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THE WEAK BASIS THEOREM FAILS IN NON-LOCALLY CONVEX F-SPACES

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W. J. Stiles showed in [**10**, Corollary 4.5] that Banach's weak basis theorem fails in the spaces l^p, $0 < p < 1$. Then, J. H. Shapiro [**9**] indicated certain general classes of non-locally convex F-spaces with the same property, and asked whether the weak basis theorem fails in *every* non-locally convex F-space with a weak basis. Our purpose is to answer this question in the affirmative. In [**3**] we observed that, essentially, the only case that remained open is that of an F-space with irregular basis (e_n), i.e. such that $s_n e_n \to 0$ for any scalar sequence (s_n). Using a very recent result of Kalton [**7**] (see Proposition 2 below), we are now able to prove that the weak basis theorem fails in this case, too. Thus our main result is

THEOREM. *Let $E = (E, |\cdot|)$ be a non-locally convex F-space. If E has a weak basis, then E has a weak basis that is not a basis (for the original topology of E).*

A basis (x_n) of a topological linear space E is called *equicontinuous* if the associated partial sum operators are equicontinuous; in this case (x_n) is a basis also for the completion of E. We denote by $\sigma(E, E')$ and $\tau(E, E')$ the weak and Mackey topology of E, respectively. For the other, unexplained, notions, the reader is referred to [**9**] and to the papers of Kalton quoted in references.

The following three propositions are essential for the proof of the Theorem. Proposition 1 is a simplified form of Proposition 3.2 in [**6**]; it was used earlier by Shapiro in the proofs of his Theorems 1 and 2 in [**9**]. Proposition 2 is Kalton's Theorem 1 in [**7**]. Proposition 3 is a stability type result for bases in F-spaces; for similar results see [**5**, Lemma 4.3] and [**2**, Theorem 2.8].

PROPOSITION 1. *Let (e_n) be a basis of an F-space E. Then (e_n) is an equicontinuous basis of $(E, \tau(E, E'))$ and, for any sequence (s_n) of scalars, $s_n e_n \to 0$ if and only if $s_n e_n \to 0$ in $\tau(E, E')$.*

PROPOSITION 2. *Let E be a non-locally convex separable F-space with separating dual. Then E has a proper closed weakly dense subspace.*

PROPOSITION 3. *Let $E = (E, ||\cdot||)$ be an F-space with a basis (e_n) which has a regular subsequence (e_{k_n}). Let (u_n) be a sequence in E such that*

$$\sum_{n=1}^{\infty} ||u_n|| < \infty \quad \textit{and} \quad u_n = 0 \quad \textit{for} \quad n \notin \{k_i : i \in \mathbf{N}\},$$

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and let $x_n = e_n + u_n$. *Then there is an* $m \geqq 1$ *such that* $(x_n)_{n \geqq m}$ *is a basic sequence equivalent to* $(e_n)_{n \geqq m}$, *and the closed linear span of* $(x_n)_{n \geqq m}$ *is of finite codimension in* E.

Proof. Let (f_n) be the sequence of functionals biorthogonal to (e_n). Since (e_{k_n}) is regular, (f_{k_n}) is equicontinuous. Hence the linear map $K: E \to E$ defined by

$$Kx = \sum_{n=1}^{\infty} f_n(x) u_n$$

is compact. Let $T = I + K$, where I is the identity map. From the Riesz theory of compact operators (see [**8**, B.IV. § 5]) we have that $T(E)$ is a closed subspace of E and $\dim T^{-1}(0) = \dim E/T(E) < \infty$. Since $\dim T^{-1}(0) < \infty$, there is an $m \geqq 1$ such that $T|\overline{\text{lin}}\ (e_n)_{n \geqq m}$ is one-one, and hence an isomorphism (see [**8**], Theorem B.IV.5.2 and its proof). This proves the proposition.

Remark. It is clear that Proposition 3 is valid also when E is an incomplete F-normed space, provided that the basis (e_n) is equicontinuous.

Proof of the Theorem. Let (e_n) be a weak basis of E. We may assume that (e_n) is a basis, for otherwise we are done. Now consider the following two mutually exclusive cases.

Case 1. There is a sequence (s_n) of non-zero scalars such that $s_n e_n \not\to 0$; without loss of generality we shall assume $s_n = 1$ for all n. Then a subsequence (e_{k_n}) of (e_n) is regular. Our further argument is similar to that used in the proof of Theorem 2 in [**9**]. Choose a sequence (a_n) of non-zero scalars such that

$$a_{k_n} e_{k_n} \to 0 \quad \text{and} \quad a_n = 1 \text{ for } n \notin \{k_i : i \in \mathbf{N}\}.$$

Since $\tau(E, E')$ is strictly weaker than the original topology of E, we may find a sequence (u_n) such that $u_n = 0$ for $n \notin \{k_i : i \in \mathbf{N}\}$, the subsequence (u_{k_n}) is regular in the original topology, and $||a_n^{-1} u_n|| \leqq 2^{-n}$ for all n, where $|| \cdot ||$ is an F-norm defining $\tau(E, E')$. By Proposition 1, (e_n) is an equicontinuous basis of $(E, || \cdot ||)$. Then, from Proposition 3 and the Remark following it, we have that for some $m \geqq 1$ the sequence $(e_n + a_n^{-1} u_n)_{n \geqq m}$ is a basis for a closed finite-codimensional subspace Y of $(E, || \cdot ||)$. (Clearly Y is closed in the original topology of E.) Let $y_n = a_n e_n + u_n$; then $(y_n)_{n \geqq m}$ is a basis of $(Y, || \cdot ||)$, hence a weak basis of $(Y, | \cdot |)$. However, since $y_{k_n} \to 0$ in $\tau(E, E')$ but *not* in the original topology of E, $(y_n)_{n \geqq m}$ is not a basis of $(Y, | \cdot |)$. This is enough to see that the Theorem 1 holds in Case 1.

Case 2. Suppose $s_n e_n \to 0$ for every scalar sequence (s_n), i.e. the basis (e_n) is irregular, in the terminology of [**3**]. Then $(E, \tau(E, E'))$ is isomorphic to a (dense) subspace of ω, where ω is the space of all scalar sequences with the topology of coordinatewise convergence, and $\sigma(E, E') = \tau(E, E')$ (see [**3**]). By Proposition 2, there exists a proper closed weakly dense subspace Z of E.

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$(Z, \sigma(E, E')|Z)$ is isomorphic to a (dense) subspace of ω, and hence has an equicontinuous basis (z_n) (see [**1**, Theorem 2; **4**, Proposition 2.2]). Clearly (z_n) is also a basis for $(E, \sigma(E, E'))$, i.e. a weak basis of E. However $\overline{\text{lin}}\ (z_n) \subset Z \neq E$, and so (z_n) is not a basis of E. (Note that the basic idea of this argument is the same as in Stiles' proof of Corollary 4.5 in [**10**].)

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