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THE HOMOLOGICAL DIMENSIONS OF SIMPLE MODULES

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We prove that (a) if R is a commutative coherent ring, the weak global dimension of R equals the supremum of the flat (or $(FP-)$injective) dimensions of the simple R-modules; (b) if R is right semi-artinian, the weak (respectively, the right) global dimension of R equals the supremum of the flat (respectively, projective) dimensions of the simple right R-modules; (c) if R is right semi-artinian and right coherent, the weak global dimension of R equals the supremum of the FP-injective dimensions of the simple right R-modules.

1. INTRODUCTION

In this paper R will denote an associative ring with identity and all modules will be unitary. Following [12], the projective (respectively, injective, flat) dimension of an R-module M will be denoted by pdM (respectively, idM, fdM), and the left (respectively, the right, the weak) global dimension of R will be denoted by $\ell D(R)$ (respectively $rD(R)$, $wD(R)$).

It is well known that $\ell D(R)$ is computed by Auslander's classical formula [2] as

$$\ell D(R) = \sup\{pdM \mid M \text{ is a cyclic left } R\text{-module}\}.$$

In general, there is no analogy to Auslander's formula in terms of injective dimensions of cyclic modules, although if R is left Noetherian we do get one [10]. For special classes of rings R the number of cyclics to be checked in computing the (weak) global dimension of R may be reduced. For example if R is a commutative Noetherian ring or a right coherent and left FBN ring, then it is sufficient to check the projective (or injective) dimensions of simple modules [11, 17]. The purpose of this paper is to prove that if R is a commutative coherent ring or a right semi-artinian ring, then we may compute the (weak) global dimension of R using just the homological dimensions of simple modules. The main results are as follows.

I. Let R be a commutative coherent ring. Then

(a) $$pdA = \sup\{n \mid \operatorname{Ext}^n(A, S) \neq 0 \text{ for some simple } R\text{-module } S\}$$
$$= \sup\{n \mid \operatorname{Tor}_n(A, S) \neq 0 \text{ for some simple } R\text{-module } S\}$$
for any finitely presented R-module A.

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(b) $wD(R) = \sup\{fdS \mid S \text{ is a simple } R\text{-module}\}$
$= \sup\{idS \mid S \text{ is a simple } R\text{-module}\}$
$= \sup\{FP\text{-}idS \mid S \text{ is a simple } R\text{-module}\}$.

II. If R is a right semi-artinian ring, then

(a) $fdA = \sup\{n \mid \mathrm{Tor}_n(S, A) \neq 0$ for some simple right R-module $S\}$ for any left R-module A.

(b) $idA = \sup\{n \mid \mathrm{Ext}^n(S, A) \neq 0$ for some simple right R-module $S\}$ for any right R-module A.

(c) $wD(R) = \sup\{fdS \mid S \text{ is a simple right } R\text{-module}\}$.

(d) $rD(R) = \sup\{pdS \mid S \text{ is a simple right } R\text{-module}\}$.

III. Let R be a right semi-artinian and right coherent ring. Then

(a) $pdA = \sup\{n \mid \mathrm{Ext}^n(A, S) \neq 0$ for some simple right R-module $S\}$ for any finitely presented right R-module A.

(b) $wD(R) = \sup\{FP\text{-}idS \mid S \text{ is a simple right } R\text{-module}\}$.

For all R-modules M, N, $\mathrm{Hom}(M, N)$ will mean $\mathrm{Hom}_R(M, N)$, and similarly $M \otimes N$ will denote $M \otimes_R N$ unless otherwise specified.

2. Preliminaries

In this section, we shall recall several known notions which we need in the later sections.

(1) An R-module M is called *FP-injective* if $\mathrm{Ext}^1(N, M) = 0$ for all finitely presented modules N. The *FP-injective dimension* of M, denoted by $FP\text{-}idM$, is defined to be the least nonnegative integer n such that $\mathrm{Ext}^{n+1}(N, M) = 0$ for all finitely presented modules N. If no such n exists, set $FP\text{-}idM = \infty$ [**15, 6**].

(2) A ring is called a *right coherent ring* if every finitely generated right ideal of R is finitely presented. For details see [**3, 8, 15**].

(3) A right R-module M is called *semi-artinian* if every non-zero quotient module of M has non-zero socle. A ring R is said to be *right semi-artinian* if it is semi-artinian as a right R-module. By [**16**, Proposition 2.5], R is right semi-artinian if and only if every right R-module is semi-artinian. A ring R is called a *right SF-ring* if all simple right R-modules are flat [**4**].

(4) Let $\mathfrak{A}$ be a nonempty collection of right ideals of a ring R. Following [**14**], a right R-module X is said to be *$\mathfrak{A}$-injective* provided each R-homomorphism $f: A \to X$ with A in $\mathfrak{A}$ can be extended to an R-homomorphism $g: R \to X$.

3. Simple modules over commutative coherent rings

The proof of the main theorem of this section depends on the following lemmas.

Lemma 3.1. *Let R be a commutative ring, M an R-module and S a simple R-module. Then*

(1) $\operatorname{Tor}_n(M, S) = 0$ *if and only if* $\operatorname{Ext}^n(M, S) = 0$ *for an integer* $n \geqslant 0$.

(2) $fdS = idS = FP\text{-}idS$.

Proof: It is easy to see that (1) implies (2). We now prove (1). Let E be the injective envelope of the direct sum of one copy of each of the simple R-modules. Thus $E = E\left(\bigoplus_{i \in I} S_i\right)$ where $\{S_i\}_{i \in I}$ is the family of all (isomorphism types) of simple R-modules and if $i \neq j$ then $S_i \ncong S_j$. Then E is an injective cogenerator by [**1**, Corollary 18.19] and $\operatorname{Hom}(S, E) \cong S$ as R-modules by the proof of [**18**, Lemma 2.6]. Since E is injective, we have an isomorphism

$$\operatorname{Ext}^n(M, \operatorname{Hom}(S, E)) \cong \operatorname{Hom}(\operatorname{Tor}_n(M, S), E),$$

that is $\operatorname{Ext}^n(M, S) \cong \operatorname{Hom}(\operatorname{Tor}_n(M, S), E)$. Therefore

$$\operatorname{Tor}_n(M, S) = 0 \text{ if and only if } \operatorname{Ext}^n(M, S) = 0$$

since E is a cogenerator. ∎

Lemma 3.2. *Let R be a commutative coherent local ring with only one maximal ideal m and M a finitely presented R-module. Then*

$$pdM \leqslant n \text{ if and only if } \operatorname{Tor}_{n+1}(M, R/m) = 0.$$

Proof: See Rotman [**12**, Lemma 9.53]. His argument remains valid in our setting. ∎

Lemma 3.3. *Let R be a commutative ring and A an R-module, then*

$$fdA = \sup\{fd_{R_m} A_m \mid m \text{ is a maximal ideal of } R\}.$$

Proof: Clear. ∎

Lemma 3.4. *Let R be a commutative coherent ring and A a finitely presented R-module. Then the following are equivalent:*

(1) $pdA \leqslant n$.

(2) $\operatorname{Tor}_{n+1}(A, S) = 0$ *for all simple R-modules S.*

PROOF: (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). For any maximal ideal $\mathfrak{m}$ of R, we have $\mathrm{Tor}_{n+1}(A, R/\mathfrak{m}) = 0$ by (2). Hence

$$\mathrm{Tor}_{n+1}^{R_\mathfrak{m}}(A_\mathfrak{m}, R_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}) \cong (\mathrm{Tor}_{n+1}(A, R/\mathfrak{m}))_\mathfrak{m} = 0.$$

Since R is commutative coherent, $R_\mathfrak{m}$ is a commutative coherent local ring with only one maximal ideal $\mathfrak{m}_\mathfrak{m}$ [8]. Then $fd_{R_\mathfrak{m}}A_\mathfrak{m} \leqslant n$ by Lemma 3.2, and hence, by Lemma 3.3,

$$pdA = fdA = \sup\{fd_{R_\mathfrak{m}}A_\mathfrak{m} \mid m \text{ is a maximal ideal of } R\} \leqslant n.$$

□

We are now in a position to prove

THEOREM 3.5. *If R is a commutative coherent ring, then*

(1) *For any finitely presented R-module A,*

$$\begin{aligned} pdA &= \sup\{n \mid \mathrm{Ext}^n(A, S) \neq 0 \text{ for some simple } R\text{-module } S\} \\ &= \sup\{n \mid \mathrm{Tor}_n(A, S) \neq 0 \text{ for some simple } R\text{-module } S\}. \end{aligned}$$

(In case there are no such n, the supremum is zero.)

(2)
$$\begin{aligned} wD(R) &= \sup\{fdS \mid S \text{ is a simple } R\text{-module}\} \\ &= \sup\{idS \mid S \text{ is a simple } R\text{-module}\} \\ &= \sup\{FP\text{-}idS \mid S \text{ is a simple } R\text{-module}\}. \end{aligned}$$

PROOF: (1) By Lemma 3.1, it suffices to prove the equality

$$pdA = \sup\{n \mid \mathrm{Tor}_n(A, S) \neq 0 \text{ for some simple } R\text{-module } S\}.$$

First, we assume $pdA = m < \infty$. Then it is easily seen that

$$\sup\{n \mid \mathrm{Tor}_n(A, S) \neq 0 \text{ for some simple } R\text{-module } S\} \leqslant m.$$

Since $pdA = m$, $\mathrm{Tor}_m(A, S) \neq 0$ for some simple R-module S by Lemma 3.4. Then the supremum is greater than or equal to m, and hence the equality holds.

Secondly, suppose $pdA = \infty$. Then for any integer $n \geqslant 1$, there exists a simple R-module S such that $\mathrm{Tor}_n(A, S) \neq 0$ by Lemma 3.4, and hence the supremum is greater than or equal to n. Thus the supremum is infinite. So we always have

$$pdA = \sup\{n \mid \mathrm{Tor}_n(A, S) \neq 0 \text{ for some simple } R\text{-module } S\},$$

and the proof of (1) is complete.

(2) By Lemma 3.1, it is sufficient to prove

$$wD(R) = \sup\{fdS \mid S \text{ is a simple } R\text{-module}\}.$$

For any finitely presented R-module A, by (1),

$$\begin{aligned} pdA &= \sup\{n \mid \mathrm{Tor}_n(A, S) \neq 0 \text{ for some simple } R\text{-module } S\} \\ &\leqslant \sup\{fdS \mid S \text{ is a simple } R\text{-module}\}. \end{aligned}$$

Hence

$$\begin{aligned} wD(R) &= \sup\{pdA \mid A \text{ is a finitely presented } R\text{-module}\} \\ &\leqslant \sup\{fdS \mid S \text{ is a simple } R\text{-module}\} \leqslant wD(R), \end{aligned}$$

that is $wD(R) = \sup\{fdS \mid S \text{ is a simple } R\text{-module}\}$. This completes the proof. ∎

As an immediate consequence of the Theorem 3.5 above, we have

Corollary 3.6. *If R is a commutative Noetherian ring, then*

$$\begin{aligned} D(R) &= \sup\{pdS \mid S \text{ is a simple } R\text{-module}\} \\ &= \sup\{idS \mid S \text{ is a simple } R\text{-module}\}. \end{aligned}$$

(Since R is commutative, we drop the unneeded letters l and r.)

4. Simple modules over right semi-artinian rings

In Section 3, it is shown that for a commutative coherent ring R,

$$\begin{aligned} wD(R) &= \sup\{fdS \mid S \text{ is a simple } R\text{-module}\} \\ &= \sup\{FP\text{-}idS \mid S \text{ is a simple } R\text{-module}\}. \end{aligned}$$

In general, the formulae fail for right coherent rings, as shown by [**7**, p.348] and [**5**, Theorem 1.4, 2.3]. In this section, we prove that if R is right coherent and right semi-artinian, then the above formulae hold. (In fact, the first formula holds for right semi-artinian rings.)

We start with two lemmas.

Lemma 4.1. *Let R be any ring and $\mathfrak{M}$ the collection of maximal right ideals of R. Then the following are equivalent:*

(1) *Every $\mathfrak{M}$-injective right R-module is injective.*
(2) *The right R-module R/E has non-zero socle for every proper essential right ideal E of R.*

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PROOF: See Smith [14, Lemma 4].

LEMMA 4.2. *Let R be right semi-artinian. Then*

(1) *For any left R-module A,*

$$fdA \leqslant n \text{ if and only if } \operatorname{Tor}_{n+1}(S, A) = 0 \text{ for all simple right } R\text{-modules } S.$$

(2) *For any right R-module A,*

$$idA \leqslant n \text{ if and only if } \operatorname{Ext}^{n+1}(S, A) = 0 \text{ for all simple right } R\text{-modules } S.$$

PROOF: (1) It is sufficient to prove the "if" part. We proceed by induction on n.

Let $n = 0$. Assume $\operatorname{Tor}_1(S, A) = 0$ for all simple right R-modules S. For any $I \in \mathfrak{M}$, R/I is a simple right R-module, hence $\operatorname{Tor}_1(R/I, A) = 0$. Let $X^+ = \operatorname{Hom}_z(X, Q/Z)$ be the character module of an R-module X. Then we have an isomorphism

$$\operatorname{Ext}^1(R/I, A^+) \cong \operatorname{Tor}_1(R/I, A)^+,$$

and hence $\operatorname{Ext}^1(R/I, A^+) = 0$. Thus A^+ is $\mathfrak{M}$-injective, and so A^+ is injective by Lemma 4.1, that is A is flat.

For $n \geqslant 1$, let

$$\cdots \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to A \to 0$$

be a projective resolution of A with $K = \operatorname{Ker}(P_{n-1} \to P_{n-2})$. Then

$$\operatorname{Tor}_1(S, K) \cong \operatorname{Tor}_{n+1}(S, A) = 0$$

for all simple right R-modules S. The case $n = 0$ shows K is flat, whence $fdA \leqslant n$.

(2) We prove the "if" part by induction on n.

Let $n = 0$. Then $\operatorname{Ext}^1(R/I, A) = 0$ for all $I \in \mathfrak{M}$, and hence A is $\mathfrak{M}$-injective. So A is injective by Lemma 4.1.

For $n \geqslant 1$, suppose

$$0 \to A \to E^0 \to \cdots \to E^{n-1} \to E^n \to \cdots$$

be an injective resolution of A with $L = \operatorname{Im}(E^{n-1} \to E^n)$. Then

$$\operatorname{Ext}^1(S, L) \cong \operatorname{Ext}^{n+1}(S, A) = 0$$

for all simple R-modules S. The case $n = 0$ shows L is injective, and hence $idA \leqslant n$. ∎

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Theorem 4.3. *Let R be right semi-artinian. Then*

(1) $fdA = \sup\{n \mid \mathrm{Tor}_n(S, A) \neq 0$ *for some simple right R-module S*$\}$ *for all left R-modules A.*

(2) $idA = \sup\{n \mid \mathrm{Ext}^n(S, A) \neq 0$ *for some simple right R-module S*$\}$ *for all right R-modules A.*

(3) $wD(R) = \sup\{fdS \mid S$ *is a simple right R-module*$\}$.

(4) $rD(R) = \sup\{pdS \mid S$ *is a simple right R-module*$\}$.

Proof: (1) and (2) follow from Lemma 4.2.

(3) For any left R-module A, by (1),

$$\begin{aligned} fdA &= \sup\{n \mid \mathrm{Tor}_n(S, A) \neq 0 \text{ for some simple right } R\text{-module } S\} \\ &\leqslant \sup\{fdS \mid S \text{ is a simple right } R\text{-module}\}. \end{aligned}$$

Hence

$$\begin{aligned} wD(R) &= \sup\{fdA \mid A \text{ is a left } R\text{-module}\} \\ &\leqslant \sup\{fdS \mid S \text{ is simple right } R\text{-module}\} \leqslant wD(R), \end{aligned}$$

and (3) follows.

(4) For any right R-module A, by (2),

$$\begin{aligned} idA &= \sup\{n \mid \mathrm{Ext}^n(S, A) \neq 0 \text{ for some simple right } R\text{-module } S\} \\ &\leqslant \sup\{pdS \mid S \text{ is a simple right } R\text{-module}\}, \end{aligned}$$

whence

$$\begin{aligned} rD(R) &= \sup\{idA \mid A \text{ is a right } R\text{-module}\} \\ &\leqslant \sup\{pdS \mid S \text{ is a simple right } R\text{-module}\} \\ &\leqslant rD(R), \end{aligned}$$

and so (4) holds. ∎

We obtain the following result of [4] immediately from Theorem 4.3 above.

Corollary 4.4. *If R is a semi-artinian and right SF-ring, then R is a von Neumann regular ring.*

Since R is left perfect if and only if R is right semi-artinian and semi-local [**16**], we have the following result of [**13**] as a corollary.

Corollary 4.5. *If R is a left perfect ring with Jacobson radical J, then*

$$lD(R) = wD(R) = fd(R/J) \text{ and } rD(R) = pd(R/J),$$

where R/J is considered as a right R-module.

PROOF: Immediate since every simple right R-module is a direct summand of the right R-module R/J by [**9**, Theorem 9.3.4]. ∎

The proof of the next main result requires a lemma.

LEMMA 4.6. *Let R be right semi-artinian and right coherent and A a finitely presented right R-module. Then*

$$pdA \leqslant n \text{ if and only if } \operatorname{Ext}^{n+1}(A, S) = 0 \text{ for all simple right } R\text{-modules } S.$$

PROOF: It suffices to prove the "if" part.

"If" part. Let B be any right R-module. We define $\{B_\alpha\}$ inductively. Let $B_0 = 0$, $B_1 = \operatorname{Soc}(B)$. For any ordinal α, if a is not a limit ordinal, let B_α be a submodule of B such that $B_\alpha/B_{\alpha-1} = \operatorname{Soc}(B/B_{\alpha-1})$; if α is a limit ordinal, let $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$. By the transfinite construction principle, $\{B_\alpha\}$ is well-defined. Since R is right semi-artinian, B is a right semi-artinian R-module. Thus $B = B_{\alpha_0}$ for some ordinal α_0 by [**16**, p.183].

Next we use transfinite induction to prove that $\operatorname{Ext}^{n+1}(A, B_\alpha) = 0$ for all ordinals α. In fact, if $\alpha = 0$, then $B_0 = 0$. Of course, $\operatorname{Ext}^{n+1}(A, B_0) = 0$. For each ordinal $\alpha > 0$, assume $\operatorname{Ext}^{n+1}(A, B_\beta) = 0$ for all $\beta < \alpha$. If α is not a limit ordinal, then we have an exact sequence

$$0 \to B_{\alpha-1} \to B_\alpha \to B_\alpha/B_{\alpha-1} \to 0.$$

Since $B_\alpha/B_{\alpha-1} = \operatorname{Soc}(B/B_{\alpha-1})$ is semisimple, $B_\alpha/B_{\alpha-1} = \bigoplus_j S_j$, where each S_j is simple. Thus

$$\operatorname{Ext}^{n+1}(A, B_a/B_{\alpha-1}) = \operatorname{Ext}^{n+1}\left(A, \bigoplus_j S_j\right) \cong \bigoplus_j \operatorname{Ext}^{n+1}(A, S_j) = 0$$

by [**15**, Theorem 3.2]. But $\operatorname{Ext}^{n+1}(A, B_{\alpha-1}) = 0$ by induction hypothesis, and so

$$\operatorname{Ext}^{n+1}(A, B_\alpha) = 0.$$

If α is a limit ordinal, then $B_\alpha = \bigcup_{\beta<\alpha} B_\beta = \varinjlim B_\beta$, and hence

$$\operatorname{Ext}^{n+1}(A, B_\alpha) \cong \varinjlim \operatorname{Ext}^{n+1}(A, B_\beta) = 0$$

again by [**15**, Theorem 3.2]. Thus $\mathrm{Ext}^{n+1}(A, B_\alpha) = 0$ for all α, in particular,

$$\mathrm{Ext}^{n+1}(A, B) = 0 \quad (\text{for } B = B_{\alpha_0}),$$

whence $pdA \leqslant n$. □

THEOREM 4.7. *Let R be right semi-artinian and right coherent. Then*

(1) *$pdA = \sup\{n \mid \mathrm{Ext}^n(A, S) \neq 0$ for some simple right R-module $S\}$ for all finitely presented right R-modules A.*

(2) *$wD(R) = \sup\{FP\text{-}idS \mid S$ is a simple right R-module$\}$.*

PROOF: (1) follows from Lemma 4.6.

(2) For any finitely presented right R-module A, by (1),

$$\begin{aligned} pdA &= \sup\{n \mid \mathrm{Ext}^n(A, S) \neq 0 \text{ for some simple right } R\text{-module } S\} \\ &\leqslant \sup\{FP\text{-}idS \mid S \text{ is a simple right } R\text{-module}\}. \end{aligned}$$

Then

$$\begin{aligned} wD(R) &= \sup\{pdA \mid A \text{ is a finitely presented right } R\text{-module}\} \\ &\leqslant \sup\{FP\text{-}idS \mid S \text{ is a simple right } R\text{-module}\} \\ &\leqslant \sup\{FP\text{-}idN \mid N \text{ is a right } R\text{-module}\} = wD(R) \end{aligned}$$

by [**15**, Theorem 3.3], and so (2) follows. □

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