P_i}$ where P_i is the projective cover of $\Omega^i U$. Now

$$\chi_U = \chi_V - \sum_{i=1}^{n-1} \chi_{\Omega^i U}$$
$$= \chi_P - \sum_{i=1}^{(n-1)/2} \left(\chi_{\Omega^{2i-1} U} + \chi_{\Omega^{2i} U}\right)$$
$$= \chi_P - \sum_{i=1}^{(n-1)/2} \chi_{P_{2i-1}}.$$

Let us now turn from periodic RG-lattices to periodic FG-modules. For an FG-module M, $[M]$ denotes the corresponding element in the Grothendieck group (see [10]).

3.6. PROPOSITION. *Let M be a periodic FG-module, $0 \to M \to P_n \to \cdots \to P_1 \to M \to 0$ an exact sequence with projective FG-modules P_i, $1 \leqslant i \leqslant n$.*

(a) *If n is odd, then $2 \cdot [M] = \sum_{i=1}^{n} (-1)^{i-1} [P_i]$ (in the Grothendieck group). In particular, this implies $|G|_p | 2 \cdot \dim_F M$.*

(b) *If n is even, then $\oplus_{i=1}^{n/2} P_{2i-1} \simeq \oplus_{i=1}^{n/2} P_{2i}$.*

The proof is similar to the proof of 3.1, except that for (b) we use the injectiveness of the map c in the cde-triangle [10, Corollary 1 to Theorem 35].

3.7. REMARK. For a periodic FG-module M of odd period we do not in general have $|G|_p | \dim_F M$. Take $G = \mathbf{Z}_2$ and $M = F_G$ as an example, M is of period 1.

3.8. COROLLARY. *Let U be a periodic AG-module of period 2. Then there exists an exact sequence $0 \to U \to P \to P \to U \to 0$, where P is the projective cover of U. Especially, for $A = F$ the socle and the head of U are isomorphic.*

PROOF. The first statement follows directly from 3.1(b) and 3.6(b). For the additional assertion observe that in case $A = F$

$$\begin{aligned} \text{socle}(U) &\simeq \text{socle}(P) \text{ (as } P \text{ is by the above also the injective envelope of } U\text{)}, \\ &\simeq \text{head}(P) \text{ (as } P \text{ is projective)}, \\ &\simeq \text{head}(U) \text{ (as } P \text{ is the projective cover of } U\text{)}. \end{aligned}$$

COROLLARY. *Let G be a p-group.*

(a) *If U is a periodic non-projective cyclic RG-lattice, then U is of even period.*

(b) *If $p \neq 2$ and M is a periodic non-projective cyclic FG-module, then M has even period.*

PROOF. (a) If U has odd period, then by 3.2(c), $|G| \mid \text{rank}_R U$. But as U is cyclic, U is an epimorphic image of RG, and hence $U \simeq RG$, a contradiction.

(b) Similar to (a), using 3.6(a) instead of 3.2(c), but just for $p \neq 2$.

3.10. REMARK. As we have seen in 3.7, for $p = 2$ we do have periodic non-projective cyclic FG-modules of odd period.

The above example is essentially the only one with a simple FG-module M, as long as we are dealing with p-solvable groups. To be more precise, we have

3.11. PROPOSITION. *Let F be algebraically closed. Let G be a p-solvable group, M a simple periodic non-projective FG-module of odd period. Then $p = 2$, a vertex D of M is isomorphic to $\mathbf{Z}_2$ and F_D is a source of M.*

PROOF. As G is p-solvable, by [6, X, 1.8] $(\dim_F M)_p = |G : D|_p$. On the other hand, by 3.6 $|G|_p \mid 2 \cdot \dim_F M$. As M is non-projective, this says $|D| = 2$. Now F_D must be a source of M, since it is the only non-projective indecomposable FD-module.

3.12. REMARK. For non-p-solvable groups we do have other examples of simple periodic modules with odd period. Let F be algebraically closed of characteristic 2 and let G be the group $\text{PSL}(2, q)$, where $q \equiv 5 \pmod 8$; then there is a simple periodic FG-module M of period 3 which has a Klein four group as a vertex [5].

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