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Reduced norm map of division algebras over complete discrete valuation fields of certain type

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Abstract. We study a ramification theory for a division algebra D of the following type: The center of D is a complete discrete valuation field K with the imperfect residue field F of certain type, and the residue algebra of D is commutative and purely inseparable over F. Using Swan conductors of the corresponding element of $\mathrm{Br}(K)$, we define Herbrand's ψ-function of D/K, and it describes the action of the reduced norm map on the filtration of D^*. We also gives a relation among the Swan conductors and the 'ramification number' of D, which is defined by the commutator group of D^*.

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1. Introduction

In this paper we develop a ramification theory of division algebras over a complete discrete valuation field K, which is analogous to the classical ramification theory of finite extensions of K. The classical ramification theory deals with a finite Galois extension L of K, under the assumption that the residue field F of K is perfect (see [5] Chapter 4, 5). There exists a good definition of 'Herbrand's function ψ', which is decided by the state of wild ramification in L/K. The classical ramification theory gives a description of the action of the norm map on the filtration of the unit groups of L and K, by using this Herbrand's function.

Now we consider a finite dimensional central division algebra D over K, instead of L/K. If F is perfect, there is no 'wild ramification' in all D/K, so the ramification theory becomes too simple in this case. Hence we now consider the case that the characteristic of F is $p > 0$ and $[F\colon F^p] = p$. We assume that the residue algebra of D is commutative and purely inseparable over F. This is the most important case; if F is separably closed, any D/K satisfies this property.

Our first main theorem is that there is a good definition of 'Herbrand's function ψ' (which is decided by the state of wild ramification in D/K) and the following holds (see Theorem 4.1).

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THEOREM A. *For any* $i = 0, 1, \ldots,$ *we have*

$$\mathrm{Nrd}(U_D^{\psi(i)}) \subset U_K^i,$$

$$\mathrm{Nrd}(U_D^{\psi(i)+1}) \subset U_K^{i+1}.$$

Here U_K^i (resp. U_D^i) *is the* i*th unit group of* K (resp. D).

Let w be the element of the Brauer group of K corresponding to D. The Swan conductor of w is an analogue to Swan conductors of characters of Galois group of K and it measures how the ramification in D/K is big. Let $s_j \in \mathbf{Z}_{\geqslant 0}$ be the Swan conductors of $p^j w (j = 0, 1, \ldots)$. Herbrand's function ψ is completely decided by the numbers s_j. The graph of Herbrand's function is the hooked line, which starts from the origin and has the slope p^{n-j} in the interval $s_j < x < s_{j-1}$. The x-coordinates of hooked points are s_j. We call $\psi(s_j)$ the ramification numbers of D/K.

In the classical case of L/K, there is a relation between the ramification numbers of L/K and valuations of $\sigma(a)/a - 1$ with $\sigma \in \mathrm{Gal}(L/K)$ and $a \in L^*$. For example, the least ramification number of L/K is equal to

$$\inf\{v_L(\sigma(a)/a - 1) | \sigma \in \mathrm{Gal}(L/K), a \in L^*\}.$$

Here v_L denotes the normalized valuation on L. Our next theorem is to give a similar relation between ramification numbers of D/K and valuations of commutators.

THEOREM B. *The least ramification number of* D/K *is equal to*

$$\inf\{v_D(aba^{-1}b^{-1} - 1) | a, b \in D^*\}.$$

Here v_D *denotes the normalized valuation on* D.

We will also give a certain description for all ramification numbers by using values $v_D(aba^{-1}b^{-1} - 1)$. But this is more complicated than the case denoted above. For details, see Theorem 5.1.

We will use the notations below:

The word 'field' means commutative fields, unless the contrary is explicitly stated.

The map Res denotes the restriction map and Cor the corestriction map of Galois cohomology.

For a complete discrete valuation field k or a finite dimensional division algebra k over a complete discrete valuation field,

v_k denotes the normalized valuation on k,

$O_k = \{x \in k | v_k(x) \geqslant 0\}$,

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$$\mathfrak{m}_k = \{x \in k | v_k(x) > 0\},$$

$$U_k = \{x \in k | v_k(x) = 0\},$$

$$U_k^i = \ker(U_k \to (O_k/\mathfrak{m}^i)^*) \quad \text{for } i = 0, 1, 2, \ldots.$$

For a complete discrete valuation field k, k_{nr} denotes the maximal unramified extension of k.

For any field k, k^{sep} denotes the separable closure of k, and $\text{Br}(k)$ denotes the Brauer group of k.

For $\theta \in \text{Br}(k)$, $D(\theta)$ denotes the division algebra over k corresponding to θ.

For any field extension k'/k and $\theta \in \text{Br}(k)$, $\theta_{k'}$ denotes $\text{Res}_{k'/k}(\theta)$.

For any Abelian group A and natural number m, ${}_mA$ denotes $\{a \in A | ma = 0\}$.

2. Basic properties of elements of Brauer group

Let K be a complete discrete valuation field and F its residue field. Suppose that the characteristic of F is $p > 0$ and $[F : F^p] = p$. Let D be a division algebra with center K and C its residue division algebra. We consider the following condition:

$$C \text{ is commutative and purely inseparable over } F. \tag{$*$}$$

Let w be the class of D in the Brauer group of K.

PROPOSITION 2.1. (i) *If* $(*)$ *holds, then*

$$[D : K]^{1/2} = [C : F] = v_D(\pi_K).$$

(ii) *The condition (*) is equivalent to the condition*

$$\text{the order of } w = \text{the order of } w_{K_{nr}}. \tag{$*$}'$$

Furthermore, if this condition holds, then the order of w *is equal to* $[D : K]^{1/2}$.

(iii) *Suppose that* $(*)$ *holds for* D. *Then* $(*)$ *also holds for* $D(p^j w)(j = 0, 1, \ldots)$ *and for* $D(w_L)$ *where* L *is an algebraic extension of* K *and satisfying either of the three conditions below*

(a) $L \subset D$,

(b) L *is unramified over* K,

(c) $p \nmid [L : K] < \infty$.

Proof. (i) Put $[D : K] = r^2$, $[C : F] = f$ and $v_D(\pi_K) = e$. It is well-known that $ef = r^2$. Take $y \in C - C^p$ so that $C = F(y)$. Take its lifting $x \in D$, then we have

$$f = [C : F] \leqslant [K(x) : K] \leqslant r$$

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(the last inequality follows from the fact $K(x)$ is a commutative subfield of D).

Next, we show that $1, \pi_D, \ldots, \pi_D^{e-1}$ are linearly independent over K. To show this, suppose that

$$a_0 + a_1\pi_D + \cdots + a_{e-1}\pi^{e-1} = 0$$

with $a_j \in K$. Since all of $v_D(a_j\pi_D^j) = ev_K(a_j) + j$ $(j = 0, 1, \ldots, e-1)$ are distinct, all of a_j must be zero. This implies

$$e \leqslant [K(\pi_D) \colon K] \leqslant r.$$

From those two inequalities, we have $r = e = f$.

(ii) The assertions '$(*)'$ implies $(*)''$' and '$(*)'$ implies the last assertion' can be shown easily by induction on the order of w, using [1] Section 4 Lemma 5 for the case that the order of w is p.

Now, we prove that $(*)$ implies $(*)'$. From (i), we have $[D\colon K]^{1/2} = [C\colon F] = v_D(\pi_K)$. Since C/F is purely inseparable, those common values are a power p^n of p. It is well-known that the order of w divides $[D\colon K]^{1/2} = p^n$. So let $p^m(m \leqslant n)$ be the order of w. We prove $m = n$ by induction on m.

We first consider the case $m = 1$. Suppose that w is split by some finite unramified Galois extension L/K. Put $G = \mathrm{Gal}(L/K)$. Let H be some p-Sylow subgroup of G and L_0 its fixed subfield. We see that the order of w_{L_0} is also p (because $\mathrm{Cor}(w_{L_0}) = [L_0\colon K]w$ and $p \nmid [L_0\colon K]$). Further, we can see $D(w_{L_0}) = D \otimes L_0$. To see this, put $p^{2r} = [D(w_L) : K]$, then it is enough to show $r = n$. Since w_{L_0} is split by some extension of L_0 of degree p^r, w is split by an extension of K of degree $[L_0\colon K]p^r$. So we have $p^n | [L_0\colon K]p^r$, and hence $r = n$. Since H is a p-group, there is a sequence of fields

$$L_0 \subset L_1 \subset \cdots \subset L_s = L,$$

such that $[L_{j+1}\colon L_j] = p(j = 0, 1, \ldots, s-1)$. Take $r \in \{0, 1, \ldots, s-1\}$ as

$$[D(w_{L_r})\colon L_r] = p^{2n} > [D(w_{L_{r+1}})\colon L_{r+1}] = p^{2n'}.$$

Take any maximal subfield M of $D(w_{L_{r+1}})$. Then w_{L_r} is split by the extension M/L_r whose degree is $p^{n'+1}$. So we have $n' + 1 = n$, and then $[M : L_r] = [D(w_{L_r}) : L_r]^{1/2}$. This shows that $D(w_{L_r})$ contains a field which is isomorphic to M. But the extension M/L_r contains the unramified extension L_{r+1}/L_r, this contradicts $(*)$ for $D(w_{L_r})$ (since $D(w_{L_r}) = D \otimes L_r$, it is clear that $(*)$ holds for $D(w_{L_r})$). This shows $w_{K_{nr}} \neq 0$.

When $m > 1$, the inductive hypothesis says that $[D(pw)\colon K] = p^{2(m-1)}$. Take a maximal commutative subfield L of $D(pw)$, then the order of w_L is p. From the case $m = 1$, w_L is split by some extension of L of degree p, and it is an extension of K of degree p^m. This completes the proof.

(iii) The case (a) is clear from the fact that $D(w_L)$ is isomorphic to the centralizer of L in D. The other parts are clear from (ii). $\square$

3. Herbrand's function ψ

From now on we assume D is a division algebra satisfying $(*)$. Let C be its residue field, w the element of $\mathrm{Br}(K)$ corresponding to D, and p^n the order of w.

Put $s_j = \mathrm{sw}(p^j w) \in \mathbf{Z}_{\geqslant 0} (j = 0, 1, \dots, n)$. Here, for any $\theta \in \mathrm{Br}(K)$, $\mathrm{sw}(\theta)$ denotes the Swan conductor of θ which is defined in [2] (see below). We have

$$s_0 > s_1 > \cdots > s_n = 0.$$

Formally put $s_{-1} = \infty$. Using those numbers, we define Herbrand's function $\psi \colon \mathbf{Z}_{\geqslant 0} \to \mathbf{Z}_{\geqslant 0}$ for D as follows

$$\psi(0) = 0,$$

$$\psi(i) = \psi(s_j) + p^{n-j}(i - s_j) \quad \text{if } s_j \leqslant i \leqslant s_{j-1}.$$

We review on Swan conductors. For any $m \in \mathbf{Z}$, the cup product induces the map

$$\begin{aligned} K^*/K^{*m} \otimes_m \mathrm{Br}(K) &= H^1(K, \mathbf{Z}/m\mathbf{Z}(1)) \otimes H^2(K, \mathbf{Z}/m\mathbf{Z}(1)) \\ &\to H^3(K, \mathbf{Z}/m\mathbf{Z}(2)) \end{aligned}$$

and taking the inductive limit on m, it induces

$$K^* \otimes \mathrm{Br}(K) \to H^3(K, \mathbf{Q}/\mathbf{Z}(2)).$$

(In the case that the characteristic of K is p, the definitions of p-primary part of $\mathbf{Z}/m\mathbf{Z}(r)$ and $\mathbf{Q}/\mathbf{Z}(r)$ are complicated. For details, see [2].) We write the image of $a \otimes \theta \in K^* \otimes \mathrm{Br}(K)$ by this map as $\{\theta, a\}$.

For any finite extension L/K, we have

$$\begin{aligned} &\mathrm{Cor}(\{\theta_L, a\}) = \{\theta, N_{L/K}(a)\} \quad \text{for any } \theta \in \mathrm{Br}(K), a \in L^*, \\ &\mathrm{Cor}(\{\theta, a\} = \{\mathrm{Cor}(\theta), a\} \qquad \text{for any } \theta \in \mathrm{Br}(L), a \in K^*. \end{aligned} \tag{1}$$

When ${}_p\mathrm{Br}(F) \neq 0$, Swan conductors can be defined as ([2] Proposition(6.5))

$$\mathrm{sw}(\theta) = \inf\{m | \ker(\{\theta, ?\}) \supset U_K^{m+1}\} \tag{2}$$

for any $\theta \in \mathrm{Br}(K)$. Remark that this definition is correct only when ${}_p\mathrm{Br}(F) \neq 0$ and $[F : F^p] = p$.

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Now, suppose ${}_p\mathrm{Br}(F) = 0$. In this case, we need more precise definition of Swan conductors, but after the proof of the next lemma, we can reduce all problems to the case ${}_p\mathrm{Br}(F) \neq 0$.

Fix $\pi_K \in K$ such that $v_K(\pi_K) = 1$. Let K_m be the fraction field of the completion of $O_K[T^{p^{-m}}]_{(\pi_K)} (m = 0, 1, \ldots)$ and K_∞ the fraction field of the completion of $\bigcup_{m=0}^{\infty} O_K[T^{p^{-m}}]_{(\pi_K)}$. Then their residue fields are $F_m = F(T^{p^{-m}})$ and $F_\infty = \cup F_m$.

LEMMA 3.1. (i) $[F_\infty : F_\infty^p] = p$ *and* ${}_p\mathrm{Br}(F_\infty) \neq 0$.

(ii) $D \otimes K_\infty$ *is a division algebra.*

(iii) *For any* $\theta \in \mathrm{Br}(K)$, *we have*

$$\mathrm{sw}(\theta) = \mathrm{sw}(\theta_{K_\infty}).$$

In particular, Herbrand's functions for D *and* $D \otimes K_\infty$ *coincide.*

(iv) $v_D = v_{D\otimes K_\infty}|_D$. *In particular, for any* $i = 0, 1, \ldots,$ *we have*

$$U_D^i = U_{D\otimes K_\infty}^i \cap D,$$

$$U_K^i = U_{K_\infty}^i \cap K.$$

(v) *The diagram*

$$\begin{array}{ccc} D & \longrightarrow & D \otimes K_\infty \\ \Big\downarrow{\scriptstyle \mathrm{Nrd}} & & \Big\downarrow{\scriptstyle \mathrm{Nrd}} \\ K & \longrightarrow & K_\infty \end{array}$$

commutes.

Proof. (iv), (v) and the first assertion of (i) are clear. Now, we prove the later part of (i). Let $\chi \in H^1(F_\infty, \mathbf{Q}/\mathbf{Z})$ be the character of $\mathrm{Gal}(F_\infty^{\mathrm{sep}}/F_\infty)$ which corresponds to the extension defined by the equation $\alpha^p - \alpha = T$. Take $a \in F - F^p$. Then the element (χ, a) of ${}_p\mathrm{Br}(F_\infty)$ is not zero, because $(\chi, a) = 0$ is equivalent to $a \in \mathrm{N}_{F_\infty(\alpha)/F_\infty}(F_\infty(\alpha)^*)$ (see [5] Chapter 14 for the definition of (χ, a)).

(iii) From [2] Proposition(6.3), we can easily see

$$\mathrm{sw}(\theta_{K_0}) \geqslant \mathrm{sw}(\theta_{K_\infty}).$$

Further, the same proposition says that, to show the opposite inequality it is enough to show that

$$\{\theta_{L_\infty}, 1 + \pi_K^{N+1}S\} = 0 \quad \text{implies } \{\theta_{L_0}, 1 + \pi_K^{N+1}S\} = 0 \quad \text{for any } N,$$

where L_m is the fractional field of the henselization of $O_{K_m}[S]_{(\pi_K)}$ and similarly L_∞. Since

$$\{\theta_{L_\infty}, 1+\pi_K^{N+1}S\} = \mathrm{Res}(\{\theta_{L_0}, 1+\pi_K^{N+1}S\})$$

in $H^3(L_\infty, \mathbf{Q}/\mathbf{Z}(2))$, $\{\theta_{L_\infty}, 1+\pi_K^{N+1}S\} = 0$ is equivalent to $\{\theta_{L_m}, 1+\pi_K^{N+1}S\} = 0$ for some m. But [2] Lemma (6.2) says

$$\mathrm{sw}(\theta) = \mathrm{sw}(\theta_{K_m}) \quad \text{for all } m = 0, 1, \ldots.$$

From this, if $\{\theta_{L_m}, 1+\pi_K^{N+1}S\} = 0$ holds for some m, then it also holds for all m, especially for $m = 0$. This completes the proof. When we have proved (ii), the later part of (iii) is clear from this.

(ii) It is enough to show $(p^{n-1}w)_{K_\infty} \neq 0$. But [2] Proposition (6.1) and (iii) say $\mathrm{sw}(p^{n-1}w_{K_\infty}) = \mathrm{sw}(p^{n-1}w) > 0$. This shows $(p^{n-1}w)_{K_\infty} \neq 0$. □

In the rest of this section, we prove some properties of Swan conductors and Herbrand's functions. If a and b are two elements of some group, we write $[a, b] = aba^{-1}b^{-1}$. For $a \in O_D$, we write $\bar{a}$ for the class of a in C.

LEMMA 3.2. *If* $n = 1$, *then we have*

$$s_0 = \inf\{v_D([a, b] - 1) | a, b \in D^*\}.$$

Proof. Let t be the right-hand side of above equation. First, we reduce to the case ${}_p\mathrm{Br}(F) \neq 0$. Using notations in Lemma 3.1, we have $s_0 = \mathrm{sw}(w_{K_\infty})$. So we should show

$$t = \inf\{v_{D\otimes K_\infty}([a, b] - 1) | a, b \in (D \otimes K_\infty)^*\}.$$

Take $\alpha \in O_D$ such that $\bar{\alpha} \notin F$, and $\pi_D \in D^*$ such that $v_D(\pi_D) = 1$. Then we also have $\bar{\alpha} \notin F_\infty$, and $v_{D\otimes K_\infty}(\pi_D) = 1$. Hence, the claim is clear from [1] Section 1 Lemma 1. Now we assume ${}_p\mathrm{Br}(F) \neq 0$. In this case, [1] Section 1 says

$$t = \inf\{m | \mathrm{Nrd}(D^*) \supset U_K^{m+1}\}.$$

Further, [4] Theorem (12.2) says that

$$\mathrm{Nrd}(D^*) = \ker(\{w, ?\}).$$

From (2), this completes the proof. □

LEMMA 3.3. *If* L/K *is a finite extension such that the residue extension is purely inseparable, then we have*

$$\mathrm{Cor}\colon H^3(L, \mathbf{Q}/\mathbf{Z}(2)) \to H^3(K, \mathbf{Q}/\mathbf{Z}(2))$$

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is isomorphic in p-primary part.

Proof. For any m, there exists an isomorphism

$$H^3(K, \mathbf{Z}/p^m\mathbf{Z}(2)) \to {}_{p^m}\mathrm{Br}(F)$$

described in [3]. Let E be the residue field of L. It is easy to see that the diagram

$$\begin{array}{ccc} H^3(K, \mathbf{Z}/p^m\mathbf{Z}(2)) & \xrightarrow{\text{iso.}} & {}_{p^m}\mathrm{Br}(F) \\ \Big\uparrow{\scriptstyle \mathrm{Cor}} & & \Big\uparrow{\scriptstyle \cong} \\ H^3(L, \mathbf{Z}/p^m\mathbf{Z}(2)) & \xrightarrow[\text{iso.}]{} & {}_{p^m}\mathrm{Br}(E) \end{array}$$

commutes, here right arrow is induced by $[E\colon F]$-th power map from E to F. □

LEMMA 3.4. *Let L/K be a field extension such that $[L\colon K]$ is prime to p. Then,*

(i) *$D \otimes L$ is a division algebra.*

(ii) *Let $e = v_L(\pi_K)$ and ψ' be Herbrand's function for $D \otimes L$. Then, for any $i = 0, 1, \ldots,$ we have*

$$\psi'(ei) = e\psi(i).$$

(iii) *For any $m = 0, 1, \ldots,$ we have*

$$U_D^i = U_{D\otimes L}^{ei} \cap D,$$

$$U_K^i = U_L^{ei} \cap K.$$

(iv) *The diagram*

$$\begin{array}{ccc} D & \longrightarrow & D \otimes L \\ {\scriptstyle \mathrm{Nrd}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Nrd}} \\ K & \longrightarrow & L \end{array}$$

commutes.

Proof. (i) It is enough to show that the order of w_L is p^n. But this is clear from the fact that the restriction map is injective in 'prime to $[L\colon K]$-part'.

(ii) It is enough to show that $\mathrm{sw}(\theta_L) = e\,\mathrm{sw}(\theta)$ for any $\theta \in \mathrm{Br}(K)$. From Lemma 3.1, we can assume ${}_p\mathrm{Br}(F) \neq 0$. Take the maximal unramified extension L' in L/K, then the extension L/L' is totally ramified (since $[L\colon K]$ is prime to

p). So it is enough to show the claim in the cases that L/K is unramified or totally ramified.

First, we consider the case L/K is totally ramified so that $e = [L : K] = v_L(\pi_K)$. Take $l, m \in \mathbf{Z}$ such that $p^n l + em = 1$. Take any $a \in U_K^i$ $(i > \mathrm{sw}(\theta_L)/e)$. Since $U_K^i \subset U_L^{\mathrm{sw}(\theta_L)+1}$, we have

$$\begin{aligned}\{\theta, a\} &= \{(p^n l + em)(\theta), a\}\\ &= \{em\theta, a\}\\ &= m\{\mathrm{Cor}(\theta_L), a\}\\ &= m\,\mathrm{Cor}(\{\theta_L, a\}) \quad \text{from (1)}\\ &= 0.\end{aligned}$$

From (2), this means $e\,\mathrm{sw}(\theta) \leqslant \mathrm{sw}(\theta_L)$. To show the opposite inequality, note that

$$N_{L/K}(U_L^{ei+1}) \subset U_K^{i+1} \quad \text{for any } i = 0, 1, \ldots.$$

This is proved by [5] Chapter 5. Take any $a \in U_L^{e\mathrm{sw}(\theta)+1}$. Then we have

$$\mathrm{Cor}(\{\theta_L, a\}) = \{\theta, N(a)\} = 0 \quad \text{from (1)}.$$

This proves the opposite inequality by using (2) and Lemma 3.3.

Next, we consider the case L/K is unramified so that $e = 1$. In this case, we have (see [5] Chapter 5)

$$N_{L/K}(U_L^i) = U_K^i \quad \text{for any } i = 0, 1, \ldots.$$

Using this fact, the inequality $\mathrm{sw}(\theta_{L'}) \geqslant \mathrm{sw}(\theta)$ can be shown by a similar way as above. We can take $a \in U_K^{\mathrm{sw}(\theta)}$ such that $\{\theta, a\} \neq 0$. There exist $b \in U_L^{\mathrm{sw}(\theta)}$ such that $N(b) = a$. Then we have

$$0 \neq \{\theta, a\} = \{\theta_L, b\} \quad \text{from (1)}.$$

From (2), this shows the opposite inequality and completes the proof.

(iii) and (iv) are trivial. □

4. The action of reduced norm on the filtration

THEOREM 4.1. *For any* $i = 0, 1, \ldots,$ *we have*

$$\mathrm{Nrd}(U_D^{\psi(i)}) \subset U_K^i,$$

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$$\mathrm{Nrd}(U_D^{\psi(i)+1}) \subset U_K^{i+1}.$$

To prove this theorem, we use induction on n. For $n = 1$, the proof is already done in [1] Section 1 and Lemma 3.2.

Assume $n > 1$. From Lemma 3.1, we can assume ${}_p\mathrm{Br}(F) \neq 0$. Our plan of the proof is as follows. Take a Galois extension L/K of degree p contained in D. Let D' be the centralizer of L in D. For $x \in D'$, we have

$$\mathrm{Nrd}_{D/K}(x) = \mathrm{N}_{L/K}(\mathrm{Nrd}_{D'/L}(x)).$$

Hence, for such x, the problem is divided into '$\mathrm{Nrd}_{D'/L}$-part' and '$\mathrm{N}_{L/K}$-part'.

First, we prove the following claim: We can assume that for any $x \in U_D$ there exists a Galois extension of K of degree p contained in D such that x is an element of the centralizer of it in D.

When the characteristic of K is p and the extension $K(x)/K$ is purely inseparable, we have $\mathrm{Nrd}(x) = x^{p^n}$ and $x \in U_D^i$ implies $x^{p^n} \in U_K^i$. Whatever the values of $\mathrm{sw}(p^j w)$ are, we have $\psi(i) \geqslant i (i = 0, 1, \ldots)$. So there is no problem in this case.

In the every other case, we can take a commutative subfield L of D containing K such that the extension L/K is not trivial and separable, and x is an element of the centralizer of L in D. We can write $L = K(y)$ for some $y \in L$. Take any pro-p-Sylow subgroup of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ and let K_1 be its fixed subfield in K^{sep}. Since a p-group is solvable, we can take a field extension $K_1(z)/K_1$ such that

$$K_1 \subset K_1(z) \subset K_1(y) = K_1 L,$$

$$[K_1(z) \colon K_1] = p.$$

Write $z = f(y)/g(y)$ where f and g are polynomials whose coefficients are in K_1. Let K_2 be the field generated by K, all coefficients of f and g, and all coefficients of the minimal equation of z over K_1. Then

$$p \nmid [K_2 \colon K] < \infty,$$

$$K_2 \subset K_2(z) \subset K_2(y),$$

$$[K_2(z) \colon K_2] = p.$$

Using Lemma 3.4, we can assume the existence of separable (not necessary Galois) extension L/K of degree p.

Now assume that a separable extension L/K of degree p is given. Take the Galois closure L' of L/K, and let K' be the fixed field of some p-Sylow subgroup of $\mathrm{Gal}(L'/K)$. Since $[L' \colon K] \leqslant p!$, we have $p \nmid [K' \colon K]$ and the extension L'/K' is Galois. Hence we have showed the claim, by using Lemma 3.4.

Now we take such a Galois extension L/K of degree p contained in D. Let D' be the centralizer of L in D. It is well-known that the class of D' in $\mathrm{Br}(L)$ is equal to w_L. The extension L/K is either a totally ramified extension or an extension with a purely inseparable residue extension of degree p. We call the first case 'totally ramified' and the latter case 'having residue extension'. Put $s'_j = \mathrm{sw}(p^{n-1-j}w_L)(j = 0, 1, \ldots, n-1)$ and let ψ' be the Herbrand's function for D'/L. Now we can use inductive hypothesis, hence we have

$$\mathrm{Nrd}_{D'/L}(U_{D'}^{\psi'(i)}) \subset U_L^i,$$

$$\mathrm{Nrd}_{D'/L}(U_{D'}^{\psi'(i)+1}) \subset U_L^{i+1}.$$

In the case 'totally ramified', we can use [5] Chapter 5. Put $t = v_L(\pi_L^\sigma/\pi_L - 1)$ where σ is a generator of $\mathrm{Gal}(L/K)$ and π_L is an element of L such that $v_L(\pi_L) = 1$. Using this, we define

$$\begin{aligned} &\rho(i) = i && \text{if } 0 \leqslant i \leqslant t,\\ &\rho(i) = t + p(i-t) && \text{if } t \leqslant i. \end{aligned}$$

Then we have

$$\mathrm{N}_{L/K}(U_L^{\rho(i)}) \subset U_K^i,$$

$$\mathrm{N}_{L/K}(U_L^{\rho(i)+1}) \subset U_K^{i+1}.$$

On the other hand, we have

$$U_{D'}^i = U_D^i \cap D',$$

$$U_K^i = U_L^{pi} \cap K.$$

So we must show

$$\psi \geqslant \psi' \circ \rho.$$

This is an easy consequence of next lemma.

LEMMA 4.2. *Use above assumptions and notations. Take m as $s_m \leqslant t < s_{m-1}$. Then we have $m \leqslant n-1$ (i.e. it does never happen that $t < s_{n-1}$), and*

$$s_{n-1} \leqslant s'_{n-2} \leqslant s_{n-2} \leqslant s'_{n-3} \leqslant \ldots$$

$$\ldots \leqslant s_m \leqslant t < s_{m-1} = \rho^{-1}(s'_{m-1})$$

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$$< s_{m-2} = \rho^{-1}(s'_{m-2})$$

$$< \cdots .$$

Proof. It is enough to show five inequalities below

$$s_{n-1} \leqslant t, \tag{3}$$

$$s_{j+1} \leqslant s'_j \quad j = 0, 1, \ldots, n-1, \tag{4}$$

$$s'_j \leqslant \rho(s_j) \quad j = 0, 1, \ldots, n-1, \tag{5}$$

$$t \leqslant s'_{m-1}, \tag{6}$$

$$\rho(s_j) \leqslant s'_j \quad j = 0, 1, \ldots, m-1. \tag{7}$$

These inequalities can be proved rather easily as follows. The key of the proof is (1) and (2).

Proof of (3): Take $a \in U_K^{t+1}$. Then we can write $a = \mathrm{N}_{L/K}(b)$ for some $b \in U_L^{t+1}$ ([5] Chapter 5). So

$$\{p^{n-1}w, a\} = \mathrm{Cor}(\{(p^{n-1}w)_L, b\}) = 0$$

and this implies (3).

Proof of (4): Take $a \in U_K^{s'_j+1}$. Then $a \in U_L^{s'_j+1}$. So

$$\{p^{j+1}w, a\} = \mathrm{Cor}(\{(p^j w)_L, a\}) = 0,$$

and this implies (4).

Proof of (5): Take $a \in U_L^{\rho(s_j)+1}$. Then $\mathrm{N}_{L/K}(a) \in U_K^{s_j+1}$. So

$$\mathrm{Cor}(\{(p^j w)_L, a\}) = \{p^j w, \mathrm{N}(a)\} = 0$$

and this implies (5) by Lemma 3.3.

Proof of (6): Since $t < s_{m-1}$, we can take $a \in U_K^{t+1}$ such that $\{p^{m-1}w, a\} \neq 0$. We can also take $b \in U_L^{t+1}$ such that $a = \mathrm{N}(b)$. So

$$0 \neq \{p^{m-1}w, a\} = \mathrm{Cor}(\{p^{m-1}w_L, b\})$$

and this implies (6).

Proof of (7): Take $a \in U_K^i$ as $\rho(i) > s'_j$. Since $t \leqslant s'_j$, we can write $a = \mathrm{N}_{L/K}(b)$ for some $b \in U_L^{s'_j+1}$. So

$$\{p^j w, a\} = \mathrm{Cor}(\{(p^j w)_L, b\}) = 0,$$

and this implies (7). □

Remark 4.3. *From this lemma, for L such that $t = s_{n-1}$, we have*

$$\psi = \psi' \circ \rho,$$

$$\rho(s_j) = s'_j \quad \textit{for all } j = 0, 1, \ldots, n-2.$$

Similar fact holds when L/K has residue extension. See below.

In the case 'having residue extension', we can use [1] Section 1: Put $t = pv_L(h^\sigma/h - 1)$ where σ is a generator of $\mathrm{Gal}(L/K)$ and h is an element of O_L such that $\bar{h} \notin F$. Using this, we define

$$\rho(i) = i/p \qquad \text{if } 0 \leqslant i \leqslant t$$

$$\rho(i) = t/p + (i - t) \qquad \text{if } t \leqslant i.$$

Then we have

$$\mathrm{N}_{L/K}(U_L^{\rho(i)}) \subset U_K^i,$$

$$\mathrm{N}_{L/K}(U_L^{\rho(i)+1}) \subset U_K^{i+1}.$$

On the other hand, we have

$$U_{D'}^i = U_D^{pi} \cap D',$$

$$U_K^i = U_L^i \cap K.$$

So we must show

$$\psi \geqslant p\psi' \circ \rho.$$

This is an easy consequence of next lemma.

LEMMA 4.4. *Use above assumptions and notations. Take m as $s_m \leqslant t < s_{m-1}$. Then we have $m \leqslant n-1$ and*

$$s_{n-1} \leqslant ps'_{n-2} \leqslant s_{n-2} \leqslant ps'_{n-3} \leqslant \cdots$$

$$\cdots \leqslant s_m \leqslant t < s_{m-1} = \rho^{-1}(s'_{m-1})$$

$$< s_{m-2} = \rho^{-1}(s'_{m-2})$$

$$< \cdots.$$

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Proof. It is enough to show five inequalities below

$$s_{n-1} \leqslant t,$$

$$s_{j+1} \leqslant s'_j < ps'_j \quad j = 0, 1, \ldots, n-1,$$

$$s'_j \leqslant \rho(s_j) \qquad j = 0, 1, \ldots, n-1,$$

$$t/p \leqslant s'_{m-1},$$

$$\rho(s_j) \leqslant s'_j \qquad j = 0, 1, \ldots, m-1.$$

The proof is very similar to 'totally ramified case', so we omit it. □

5. The ramification numbers

For any subset S of D^*, we write

$$t_D(S) = \inf\{v_D([a, b] - 1) | a, b \in S\}.$$

We can prove the following fact by just the same way as [1] Section 1 Lemma 1. If $\alpha \in O_D$ and $\pi_D \in D^*$ satisfy $\bar{\alpha} \in C - C^p$ and $v_D(\pi_D) = 1$, then

$$t_D(D^*) = v_D([\alpha, \pi_D] - 1).$$

Recall that the numbers $\psi(s_j)$ are called the ramification numbers of D/K.

THEOREM 5.1. *For* $j = 0, 1, \ldots, n-1$, *put*

$$t_j = \sup\{t_D(D'^*) | D' \textit{satisfies conditions below}\},$$

D' *is a division algebra*,

$$K \subset D' \subset D,$$

$$[D : \textit{center of } D'] = p^{2j+2},$$

$$[\textit{center of } D' : K] = p^{n-j-1}.$$

(*In particular*

$$t_{n-1} = t_D(D^*).)$$

Then we have

$$\psi(s_j) = t_j \quad \textit{for any } j = 0, 1, \ldots, n-1.$$

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First we prove two lemmas.

LEMMA 5.2. *Fix any* $\pi_D \in D$ *such that* $v_D(\pi_D) = 1$, *and put* $\pi_K = \mathrm{Nrd}(\pi_D)$. *If* $i < s_{n-1}$, *then we have*

$$\mathrm{Nrd}(1 + \pi_D^i u) \equiv 1 + \pi_K^i u^{p^n} \mod \pi_K^{i+1}$$

for any $u \in U_D$.

Proof. This can be showed easily by induction using three points below. The case $n = 1$ is proved in [1] Section 1. Similar fact for $\mathrm{N}_{L/K}: L \to K$ is proved in [5] Chapter 6 or [1] Section 1. For any totally ramified Galois extension L/K of degree p, we already proved in Lemma 4.2 that

$$s_{n-1} \leqslant t,$$

$$s_{n-1} \leqslant s'_{n-2},$$

or similar fact for a 'having residue extension' case, using notations as in the proof of Theorem 4.1. □

LEMMA 5.3. *If* K_0/K *is a finite field extension such that* $p \nmid [K_0 : K]$. *Then* $D \otimes K_0$ *is a division algebra and*

$$t_{D\otimes K_0}((D \otimes K_0)^*) = et_D(D^*).$$

Here $e = v_{K_0}(\pi_K)$.

Proof. The first part of this lemma is already proved in Lemma 3.4. Take $\alpha \in O_D$ such that $\bar{\alpha} \in C - C^p$, then we also have

$$\bar{\alpha} \in (O_{D\otimes K_0}/\mathfrak{m}_{D\otimes K_0}) - (O_{D\otimes K_0}/\mathfrak{m}_{D\otimes K_0})^p.$$

Fix $\pi_D \in D$ and $\pi_{K_0} \in K_0$ such that $v_D(\pi_D) = 1$ and $v_{K_0}(\pi_{K_0}) = 1$. Take $l, m \in$ **Z** such that $p^n l + em = 1$. Put $\pi_{D\otimes K_0} = \pi_{K_0}^l \pi_D^m$ so that $v_{D\otimes K_0}(\pi_{D\otimes K_0}) = 1$. Put $[\alpha, \pi_D] = 1 + \pi_D^r u$ with $u \in U_D$, then we have

$$\begin{aligned}[][\alpha, \pi_{D\otimes K_0}] &= [\alpha, \pi_{K_0}^l \pi_D^m] \\ &= [\alpha, \pi_D^m] \\ &= [\alpha, \pi_D](\pi_D[\alpha, \pi_D]\pi_D^{-1}) \dots (\pi_D^{m-1}[\alpha, \pi_D]\pi_D^{1-m}) \\ &\equiv 1 + \pi_D^r(u + \pi_D u \pi_D^{-1} + \dots + \pi_D^{m-1} u \pi_D^{1-m}) \mod \pi_{D\otimes K_0}^{er+1}.\end{aligned}$$

Since $u \equiv \pi_D u \pi_D^{-1} \bmod \pi_D$ and $p \nmid m$, we have

$$u + \pi_D u \pi_D^{-1} + \dots + \pi_D^{m-1} u \pi_D^{1-m} \equiv mu \not\equiv 0 \mod \pi_D.$$

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Hence we have

$$\begin{aligned} t_{D\otimes K_0}((D\otimes K_0)^*) &= v_{D\otimes K_0}([\alpha, \pi_{D\otimes K_0}]-1) = er \\ &= ev_D([\alpha,\pi_D]-1) = et_D(D^*) \end{aligned}$$

and this completes the proof. □

Now, let us begin the proof of Theorem 5.1. We again use induction on n. The case $n=1$ is already done in Lemma 3.2.

Suppose that $n>1$. First, we prove the case $j=n-1$. We have $t_{n-1}=t_D(D^*)$ and $\psi(s_{n-1})=s_{n-1}$. Since $\mathrm{Nrd}([a,b])=1$ for any $a,b\in D^*$, we can easily see $v_D([a,b]-1)\geqslant s_{n-1}$ using Lemma 5.2. Now, we must show the existence of $a,b\in D$ such that $v_D([a,b]-1)=s_{n-1}$.

The first step is to prove the following claim: We can assume an existence of a Galois extension L/K of degree p contained in D which satisfies the next condition: Let σ be a generator of $\mathrm{Gal}(L/K)$. Then,

$s_{n-1}=v_L(\sigma(\pi_L)/\pi_L-1)$	for some $\pi_L\in L$ such that $v_L(\pi_L)=1$ when L/K is totally ramified,
$s_{n-1}=pv_L(\sigma(h)/h-1)$	for some $h\in O_L$ such that $\bar{h}\notin F$ when L/K has residue extension.

If L is a maximal commutative subfield of $D(p^{n-1}w)$, then there is an inclusion $L\hookrightarrow D$ (this can be proved by the same argument as in Section 2). Hence, it is enough to show the claim in the case $n=1$. In this case, we know that there exists some $x,y\in D^*$ such that

$$s_0=v_D([x,y]-1).$$

Take some maximal commutative subfield L of D which contains $[x,y]$. Again we can assume the extension L/K is Galois. If the extension L/K is totally ramified, put $=v_L(\sigma(\pi_L)/\pi_L-1)$, using the same notation as above. Then it is clear that

$$1\neq \text{the class of } [x,y]\in \ker(\mathrm{N}\colon U_L^{s_0}/U_L^{s_0+1}\to U_K^{s_0}/U_K^{s_0+1}).$$

On the other hand, [5] Chapter 6 says that for $i<t$

$$\mathrm{N}\colon U_L^i/U_L^{i+1}\to U_K^i/U_K^{i+1}$$

is injective. This implies $s_0\geqslant t$. We already know $s_0\leqslant t$ by Lemma 4.2. This proves the claim in this case. The proof of the case that the extension L/K has residue extension goes similarly, and hence we omit it.

Now suppose that such an extension L/K is given. We use the same notations as in the proof of Theorem 4.1 for D', s'_j, ψ', t and ρ. Since the case L/K has

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residue extension can be proved by the similar way, we prove the case L/K is totally ramified. In this case, there exists $\pi_{D'} \in D'$ such that $v_D(\pi_{D'}) = 1$. Put $\pi_L = \mathrm{Nrd}_{D'/L}(\pi_D)$ and $\pi_K = \mathrm{Nrd}_{D/K}(\pi_D)$. From the definition, we have $s_{n-1} = t = v_L(\sigma(\pi_L)/\pi_L - 1)$. From general theories of central simple algebras, there exists $\alpha \in D^*$ such that the restriction of the inner automorphism

$$x \mapsto \alpha x \alpha^{-1}$$

on D to L is equal to σ. We have

$$\begin{aligned} s_{n-1} = t &= v_L(\sigma(\pi_L)/\pi_L - 1) \\ &= v_L([\alpha, \pi_L] - 1) \\ &= v_L([\alpha, \mathrm{Nrd}_{D'/L}(\pi_D)] - 1) \\ &= v_L(\mathrm{Nrd}_{D'/L}([\alpha, \pi_D]) - 1) \end{aligned}$$

Since $s_{n-1} = t$, Lemma 4.2 says $t < s'_{n-2}$. Applying Lemma 5.2 to $[\alpha, \pi_D]$ on D'/L, we have

$$v_L(\mathrm{Nrd}_{D'/L}([\alpha, \pi_D]) - 1) = v_D([\alpha, \pi_D] - 1).$$

This completes the proof.

Next, we consider the case $j < n - 1$. First, we prove the existence of D_0 such that $t_D(D_0^*) = s_j$. We use the same L as in the proof of the case $j = n - 1$. Again, we only deal with the case L/K is totally ramified, because the proof of the case L/K has residue extension goes similarly. In this case, we have $\psi = \psi' \circ \rho$ and $t_D(S) = t_{D_0}(S)$ for any $S \subset D_0{}^*$. Using the inductive hypothesis, there exists a sub-division algebra $D_0 \subset D'$ such that

$$[D_0 \colon \text{the center of } D_0] = p^{2j+2},$$

$$[\text{the center of } D_0 \colon L] = p^{n-2-j},$$

$$\psi'(s'_j) = t_{D_0}(D_0^*).$$

Then we have

$$t_D(D_0^*) = \psi'(s'_j) = \psi'(\rho(s_j)) = \psi(s_j).$$

This is what we wanted.

Next, take any sub division algebra $D_0 \subset D$ such that

$$[D_0 \colon \text{the center of } D_0] = p^{2j+2} \quad \text{and} \quad [\text{the center of } D_0 \colon K] = p^{n-1-j},$$

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and we begin to prove $\psi(s_j) \geqslant t_D(D_0^*)$. Let L_0 be the center of D_0. First, we consider the case L_0/K is not purely inseparable. In this case, we can assume that there exists L such that $K \subset L \subset L_0$ and L/K is a Galois extension of degree p by using Lemma 3.4 and 5.3. Let D' be the centralizer of L in D so that $D_0 \subset D'$. We will use the same notations as before. Using the inductive hypothesis, we have

$$t_{D'}(D_0^*) \leqslant \psi'(s_j').$$

Again, we only prove in the case L/K is totally ramified. Lemma 4.2 says that we can choose $m \in \{0, \ldots, n-1\}$ so that

$$\begin{aligned} & s_{n-1} \leqslant s'_{n-2} \leqslant s_{n-2} \leqslant s'_{n-3} \leqslant \cdots \\ & \cdots \leqslant s_m \leqslant t < s_{m-1} = \rho^{-1}(s'_{m-1}) \\ & < s_{m-2} = \rho^{-1}(s'_{m-2}) \\ & < \cdots . \end{aligned}$$

Hence we have

$$\psi(s_j) \geqslant \psi'(\rho(s_j)) \geqslant \psi'(s_j') \geqslant t_{D'}(D_0^*) = t_D(D_0^*).$$

This proves the inequality.

When L_0/K is purely inseparable, we can prove the inequality more easily. Put $v_{L_0}(\pi_K) = p^e$. Then we have

$$\begin{aligned} v_{L_0}(a) &= p^e v_K(a) && \text{for any } a \in K, \\ v_D(a) &= p^{n-j-1-e} v_{D_0}(a) && \text{for any } a \in D_0. \end{aligned}$$

Using this, we have

$$\begin{aligned} t_D(D_0^*) &= p^{n-j-1-e} t_{D_0}(D_0^*) \\ &= p^{n-j-1-e} \mathrm{sw}(w_{L_0}) && \text{by the inductive hypothesis} \\ &\leqslant \mathrm{sw}(p^j w) && \text{see below} \\ &\leqslant \psi(s_j) && \text{because } i \leqslant \psi(i) \text{ for any } i. \end{aligned}$$

Now, let us show $p^{n-j-1-e}\,\mathrm{sw}(w_{L_0}) \leqslant \mathrm{sw}(p^j w)$. Take $a \in U_{L_0}^i$ with $i > p^{e+1+j-n} s_j$. Noting that

$$\mathrm{N}_{L_0/K}(a) = a^{p^{n-j-1}} \in U_{L_0}^{ip^{n-j-1}} \cap K \subset U_K^{ip^{n-j-1-e}} \subset U_K^{s_j+1},$$

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we have

$$\{w_{L_0}, a\} = \{w, a^{p^{n-j-1}}\} = 0 \quad \text{from } (1).$$

From (2), this proves the inequality. And hence, we have just proved Theorem 5.1. □

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