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MULTIPLIERS OF INVARIANT SUBSPACES IN THE BIDISC

by TAKAHIKO NAKAZI*

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For any nonzero invariant subspace M in $H^2(T^2)$, set $M^x=[\bigcup_{n=0}^{\infty}\bar{z}^n M]\cap[\bigcup_{n=0}^{\infty}\bar{w}^n M]$. Then M^x is also an invariant subspace of $H^2(T^2)$ that contains M. If M is of finite codimension in $H^2(T^2)$ then $M^x=H^2(T^2)$ and if $M=qH^2(T^2)$ for some inner function q then $M^x=M$. In this paper invariant subspaces with $M^x=M$ are studied. If $M=q_1H^2(T^2)\cap q_2H^2(T^2)$ and q_1,q_2 are inner functions then $M^x=M$. However in general this invariant subspace may not be of the form: $qH^2(T^2)$ for some inner function q. Put $\mathscr{M}(M)=\{\phi\in L^\infty:\phi M\subseteq H^2(T^2)\}$; then $\mathscr{M}(M)$ is described and $\mathscr{M}(M)=\mathscr{M}(M^x)$ is shown. This is the set of all multipliers of M in the title. A necessary and sufficient condition for $\mathscr{M}(M)=H^\infty(T^2)$ is given. It is noted that the kernel of a Hankel operator is an invariant subspace M with $M^x=M$. The argument applies to the polydisc case.

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1. Introduction

Let T^2 be the torus that is the Cartesian product of 2 unit circles in $\mathbb{C}$. The usual Lebesgue spaces, with respect to the Haar measure m of T^2, are denoted by $L^p=L^p(T^2)$, and the Hardy spaces $H^p=H^p(T^2)$ are spaces of all $f\in L^p(T^2)$ whose Fourier coefficients

$$\hat{f}(j,\ell)=\int_{T^2} f(z,w)\bar{z}^j\bar{w}^\ell\,dm(z,w)$$

are 0 as soon as at least one component of (j,ℓ) is negative, where $1\leqq p\leqq\infty$. Let U^2 be the unit bidisc that is the Cartesian product of 2 open unit discs in $\mathbb{C}$. Any function f in H^p has an analytic extension to U^2 which is also denoted by f.

A closed subspace M of H^2 is said to be invariant if

$$zM\subset M \quad\text{and}\quad wM\subset M.$$

Put

$$M_1=\left[\bigcup_{M=0}^{\infty}\bar{z}^n M\right] \quad\text{and}\quad M_2=\left[\bigcup_{M=0}^{\infty}\bar{w}^n M\right] \quad\text{where}\quad \left[\bigcup_{M=0}^{\infty}\bar{z}^n M\right]$$

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is the closed linear span of

$$\bigcup_{n=0}^{\infty} \bar{z}^n M$$

in L^2. Set

$$M^x = M_1 \cap M_2.$$

Then M is also an invariant subspace of H^2. Put

$$\mathbf{H}_1 = \left[\bigcup_{n=0}^{\infty} \bar{z}^n H^2\right] \quad \text{and} \quad \mathbf{H}_2 = \left[\bigcup_{n=0}^{\infty} \bar{w}^n H^2\right].$$

Then $H^2 = \mathbf{H}_1 \cap \mathbf{H}_2$ and hence $(H^2)^x = H^2$. Therefore it is desirable to know invariant subspaces M which have the following property: $M^x = H^2$ or $M^x = M$. An invariant subspace M of H^2 has full range if $M_1 = \mathbf{H}_1$ and $M_2 = \mathbf{H}_2$. Such an invariant subspace has been studied by Agrawal, Clark and Douglas [2]. It is clear that M has full range if and only if $M^x = H^2$. An invariant subspace M with $M^x = M$ has not been studied. In this paper we study invariant subspaces with $M^x = M$. For an invariant subspace M of H^2 set

$$\mathcal{M}(M) = \{\phi \in L^\infty : \phi M \subseteq H^2\}.$$

It is essentially known [2] that if $M^x = M$ then $\mathcal{M}(M) = H^\infty$. In this paper we give a necessary and sufficient condition for $\mathcal{M}(M) = H^\infty$.

Let K_0^2 denote the orthogonal complement of $\bar{H}^2 = \{\bar{f} : f \in H^2\}$ in L^2. The invariant subspace qH^2 for an inner function q is called a Beurling subspace.

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2. Intersection of Beurling subspaces

A Beurling subspace M satisfies $M^x = M$ trivially. In this section we show that the intersections of Beurling subspaces have this property.

Proposition 1. *If M is a nonzero invariant subspace of H^2 then there exist two unimodular functions Q_1 in $\mathbf{H}_1$ and Q_2 in $\mathbf{H}_2$ such that $M_1 = Q_1\mathbf{H}_1$ and $M_2 = Q_2\mathbf{H}_2$, and hence*

$$M^x = Q_1\mathbf{H}_1 \cap Q_2\mathbf{H}_2.$$

Proof. By [7, pp. 164–165], $[\bigcup_{M=0}^{\infty} \bar{z}^n M] = \chi_E Q_1 \mathbf{H}_1 + (1-\chi_E)\, L^2$ where χ_E is a characteristic function of some measurable set E in T^2, $\chi_E \in \mathbf{H}_1$ and $|Q_1| = 1$ a.e. Since

$M \subset H^2$ and $\mathbf{H}_1$ has no reducing invariant subspaces under the multiplication of w, this implies that $M_1 = Q_1\mathbf{H}_1$ and $Q_1 \in \mathbf{H}_1$ because $M_1 \subset \mathbf{H}_1$. The same argument shows that $M_2 = Q_2\mathbf{H}_2$ and $Q_2 \in \mathbf{H}_2$.

Theorem 2. *Let ℓ be a finite positive integer. If $M = \bigcap_{j=1}^{\ell} q_j H^2$ and $\{q_j\}_{j=1}^{\ell}$ are inner functions then $M^x = M$. Moreover*

$$M^x = Q_1\mathbf{H}_1 \cap Q_2\mathbf{H}_2$$

where $\{Q_i\}_{i=1}^2$ are unimodular functions in $\{\mathbf{H}_i\}_{i=1}^2$ and

$$M_i = Q_i\mathbf{H}_i = \bigcap_{j=1}^{\ell} q_j\mathbf{H}_i \quad (i=1,2).$$

Proof. Let $N = \bar{M}^\perp$, that is, N is the orthogonal complement of $\bar{M} = \{\bar{f} : f \in M\}$. Then $N = [\sum_{j=1}^{l} q_j K_0^2]$ because $(\bar{H}^2)^\perp = K_0^2$. Put $N_1 = [\sum_{j=1}^{\ell} q_j w\mathbf{H}_1]$ and $N_2 = [\sum_{j=1}^{\ell} q_j z\mathbf{H}_2]$; then $\{N_i\}_{i=1}^2$ are invariant subspaces and $N = [N_1 + N_2]$ because $K_0^2 = w\mathbf{H}_1 + z\mathbf{H}_2$. It is clear that $zN_1 = N_1$ and $wN_2 = N_2$. If $wN_1 = N_1$ then $N_1 \supset \bar{q}_j \bar{w}^n \mathbf{H}_1$ for any positive n and so $N_1 = L^2$. This contradicts that $M \supset (\prod_{j=1}^{l} q_j)\ H^2$. Thus $wN_1 \neq N_1$ and similarly $zN_2 \neq N_2$. By [7] there exist unimodular functions Q_1 and Q_2 such that $N_1 = Q_1 w\mathbf{H}_1$ and $N_2 = Q_2 z\mathbf{H}_2$. Since $N = [N_1 + N_2]$,

$$M = Q_1\mathbf{H}_1 \cap Q_2\mathbf{H}_2.$$

Since Q_iH_i is the orthogonal complement of N_i and $N_i = (\bar{M}_i)^\perp$ for $i = 1, 2$,

$$M_i = Q_i\mathbf{H}_i = \bigcap_{j=1}^{l} q_j\mathbf{H}_i \quad (i=1,2).$$

In the case of one variable an intersection of two Beurling subspaces is also a Beurling subspace. This is not true in the case of two variables by [**11**, Theorem 2 and its proof]. In fact X_1 and X_2 in [**11**, Theorem 2] are intersections of two Beurling subspaces and $X_j \subseteq H^2$ for $j = 1, 2$. If $X_1 = q_1 H^2$ and $X_2 = q_2 H^2$ for some inner functions q_1, q_2 then $\bar{q}_1 X_1 = \bar{q}_2 X_2 = H^2$. This contradicts [**11**, Theorem 2]. Hence our Theorem 2 is not trivial. If an invariant subspace M is determined by vanishing conditions at finitely many points of U then $M^x = M$ because M is a finite co-dimensional subspace of H^2. We are interested in an invariant subspace determined by vanishing conditions at infinitely many points of U. Let s be an analytic function on U such that $s(U) \subset U$. Put

$$M_s = \{f \in H^2 : f(z, s(z)) = 0 \quad \text{for all } z \in U\}$$

then M_s is an invariant subspace of H^2.

Proposition 3. *Let s be an analytic function on U and $s(U) \subset U$. Then*

$$M_s^x = Q\mathbf{H}_1 \cap \mathbf{H}_2$$

where $Q = w - s/1 - \bar{s}w$. (1) *If* $|s| = 1$ *a.e. on* T *then* $M_s^x = H^2$. (2) *If* $|s| = r < 1$ *a.e. on* T *for some constant* r *then*

$$M_s^x = M_s = Q'\mathbf{H}_1 \cap \mathbf{H}_2 = Q'H^2 \cap H^2$$

where $Q' = w - s/sr^{-1} - rw$.

Proof. If $f \in M_s$ then, for a.e. $z \in T$,

$$\frac{1 - \bar{s}w}{w - s} f(z, w)$$

is w-analytic and hence f belongs to $Q\mathbf{H}_1$. Therefore $M_{s,1} \subset Q\mathbf{H}_1$. Since $(1 - r\bar{s}w)^{-1} \in \mathbf{H}_1$ for any constant r with $0 < r < 1$ and $w - s \in M_s$, $(w - s)(1 - r\bar{s}w)^{-1}$ belongs to $M_{s,1}$. Since

$$\left|\frac{w - s}{1 - r\bar{s}w} - \frac{w - s}{1 - \bar{s}w}\right| \leqq \left|\frac{1 - r}{1 - r\bar{s}w}\right| \to 0 \quad (\text{as } r \to 1)$$

because $m\ \{(z, w) \in T^2 : \bar{s}(z)w = 1\} = 0$, Q belongs to $M_{s,1}$ and hence $M_{s,1} = QH_1$. Since $(w - rs)^{-1} \in \mathbf{H}_2$ for any r with $0 < r < 1$ and $w - s \in M_s$, $(w - s)(w - rs)^{-1} \in M_{s,2}$. As $r \to 1$ the constant 1 belongs to $M_{s,2}$ and hence $M_{s,2} = \mathbf{H}_2$. This implies $M_s^x = Q\mathbf{H}_1 \cap \mathbf{H}_2$. (1) is clear because $s\mathbf{H}_1 = \mathbf{H}_1$. (2) Since

$$\frac{w - s}{1 - \bar{s}w} = \frac{sr^{-1}(w - s)}{sr^{-1} - rw}, \quad M_s = Q'\mathbf{H}_1 \cap \mathbf{H}_2.$$

Put $N = Q'H^2 \cap H^2$. If $Q'f = g$ for some f and g in H^2 then

$$(w - s)f(z, w) = (sr^{-1} - rw)g(z, w) \quad (z, w) \in U^2.$$

Thus g belongs to M_s and hence $M_s \supset N$. It is clear that $N_1 \subset Q'\mathbf{H}_1$ and $N_2 \subset \mathbf{H}_2$. The Q' belongs to N_1 because $(sr^{-1} - rw)^{-1} \in \mathbf{H}_1$ and $w - s \in N$. The constant 1 belongs to N_2 because $(w - s)^{-1} \in \mathbf{H}_2$ and $w - s \in N$. Thus $N = Q'\mathbf{H}_1 \cap \mathbf{H}_2$. Since $N^x = N$ by the proof of Theorem 2, $M_s^x = M_s = Q'\mathbf{H}_1 \cap \mathbf{H}_2 = Q'H^2 \cap H^2$.

3. Multipliers of an invariant subspace

An invariant subspace M of H^2 is said to be podal if every invariant subspace N of H^2 which is unitarily equivalent to M is a subspace of M (cf. [3]). Agrawal, Clark and Douglas [2] showed that if an invariant subspace of H^2 has full range then it is podal. The following is a generalization of that. For if N is full range then $N^x = H^2$.

Proposition 4. *Let M and N be invariant subspaces of H^2 with $M \subset N^x$. If M is unitarily equivalent to N, then $M \subset N$.*

Proof. By [2, Lemma 1] $M = qN$ for some unimodular function q. Then $M^x = qN^x$ and $M^x \subset N^x$. By [2, Proposition 3] the q is an inner function.

Proposition 5. *Let M* and *N be invariant subspaces of H^2 with $M^x = N^x$. Then M is unitarily equivalent to N if and only if $M = N$.*

Proof. If M is unitarily equivalent to N then $M = qN$ for some unimodular function q. Then $M^x = qN^x$ and hence $M^x = qM^x$. Thus q is an inner function. By the same argument $N^x = \bar{q}N^x$ and hence $\bar{q}$ is also an inner function. Thus q is constant and $M = N$.

If M and N are invariant subspaces of finite codimension in H^2, then we can show $M^x = N^x$ and hence by Proposition 5 $M = N$. This is Corollary 3 in [2] (see [5, Corollary 6]). Douglas and Yan [3] asked the following question. Can one characterize podal invariant subspaces?

Proposition 6. *Let M be a nonzero invariant subspace of H^2. M is podal if and only if any unimodular functions in $\mathcal{M}(M)$ belong to H^∞.*

Proof. If ϕ is a unimodular function in $\mathcal{M}(M)$ then $\phi M \subset H^2$ and ϕM is unitarily equivalent to M. If M is podal then $\phi M \subset M$ and hence $\phi \in H^\infty$ by [2, Proposition 3]. Suppose any unimodular functions in $\mathcal{M}(M)$ belong to H^∞. If M is unitarily equivalent to N which is an invariant subspace in H^2 then $N = \phi M$ for some unimodular function ϕ. Since $\phi \in \mathcal{M}(M)$, by the hypothesis ϕ belongs to H^∞ and so M is podal.

By the proposition above, if $\mathcal{M}(M) = H^\infty$ then M is podal. We will characterize an invariant subspace M with $\mathcal{M}(M) = H^\infty$. In general $\mathcal{M}(M)$ is an invariant subspace of L^∞ which contains H^∞. The strusture of $\mathcal{M}(M)$ is simpler than that of M.

Theorem 7. *If M is a nonzero invariant subspace of H^2 then*

$$\mathcal{M}(M) = \bar{Q}_1 \mathbf{H}_1 \cap \bar{Q}_2 \mathbf{H}_2 \cap L^\infty$$

where $M^x = Q_1 \mathbf{H}_1 \cap Q_2 \mathbf{H}_2$ and Q_i is a unimodular function in $\mathbf{H}_i$ for $i = 1, 2$. Hence $\mathcal{M}(M) = H^\infty$ if and only if

$$\bar{Q}_1 \mathbf{H}_1 \cap \bar{Q}_2 \mathbf{H}_2 \cap L^\infty = H^\infty.$$

Proof. If $\phi \in \mathcal{M}(M)$ then $\phi M \subset H^2$. Hence for $i = 1, 2$

$$\phi M_i \subset \mathbf{H}_i \quad \text{and} \quad M_i = Q_i \mathbf{H}_i$$

where Q_i is a unimodular function in $\mathbf{H}_i$. Thus $\phi \in \bar{Q}_1\mathbf{H}_1 \cap \bar{Q}_2\mathbf{H}_2 \cap L^\infty$ and hence $\mathcal{M}(M) \subset \bar{Q}_1\mathbf{H}_1 \cap \bar{Q}_2\mathbf{H}_2 \cap L^\infty$. Conversely let ϕ be in $\bar{Q}_1\mathbf{H}_1 \cap \bar{Q}_2\mathbf{H}_2 \cap L^\infty$. Then

$$\phi = \bar{Q}_1\phi_1 = \bar{Q}_2\phi_2$$

where ϕ_i is in $\mathbf{H}_i$ for $i = 1, 2$. If $f \in M$ then $f \in M^x$ and hence

$$f = Q_1 f_1 = Q_2 f_2$$

where f_i is in $\mathbf{H}_i$ for $i = 1, 2$. Therefore $\phi f = \phi_1 f_1 = \phi_2 f_2$. This implies that $\phi f \in \mathbf{H}_1 \cap \mathbf{H}_2 = H^2$ and hence $\phi \in \mathcal{M}(M)$. Thus

$$\bar{Q}_1\mathbf{H}_1 \cap \bar{Q}_2\mathbf{H}_2 \cap L^\infty \subset \mathcal{M}(M).$$

This establishes the theorem.

For $\phi \in L^\infty$, the Hankel operator determined by ϕ is

$$H_\phi = (1 - P)M_\phi \,\big|\, H^2$$

where P is the orthogonal projection from L^2 to H^2.

Proposition 8. *The kernel of a Hankel operator is an invariant subspace M with $M^x = M$.*

Proof. Suppose H_ϕ is the Hankel operator defined by $\phi \in L^\infty$ and let M be its kernel. It is clear that M is an invariant subspace. Since M is the kernel of H_ϕ, $\phi M \subset H^2$. By Theorem 7, $\mathcal{M}(M) = \mathcal{M}(M^x)$ and $\phi M^x \subset H^2$. This implies $M = M^x$ because $M = \{f \in H^2 : \phi f \in H^2\}$.

REFERENCES

1. P. R. Ahern and D. N. Clark, Invariant subspaces and analytic continuation in several variables. *J. Math. Mech.* **19** (1970), 963–969.

2. O. P. Agrawal, D. N. Clark and R. G. Douglas, Invariant subspaces in the polydisk, *Pacific J. Math.* **121** (1986), 1–11.

3. R. G. Douglas and K. Yan, On the rigidity of Hardy submodules, *Integral Equations Operator Theory* **13** (1990), 350–363.

4. K. Hoffman, *Banach Spaces of Analytic Functions*. (Prentice-Hall, 1962).

5. K. Izuchi, Unitary equivalence of invariant subspaces in the polydisk, *Pacific J. Math.* **130** (1987), 351–358.

6. T. Nakazi, Nonmaximal weak-* Dirichlet algebra, *Hokkaido Math. J.* **5** (1976), 88–96.

7. T. Nakazi, Invariant subspaces of weak-* Dirichlet algebras. *Pacific J. Math.* **69** (1977), 151–167.

8. T. NAKAZI, Certain invariant subspaces of H^2 and L^2 on a bidisc, *Canad. J. Math.* **40** (1988), 1272–1280.

9. T. NAKAZI, Invariant subspaces in the bidisc and commutators, *J. Austral. Math. Soc.*, to appear.

10. T. NAKAZI, Homogeneous polynomials and invariant subspaces in the polydiscs. *Arch. Math.* **58** (1992). 56–63.

11. W. RUDIN, Invariant subspaces of H^2 on a torus, *J. Funct. Anal.* **61** (1985), 378–384.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
HOKKAIDO UNIVERSITY
SAPPORO 060
JAPAN