UNIVERSAL VARIETIES OF DISTRIBUTIVE DOUBLE p-ALGEBRAS

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(Received 10 January, 1984)

1. Introduction. An algebra $(L; \vee, \wedge, {}^*, {}^+, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ is a *distributive double p-algebra* provided $(L; \vee, \wedge, 0, 1)$ is a distributive $(0, 1)$-lattice, and *, $^+$ are unary operations of pseudocomplementation, or dual pseudocomplementation, respectively: the operation * satisfies $x \leqslant a^*$ if and only if $x \wedge a = 0$, while $x \geqslant a^+$ holds if and only if $x \vee a = 1$.

A category C is *universal* (or *binding*) if any full category of algebras is isomorphic to a full subcategory of C. In particular, see **[10]** or **[18]**, for every monoid M there is a proper class K of pairwise non-isomorphic objects in C such that M is isomorphic to the endomorphism monoid $\mathrm{End}(X)$ for every $X \in K$; hence any binding class C of algebras contains arbitrarily large members with endomorphism monoids isomorphic to M.

In **[3]** it was shown that the category DP of all homomorphisms of distributive double p-algebras contains infinitely many non-isomorphic finite algebras X_i with $\mathrm{End}(X_i) \cong M$ for any finite monoid M, and that there exist arbitrarily large double p-algebras with one-element endomorphism monoid. In the present paper we exhibit a finitely generated universal subvariety V of DP; this strengthens the results of **[3]** and also answers a question, raised by Ervin Fried, of whether a finitely generated congruence distributive variety can be universal as a category. Since any nontrivial finitely generated variety of distributive double p-algebras contains a finite simple algebra **[4]**, **[6]**, it is natural to ask whether a variety generated by finitely many finite simple double p-algebras can be universal. We show that this is not the case; in fact, there are groups not representable as automorphism groups of algebras in such a variety.

A category E is an *iso-category* if all morphisms of E are invertible; thus the category $\mathrm{Iso}(C)$ of all isomorphisms of any category C is an iso-category. We say that a category C is *iso-universal* if $\mathrm{Iso}(A)$ is isomorphic to a full subcategory of $\mathrm{Iso}(C)$ for any full category A of algebras. Consequently **[18]**, if C is an iso-universal category of algebras then, for any group G, there is a proper class K of algebras in C such that $\mathrm{Aut}(X) \cong G$ for every $X \in K$. In this terminology, we show that a finitely generated variety V of double p-algebras is iso-universal if and only if it contains a subdirectly irreducible algebra which is not simple, that is, if and only if V is not a congruence permutable variety. On the other hand, the variety generated in DP by all simple algebras of range two is congruence permutable and iso-universal. We are thus led to the following modifications of Fried's original question.

† The support of NSERC is gratefully acknowledged.

PROBLEM 1. *Are there any iso-universal or universal finitely generated congruence permutable varieties of any type?*

In connection with this question, it should be pointed out that congruence permutability and universality do not exclude each other in general: adding a new nullary operation to Heyting algebras constructed in [**2**] produces a universal congruence permutable variety.

PROBLEM 2. *Characterize finitely generated universal varieties of double p-algebras.*

It is a pleasure to acknowledge stimulating and extensive correspondence with R. Beazer on varieties of double p-algebras, as well as numerous fruitful discussions with M. E. Adams concerning Priestley's duality.

2. Preliminaries. Priestley's duality [**14**], [**15**] for distributive lattices is used extensively throughout this paper. We recall briefly its basic properties; for details, the reader is referred to B. Davey [**6**], H. A. Priestley [**14**], [**15**], or to overview articles by B. Davey and D. Duffus [**7**], H. A. Priestley [**17**].

An *ordered space* $(X; \tau, \leqslant)$ consists of a poset $(X; \leqslant)$ and a topology τ on X. A subset A of X is *decreasing* if $x \leqslant a \in A$ implies $x \in A$ for every $x \in X$; *increasing* set is defined dually. An ordered space $(X; \tau, \leqslant)$ is *totally order disconnected* if $x \not\geqslant y$ implies the existence of a clopen decreasing set $A \subseteq X$ with $x \in A$ and $y \notin A$ for any $x, y \in X$. Let CT denote the category of all continuous order preserving maps between compact totally order disconnected spaces, and let D be the category of all $(0, 1)$-homomorphisms of distributive $(0, 1)$-lattices.

THEOREM 2.1 (H. A. Priestley [**14**]). *There is a contravariant isofunctor $\Delta : CT \to D$. The lattice $\Delta(X; \tau, \leqslant)$ is the inclusion-ordered set of all clopen decreasing subsets of $(X; \tau, \leqslant)$. For any CT-morphism $f : S \to S'$ and every clopen decreasing subset Y' of S', the lattice homomorphism $\Delta(f)$ is given by $\Delta(f)(Y') = f^{-1}(Y')$.*

Let $\mathrm{Min}(x)$ be the set of all minimal elements of $(X; \leqslant)$ below $x \in X$ and, for any $A \subseteq X$, let $\mathrm{Min}(A)$ be the union $\bigcup(\mathrm{Min}(a) : a \in A)$; define $\mathrm{Max}(x)$ and $\mathrm{Max}(A)$ dually. The sets $\mathrm{Min}(x)$ and $\mathrm{Max}(x)$ are nonvoid for any element x of a compact totally order disconnected space $(X; \tau, \leqslant)$, called also a *Priestley space*. For any $a \in X$, let $(a] = \{x \in X : x \leqslant a\}, [a) = \{x \in X : a \leqslant x\}$; given a subset A of X, set $(A] = \bigcup((a] : a \in A)$ and $[A) = \bigcup([a) : a \in A)$. Clearly $[A) = [\mathrm{Min}(A))$ for any decreasing subset A of X, and, dually, $(A] = (\mathrm{Max}(A)]$ if A is increasing. The following result describes the restriction of Priestley's duality to the category DP of distributive double p-algebras [**15**]; see also [**17**] or [**6**].

THEOREM 2.2 (H. A. Priestley). *If $S = (X; \tau, \leqslant)$ is a Priestley space then $\Delta(S)$ lies in DP if and only if $[A)$ is clopen for any clopen decreasing set A of S, and $(B]$ is clopen for any clopen increasing set B of S.*

Let $f : S' \to S$ be a CT-morphism with $\Delta(S), \Delta(S') \in DP$. Then $\Delta(f) : \Delta(S) \to \Delta(S')$ is a DP-morphism if and only if $f(\mathrm{Min}(x)) = \mathrm{Min}(f(x))$ and $f(\mathrm{Max}(x)) = \mathrm{Max}(f(x))$ for every

$x \in X$. Furthermore, $\Delta(f)$ is one-to-one (onto) if and only if f is onto (f is a homeomorphism and order isomorphism of S' into S).

Next we recall a characterization of finite subdirectly irreducible distributive double p-algebras due to B. Davey [**6**].

THEOREM 2.3 (B. Davey). *Let $S = (X; \tau, \leqslant)$ be a Priestley space. Then*

(a) *$\Delta(S)$ is a finite simple double p-algebra if and only if $(X; \leqslant)$ is a finite connected poset such that $X = \mathrm{Min}(X) \cup \mathrm{Max}(X)$;*

(b) *$\Delta(S)$ is a finite subdirectly irreducible double p-algebra if and only if $(X; \leqslant)$ is a finite connected poset and X has at most one element outside* $\mathrm{Min}(X) \cup \mathrm{Max}(X)$.

Given a finite simple distributive double p-algebra F, let $V(F)$ denote the subvariety of DP generated by all finite subdirectly irreducible algebras having F as a homomorphic image. Theorems 2.2 and 2.3 combine to give Proposition 2.4 below.

PROPOSITION 2.4. *The variety $V(F)$ is finitely generated.*

The following sufficient condition for simplicity is easily obtained from Theorem 2.4 of B. Davey [**6**].

PROPOSITION 2.5. *If $\Delta(X; \tau, \leqslant) \in DP$ is such that $(X; \leqslant)$ is a connected poset with $X = \mathrm{Min}(X) \cup \mathrm{Max}(X)$ then $\Delta(X; \tau, \leqslant)$ is a simple distributive double p-algebra.*

Let L_3 denote the category whose objects $(L; a, b, c)$ consist of a distributive $(0, 1)$-lattice and three constants $a, b, c \in L \backslash \{0, 1\}$ satisfying $a \vee b = 1 > a \vee c$, $c < b$, and $a \wedge c = 0 < a \wedge b$; the morphisms of L_3 are all lattice homomorphisms preserving these three constants. It was shown in [**1**] that L_3 is a universal category. Since every lattice homomorphism preserving the three constants is a $(0, 1)$-homomorphism, Priestley's duality can be used to reformulate Theorem 1.1 of [**1**] as follows.

PROPOSITION 2.6 [**1**]. *The category T^3 of all Priestley spaces X with three distinguished clopen decreasing sets A, B, C satisfying $A \cap C = \varnothing$, $C \subseteq B$, $A \cup B = X$, and of all continuous order preserving maps $f: (X; A, B, C) \to (X'; A', B', C')$ of X into X' such that $f(A \backslash B) \subseteq A' \backslash B'$, $f(A \cap B) \subseteq A' \cap B'$, $f(B \backslash (A \cup C)) \subseteq B' \backslash (A' \cup C')$, and $f(C) \subseteq C'$ is dually isomorphic to a universal category.*

Any totally order disconnected space $(X; \tau, \leqslant)$ whose topology τ has a subbase consisting of clopen decreasing sets and their complements (that is, clopen increasing sets) has an open base formed by sets of the form $C \cap D$ with C clopen increasing and D clopen decreasing; any such space will be called *order regular*. Clearly, any subspace of a Priestley space S whose order is the restriction of that of S is an order regular space; on the other hand, every compact order regular space is a Priestley space.

For an order regular space $(X; \tau, \leqslant)$, let $K(X)$ be the Boolean algebra of all clopen sets of $(X; \tau)$, and let βX denote the set of all prime filters of $K(X)$. For every $W \in K(X)$ set $\mathrm{cl}(W) = \{a \in \beta X : W \in a\}$, and let $B = \{\mathrm{cl}(W) : W \in K(X)\}$. It is easy to see that B is closed under finite unions; there is a topology σ of βX having B as its base for closed sets.

Standard arguments show that $(\beta X; \sigma)$ is a compact totally disconnected space containing $(X; \tau)$ as a dense subspace, once every $x \in X$ is identified with the prime filter $p(x) = \{W \in K(X) : x \in W\}$ of $K(X)$. Every continuous map f of $(X; \tau)$ into a compact totally disconnected space Y extends uniquely to a continuous map g defined on $(\beta X; \sigma)$. Define a relation $e \subseteq \beta X \times \beta X$ as the set of all pairs (a, b) such that $b \in \mathrm{cl}(W)$ implies $a \in \mathrm{cl}(W)$ for every decreasing $W \in K(X)$. It is easily seen that e is a preorder on βX. If $x, y \in X$ then $(x, y) \in e$ is equivalent to $x \leq y$ since $(X; \tau, \leq)$ is totally order disconnected; furthermore, (a, x), $(x, a) \in e$ can hold only if $a = x$.

If $\varphi : \beta X \to \alpha X$ is an onto map such that $\varphi(a) = \varphi(b)$ if and only if (a, b), $(b, a) \in e$, then the φ-quotient $\leq$ of e is a partial order and the quotient topology ν is compact and totally order disconnected with respect to $\leq$. Thus $(\alpha X; \nu, \leq)$ is a Priestley space; its existence was first proved by L. Nachbin [**13**] (see also H. A. Priestley [**14**]). It is easy to show that the continuous extension g of an order preserving continuous mapping f of $(X; \tau, \leq)$ into a Priestley space Y has the form $g = h \circ \varphi$ for some continuous order preserving mapping h.

PROPOSITION 2.7 (L. Nachbin, H. A. Priestley). *For every order regular space $(X; \tau, \leq)$ there exists a uniquely determined Priestley space $(\alpha X; \nu, \leq)$ containing $(X; \tau, \leq)$ as an ordered dense subspace such that every continuous order preserving mapping f of $(X; \tau, \leq)$ into a Priestley space $(Y; \rho, \leq)$ extends to a continuous order preserving mapping $h : (\alpha X; \nu, \leq) \to (Y; \rho, \leq)$.*

If $S_i = (X_i; \tau_i, \leq_i)$ is a Priestley space for every $i \in I$, set X to be the disjoint union of $(X_i : i \in I)$, and define $x \leq y$ iff $x \leq_i y$ for some $i \in I$. Equipped with the union topology τ, the space $\sum (S_i : i \in I) = (X; \tau, \leq) = S$ is order regular; the inclusion map e_i of X_i into X is a homeomorphism of S_i into S and also an order embedding for every $i \in I$. Hence e_i is also a continuous order embedding of S_i into $\alpha S = (\alpha X; \nu, \leq)$. If Y is a Priestley space and $f_i : S_i \to Y$ is a continuous order preserving mapping for every $i \in I$, then the joint extension $f : S \to Y$ of all f_i preserves order and is continuous. The continuous extension h of f from Proposition 2.7 preserves order and $h \circ e_i = f_i$ for every $i \in I$. Since h is the only continuous map with the latter property, it follows from Theorem 2.1 that $\Delta(\alpha S)$ is isomorphic to the product $\prod (\Delta(S_i) : i \in I)$ of distributive lattices $\Delta(S_i)$. The product of double p-algebras is again a double p-algebra; this completes the proof of the claim below.

PROPOSITION 2.8. *For any system $(\Delta(S_i) : i \in I)$ of double p-algebras, the product $\prod (\Delta(S_i) : i \in I)$ is isomorphic to the dual of $\alpha \sum (S_i : i \in I)$.*

3. A finitely generated universal variety. Theorem 4.6 of the next section shows that any universal variety of distributive double p-algebras must contain a subdirectly irreducible algebra which is not simple. The universal variety V exhibited here will be generated by four finite subdirectly irreducible algebras whose quotients modulo their respective critical congruences are isomorphic to a simple algebra F.

To prove the universality of such a variety V, a full embedding Φ of the category T^3

described in Proposition 2.6 into the class of Priestley spaces dual to algebras in V will be constructed as follows.

Let $T=(X; A, B, C)$ be an object of T^3. Extending X by the set $S=\{a_0, c_0, p_0, q_0, a_1, b_1, p_1, q_1\}$ of eight isolated points preserves compactness. The order of the compact space $\Phi(T)=(X\cup S; \tau, \leqslant)$ is defined by

(i) for any $x, y\in X$, $x\leqslant y$ in $\Phi(T)$ if and only if $x\leqslant y$ in X,

(ii) $\mathrm{Min}(\Phi(T))=\{a_0, c_0, p_0, q_0\}$ and $\mathrm{Max}(\Phi(T))=\{a_1, b_1, p_1, q_1\}$,

(iii) $\mathrm{Min}(a_1)=\{p_0, c_0\}$, $\mathrm{Max}(c_0)=\{a_1, q_1, b_1\}$, $\mathrm{Min}(b_1)=\{c_0, q_0, a_0\}$, $\mathrm{Max}(a_0)=\{b_1, p_1\}$, and $q_0\leqslant q_1$;

furthermore, for any $x\in X$,

(iv) $c_0\leqslant x$ (if and only if $x\in X\backslash C$; $a_0\leqslant x$ if and only if $x\in X\backslash A$;

(v) $x\leqslant a_1$ if and only if $x\in A$; $x\leqslant b_1$ if and only if $x\in B$.

It is easily verified that (i)–(v) indeed define a partial order on $X\cup S$; since the poset X is convex in $\Phi(T)$, the ordered space $\Phi(T)$ is totally order disconnected. To see that $\Phi(T)$ is a space dual to a double p-algebra, in view of (ii) it suffices to observe that $[a_0)\cap X=X\backslash A, [c_0)\cap X=X\backslash C$, $[p_0)\cap X=[q_0)\cap X=\varnothing$ are clopen in $\Phi(T)$, and so are the sets $(a_1]\cap X=A$, $(b_1]\cap X=B$, and $(p_1]\cap X=(q_1]\cap X=\varnothing$.

The conditions (ii) and (iii) show that $(S;\leqslant)$ is a connected poset with $S=\mathrm{Min}(S)\cup\mathrm{Max}(S)$; by Theorem 2.3(a), $\Delta(S)$ is a simple double p-algebra. If $\Delta(g)$ is an endomorphism of $\Delta(S)$ then the finiteness of S implies that g is a bijection of S onto itself. Since $p_0\in S$ is the only minimal element of S with exactly one maximal element above it, $g(p_0)=p_0$ follows by Theorem 2.2. Furthermore, $g(c_0)=c_0$ because $|\mathrm{Max}(s)|=3$ if and only if $s=c_0$. Arguments dual to these show that $g(p_1)=p_1$ and $g(b_1)=b_1$. From $\mathrm{Min}(p_1)=\{a_0\}$, $\mathrm{Max}(p_0)=\{a_1\}$ it follows that $g(a_0)=a_0$, $g(a_1)=a_1$, respectively. Now $g(q_i)=q_i$ for $i=0, 1$ since g permutes $\mathrm{Max}(S)$, $\mathrm{Min}(S)$. Thus the algebra $\Delta(S)$ is *rigid*, that is, its endomorphism monoid is trivial.

Let $f:(X; A, B, C)\to(X'; A', B', C')=T'$ be a morphism of T^3, see Proposition 2.6. Using Theorem 2.2 it is routine to verify that the extension $\Phi(f):\Phi(T)\to\Phi(T')$ of f by the identity of S is a dual of a double p-algebra homomorphism. This also shows that Φ is a faithful functor.

To show that the image of Φ is a full subcategory in the category of all Priestley spaces dual to double p-algebras, let $g:\Phi(T)\to\Phi(T')$ be such that $\Delta(g)\in DP$. Since $\mathrm{Min}(\Phi(T))\cup\mathrm{Max}(\Phi(T))=S=\mathrm{Min}(\Phi(T'))\cup\mathrm{Max}(\Phi(T'))$, the restriction of g to S is an endomorphism of S; since S is rigid, g preserves S pointwise. From (iii), (iv), and (v) it follows that $\mathrm{Min}(z)=\{c_0\}$ and $\mathrm{Max}(z)=\{a_1\}$ hold simultaneously for any $z\in\Phi(T)$ if and only if $z\in X\backslash B=A\backslash B$. Thus $g(A\backslash B)\subseteq A'\backslash B'$ by Theorem 2.2. An element z of $A\cap B$ is characterized by $\mathrm{Min}(z)=\{c_0\}$ and $\mathrm{Max}(z)=\{a_1, b_1\}$, so that $g(A\cap B)\subseteq A'\cap B'$. Elements of $X\backslash(A\cup C)=B\backslash(A\cup C)$ are singled out as those with $\mathrm{Min}(z)=\{a_0, c_0\}$, $\mathrm{Max}(z)=\{b_1\}$, while C is the set of all z satisfying $\mathrm{Min}(z)=\{c_0\}$ and $\mathrm{Max}(z)=\{b_1\}$; the inclusions $g(B\backslash(A\cup C))\subseteq B'\backslash(A'\cup C')$ and $g(C)\subseteq C'$ thus follow analogously to the previous two cases. Proposition 2.6 concludes the proof of the claim below.

LEMMA 3.1. *The full subcategory of DP determined by double p-algebras dual to spaces $\Phi(T)$ with $T\in T^3$ is universal.*

For any $T \in T^3$, the subspace $\mathrm{Min}(\Phi(T)) \cup \mathrm{Max}(\Phi(T)) = S$ of $\Phi(T)$ represents a finite simple double p-algebra F; in fact, F is the quotient of $\Delta(\Phi(T))$ modulo the *determination congruence* δ, defined by $x\delta y$ if and only if $x^* = y^*$ and $x^+ = y^+$, see R. Beazer [**4**]. Let $Q(F)$ denote the class of all algebras A whose quotient by the determination congruence is isomorphic to F, or, equivalently, such that the dual X of A satisfies $\mathrm{Min}(X) \cup \mathrm{Max}(X) = S$. To show that the class $\Phi(T^3)$ is contained in a finitely generated variety, we need the following claim.

LEMMA 3.2. *If F is a finite simple double p-algebra and $B \in Q(F)$ then B is a subdirect product of subdirectly irreducible members of $Q(F)$.*

Proof. Let Y, S be the dual spaces of B, F, respectively. For any $y \in Y \backslash (\mathrm{Min}(Y) \cap \mathrm{Max}(Y))$, the subspace $S(y) = S \cup \{y\}$ represents a finite subdirectly irreducible double p-algebra $A(y)$, see Theorem 2.3. From Theorem 2.2, it follows that $A(y) \in Q(\Delta(S))$. The dual $\Delta(\xi_y)$ of the embedding $\xi_y : S(y) \to Y$ is a double p-algebra homomorphism for every $y \in Y \backslash S$. Since the embeddings ξ_y are collectively onto, the algebra $\Delta(Y)$ is embedded into the direct product of algebras $\Delta(S(y)) = A(y) \in Q(F)$.

Lemmata 3.1 and 3.2 combine as follows.

THEOREM 3.3. *There exists a finitely generated universal variety of distributive double p-algebras, namely the variety $V(F)$ generated by all subdirectly irreducible algebras whose quotient modulo the critical congruence is isomorphic to the finite simple distributive double p-algebra F represented by the poset S defined by* (i)–(iii) *above.*

It is easy to see that the four subdirectly irreducible algebras represented by finite posets containing $S \subseteq \Phi(T)$ whose respective nonextremal elements represent the relative order of $A \backslash B$, $A \cup B$, $B \backslash (A \cup C)$, and of C to elements of S already generate a variety containing the class $\Phi(T^3)$. Furthermore, every $\Phi(T)$ is dual to an algebra of *range three*, that is, $V(F)$ satisfies the identity $x^{4(+*)} = x^{3(+*)}$.

4. Iso-universal varieties.

To investigate iso-universality of varieties generated by simple algebras we consider compactifications of *split* spaces, defined as order regular spaces $(X; \tau, \leqslant)$ such that $X = \mathrm{Min}(X) \cup \mathrm{Max}(X)$ and both $m = \mathrm{Min}(X) \backslash \mathrm{Max}(X)$ and $M = \mathrm{Max}(X) \backslash \mathrm{Min}(X)$ are τ-clopen.

LEMMA 4.1. *If $S = (X; \tau, \leqslant)$ is a split space then $\alpha S = \beta S$ is a compact split space with* $\mathrm{cl}(\mathrm{Max}(X) \backslash \mathrm{Min}(X)) = \mathrm{Max}(\beta S) \backslash \mathrm{Min}(\beta S)$ *and* $\mathrm{cl}(\mathrm{Min}(X) \backslash \mathrm{Max}(X)) = \mathrm{Min}(\beta S) \backslash \mathrm{Max}(\beta S)$.

Proof. The space S is a disjoint union of clopen sets M, m, and $I = \mathrm{Max}(X) \cap \mathrm{Min}(X)$. Suppose that $a \neq b$ and $(a, b) \in e$. If $m \in b$ then, for any $Z \in b$, the set $Z \cap m \in b$ is decreasing, so that $(a, b) \in e$ implies $Z \cap m \in a$. Thus $b \subseteq a$; since a, b are prime filters, a contradictory $a = b$ is obtained. Therefore $m \notin b$, and, similarly, $I \notin b$; it follows that $M \in b$. Since $(a, b) \in e$ if and only if $W \in b$ for any clopen increasing $W \in a$, an argument dual to the previous one shows that $m \in a$. Thus e is a partial order containing no three-element chain and, consequently, $\alpha S = \beta S$ is a compact split space such that $\mathrm{Max}(\beta S) \backslash \mathrm{Min}(\beta S) \subseteq$

cl(M) and $\mathrm{Min}(\beta S)\backslash\mathrm{Max}(\beta S)\subseteq \mathrm{cl}(m)$. Since $\mathrm{cl}(\mathrm{Min}(X))=\mathrm{Min}(\beta S)$ and $\mathrm{cl}(\mathrm{Max})(X))=\mathrm{Max}(\beta S)$, from the definition of M and m it follows that $\mathrm{cl}(M)\subseteq \mathrm{Max}(\beta S)\backslash\mathrm{Min}(\beta S)$ and $\mathrm{cl}(m)\subseteq \mathrm{Min}(\beta S)\backslash\mathrm{Max}(\beta S)$.

The following claim describes the order of βS.

LEMMA 4.2. *If $S=(X;\tau,\leqslant)$ is a discrete split space then, for any $x\in X$ and $a\in\beta X\backslash X$, $x\leqslant a$ if and only if $x\in\mathrm{Min}(X)$ and $\mathrm{Max}_S(x)\in a$. In particular, if $\mathrm{Max}_S(x)$ is finite then $\mathrm{Max}_{\beta X}(x)\subseteq X$ for every $x\in X$.*

Proof. If $Z\in a$ is decreasing and $\mathrm{Max}_S(x)\in a$, then $Z\cap\mathrm{Max}_S(x)\in a$ is nonvoid and hence $x\in Z$; thus $x\leqslant a$. Conversely, $[x)\in a$ follows from $x\leqslant a$. Since x, a are distinct, $\mathrm{Max}_S(X)\in a$ and $x\in\mathrm{Min}_S(X)$ by Lemma 4.1, so that $\mathrm{Max}_S(x)=[x)\cap\mathrm{Max}(X)\in a$.

Let (V,E) be an undirected graph, that is, let the set E of *edges* consist of two-element subsets of the *vertex* set V. The graph (V,E) will be *connected*: for every pair $v,w\in V$ there exist edges $e_0,\dots,e_n$ such that $v\in e_o$, $w\in e_n$, and $e_i\cap e_{i+1}\neq\varnothing$ for $i=0,\dots,n-1$.

For a connected graph (V,E) set $X=V\cup E\cup\{p,q\}$, where p, q are distinct elements not contained in $V\cup E$. Define a partial order on X by requiring $\mathrm{Max}(X)=E\cup\{q\}$, $\mathrm{Min}(X)=V\cup\{p\}$, $v\leqslant e$ if and only if $v\in e$, $v\leqslant q$ for every $v\in V$, and $p\leqslant e$ for every $e\in E$. If δ is the discrete topology then $S=(X;\delta,\leqslant)$ is a split discrete space whose order is connected. Set $\Phi(V,E)=\alpha S$. Then $\Phi(V,E)$ is a Priestley space, see Proposition 2.7; by Lemma 4.1, $\Phi(V,E)$ is defined on the set βX of all ultrafilters on X. If $\mathrm{Max}(X)\in a$ and $\{q\}\in a$ then $a\in X$; otherwise $E\in a$, and $p\leqslant a$ follows from $E=\mathrm{Max}(p)$ by Lemma 4.2. In particular, $a\in\mathrm{Max}(\alpha S)\backslash\mathrm{Min}(\alpha S)$. If, on the other hand, $\mathrm{Max}(X)$ does not lie in a, then $\mathrm{Min}(X)=X\backslash\mathrm{Max}(X)\in a$. If $a\in\beta X\backslash X$ then $V=\mathrm{Min}(X)\backslash\{p\}\in a$, and $a\leqslant q$ is obtained from a statement dual to that of Lemma 4.2. Thus $a\in\mathrm{Min}(\alpha S)\backslash\mathrm{Max}(\alpha S)$ in this case. We see that αX is a compact split space with connected order.

To show that $\Phi(V,E)$ is a space dual to a simple double p-algebra, the claim below is needed, see Theorem 2.2.

LEMMA 4.3. *If the compactification $(\beta X;\sigma,\leqslant)$ of $S=(X;\tau,\leqslant)$ is partially ordered, and if the smallest increasing subset $\uparrow Y$ of X containing Y is τ-clopen for any clopen decreasing $Y\subseteq X$, then $[C)$ is σ-clopen for every σ-clopen decreasing $C\subseteq\beta X$.*

Proof. Clearly $C\cap X\subseteq\uparrow(C\cap X)\subseteq[C)$, where $\uparrow(C\cap X)$ is τ-clopen by hypothesis; since $[C)$ is closed in any Priestley space, $\mathrm{cl}(C\cap X)\subseteq\mathrm{cl}(\uparrow(C\cap X))\subseteq[C)$. From the definition of $\leqslant$ on βX it is clear that $\mathrm{cl}(\uparrow(C\cap X))$ is an increasing set; since $C=\mathrm{cl}(C\cap X)$, the claim follows because $[C)$ is the smallest increasing set containing C and $\mathrm{cl}(Z)$ is σ-clopen for any τ-clopen $Z\subseteq X$.

Lemma 4.3 and its dual show that $\Phi(V,E)$ represents a double p-algebra. Since $\Phi(V,E)$ is an order connected split space, the double p-algebra represented by $\Phi(V,E)$ is simple for any connected undirected graph (V,E) by Proposition 2.5.

Let $f:(V,E)\to(V',E')$ be a graph isomorphism, that is, a bijection of V onto V' such that $\{v,w\}\in E$ if and only if $\{f(v),f(w)\}\in E'$. The extension f' of f to $V\cup E\cup\{p,q\}$

by $f'(p)=p$, $f'(q)=q$, and $f'\{v, w\}=\{f(v), f(w)\}$ for all $\{v, w\}\in E$ is an order preserving isomorphism of $S=(X; \delta, \leqslant)$ onto S'. By Proposition 2.7, there is a unique order preserving homeomorphism $\Phi(f)=\beta(f')$ of $\Phi(V, E)$ onto $\Phi(V', E')$. It is easily seen that $\Phi(f)(\mathrm{Max}(a))=\mathrm{Max}(\Phi(f)(a))$ and $\Phi(f)(\mathrm{Min}(a))=\mathrm{Min}(\Phi(f)(a))$ for every $a\in\beta X$. Clearly, Φ is a one-to-one functor of the iso-universal category Iso(G) of all isomorphisms of connected graphs with more than two vertices [**8**] into the category of Priestley spaces dual to the variety Q generated by all simple double p-algebras $\Phi(V, E)$. Since the dual of an iso-universal category is iso-universal itself, the iso-universality of Q will be demonstrated once it is shown that every order preserving homeomorphism g of $\Phi(V, E)$ onto $\Phi(V', E')$ has the form $g=\Phi(f)$ for some invertible $f:(V, E)\to(V', E')$.

Let $g:\Phi(V, E)\to\Phi(V', E')$ be an order preserving continuous invertible mapping. A singleton set $\{a\}\in\beta X$ is σ-clopen if and only if $a\in X=V\cup E\cup\{p, q\}$; the mapping g also preserves maximality and minimality. Hence $g(E\cup\{q\})=E'\cup\{q\}$ and $g(V\cup\{p\})=V'\cup\{p\}$. The minimal elements of βX below any $e=\{v, w\}\in E$ are just v and w, by the dual of Lemma 4.2; if $g(e)=q$ then $\mathrm{Min}(q)\supseteq V'$ can have only two elements. For $|V|$, $|V'|>2$ we thus conclude $g(E)=E'$ and $g(q)=q$. Since p is the only minimal element not below q in either space, $g(p)=p$ and $g(V)=V'$ follow. Let f denote the restriction of g to the set V. If $\{v, w\}\in E$ then $g\{v, w\}\in E'$ is an upper bound of $\{f(v), f(w)\}\subseteq V'$, and $g\{v, w\}=\{f(v), f(w)\}$ follows by the definition of the order. Therefore the restriction of g to $(X; \delta, \leqslant)$ coincides with f' for some isomorphism f of (V, E) onto (V', E'); hence $g=\beta(f')=\Phi(f)$ as was to be shown.

Let us consider the variety Q generated by all algebras $\Phi(V, E)$ in more detail. Recall that, for $c\in\Delta(T)\in DP$ represented by the clopen decreasing set C, the pseudocomplement c^* of c is represented by $C^*=T\backslash[C)$ and the dual pseudocomplement c^+ of c corresponds to $C^+=(T\backslash C]$. If C is a proper clopen decreasing subset of $T=\Phi(V, E)=\beta X$, then the clopen set $(T\backslash C)\cap\mathrm{Max}(T)$ is nonvoid and hence it contains some $x\in\mathrm{Max}(X)=E\cup\{q\}$. Consequently there is $v\in V$ such that $v\in C^+=(T\backslash C]$. Since (V, E) is a connected graph, there exists some $e\in E$ in $[C^+)$; note that also $q\in[C^+)$. The set $C^{+*}=T\backslash[C^+)$ thus contains neither e nor q, so that $\{e, q\}\subseteq(T\backslash C^{+*}]=C^{+*+}$. Every minimal element of T is below e or below q; thus $\mathrm{Min}(T)\subseteq C^{+*+}$, and $C^{+*+*}=\varnothing$ follows. Hence, in every generating algebra $\Phi(V, E)$ of Q, the polynomial $p(x)=x^{+*+*}=x^{2(+*)}$ satisfies $p(1)=1$ and $p(x)=0$ for all $x<1$. Consequently, $p(x)^{+*}=p(x)$, that is, every algebra of Q is of range two.

LEMMA 4.4 (R. Beazer [**5**]). *Any variety V of distributive double p-algebras generated by a class of simple algebras is congruence permutable.*

Proof. The determination congruence on any simple algebra must be trivial, that is to say, every simple distributive double p-algebra is regular. Regular algebras form a subvariety R of DP, see J. Varlet [**19**]. Since R is congruence permutable [**4**], so is its subvariety V.

Lemma 4.4 and the arguments presented earlier combine as follows.

THEOREM 4.5. *The class of all simple distributive double p-algebras of range two, and* *hence also the congruence permutable variety $S(2)$ generated by simple algebras of range two are iso-universal.*

Let V be a finitely generated subvariety of DP. If V has only simple algebras in its generating set then V is congruence permutable by Lemma 4.4. Therefore any finitely generated variety V which is not congruence permutable must contain a finite subdirectly irreducible algebra T which is not simple. The dual of T is a finite connected poset D which is the disjoint union of $\mathrm{Max}(D)$, $\mathrm{Min}(D)$, and a singleton $\{d\}$, see Proposition 2.3. Clearly, the three-element chain $C_3 = \{0, c, 1\}$ is a quotient of D via an onto map whose dual is a one-to-one homomorphism of $\Delta(C_3)$ into $\Delta(T)$. Hence the four-element chain $\Delta(C_3)$ is a subdirectly irreducible algebra in V; as a result, V contains the variety S^2 of distributive double Stone algebras, see T. Katriňák **[12]**. The variety S^2 contains the five-element chain which, as a double p-algebra, is not congruence permutable. This establishes the equivalence of (i), (ii), and (iii) in Theorem 4.6 below.

THEOREM 4.6. *The following conditions are equivalent for any finitely generated variety V of distributive double p-algebras:*

(i) *V is not congruence permutable,*
(ii) *V contains a subdirectly irreducible algebra which is not simple,*
(iii) *the variety S^2 of distributive double Stone algebras is contained in V,*
(iv) *V is iso-universal.*

To show that (iii) implies (iv), define $\Psi(V, E)$ as the disjoint union $\Phi(V, E) \cup \{0, 1\}$, where 0, 1 are isolated points such that $0 \leqslant z \leqslant 1$ for all z in $\Psi(V, E)$. It is clear that $\Psi(V, E)$ is a dual of a double p-algebra for any graph (V, E), and that, for every graph isomorphism f, the mapping $\Psi(f)$ defined as the extension of $\Phi(f)$ by the identity mapping of $\{0, 1\}$ is order preserving, continuous, and invertible. Conversely, any invertible order preserving homeomorphism $g : \Psi(V, E) \to \Psi(V', E')$ satisfies $g(\Phi(V, E)) = \Phi(V', E')$ and preserves 0 and 1; thus $g \restriction \Phi(V, E) = \Phi(f)$ for some graph isomorphism as in Theorem 4.5 and, as a result, $g = \Psi(f)$. From Lemma 3.2, it follows that $\Delta \circ \Psi$ maps the category $\mathrm{Iso}(G)$ into the variety S^2 generated by the non-simple subdirectly irreducible four-element chain $\Delta(C_3)$. Thus (iii) implies (iv).

To prove the converse implication, let S be a finite set of finite simple double p-algebras generating a variety V; without a loss of generality we may assume that S is hereditary. Translated by the duality, S is represented by a set $F = \{Y_1, \ldots, Y_n\}$ of finite connected posets with discrete topology such that $Y = \mathrm{Min}(Y) \cup \mathrm{Max}(Y)$ for every $Y \in F$. Since S is hereditary, F is closed under quotients; apart from the one-element poset, any $Y \in F$ is a disjoint union of $\mathrm{Max}(Y)$ and $\mathrm{Min}(Y)$. Any algebra from V is then dual to a quotient T of some Priestley space $\alpha \sum (X_i : i \in I)$, where every X_i is a finite poset isomorphic to a member of F, see Propositions 2.2 and 2.8.

To be more definite, let $k : I \to \{1, \ldots, n\}$ be an arbitrary mapping, and let S be the union order on the set $X = \bigcup (\{i\} \times Y_{k(i)} : i \in I)$. Then S is a split space; by Lemma 4.1, the dual αS of the product is the split space βS of all ultrafilters on X; order components of βS are completely contained either in S or in $\beta S \backslash S$ by Lemma 4.2. For every proper ultrafilter $a \in \beta X$ there is exactly one $j \in \{1, \ldots, n\}$ such that $k^{-1}\{j\} \in a$. The set $k^{-1}\{j\}$ is a disjoint union of finitely many sets of the form $k^{-1}\{j\} \times \{y\}$ with $y \in Y_j$. Thus every $a \in \beta X$ uniquely determines $j \in \{1, \ldots, n\}$ and $y \in Y_j$ such that $a \in \mathrm{cl}(k^{-1}\{j\} \times \{y\})$. Since $\mathrm{cl}(X')$ properly contains $X' \subseteq X$ whenever X' is infinite, it is easily seen that any order component of $\beta S \backslash S$ containing some $a \in \mathrm{cl}(k^{-1}\{j\})$ is isomorphic to Y_j. If T is a Priestley space representing a subalgebra of $\prod (\Delta(Y_{k(i)}) : i \in I)$ then every order component of T is a quotient of an order component in βS; as a result, the cardinalities of order components of T are finite and bounded. Let θ be an equivalence on T consisting of all pairs (t, u) such that $t, u \in Q$ for some order component Q of T, and let B be the quotient of T modulo θ. The space B is clearly Boolean; let $h : T \to B$ denote the onto map with $\mathrm{Ker}(h) = \theta$.

If $\mathrm{Aut}(T)$ is the group of all order preserving continuous invertible maps of T into itself, then $g(Q)$ is an order component of T for any order component Q and any $g \in \mathrm{Aut}(T)$. Assume that $\mathrm{Aut}(T)$ contains an element g of a prime order $p > |Y|$ for every $Y \in F$. For some $t \in T$ the elements $t_r = g^r(t)$ with $r \in p = \{0, 1, \ldots, p-1\}$ are pairwise distinct, so that there are disjoint order components $Q_0, \ldots, Q_{p-1}$ of T such that $g(Q_r) = Q_{r+1}$ (where the addition is performed modulo p) for all $r \in p$. Since the image $g(Q)$ of any order component Q is order connected, there is an automorphism g' of B satisfying $g' \circ h = h \circ g$; if $\{b_r\} = h(Q_r)$, then $g'(b_r) = b_{r+1}$ and the elements $b_0, \ldots, b_{p-1}$ of B are distinct. Since B is totally disconnected there exists a clopen set A_0 such that $\{b_r : r \in p\} \cap A_0 = \{b_0\}$; clearly, $b_s \in A_r = (g')^r(A_0)$ if and only if $s = r$. The sets $B_r = A_r \backslash \bigcup (A_s : s \neq r)$ are pairwise disjoint, clopen, and such that $g'(B_r) = B_{r+1}$ and $b_r \in B_r$ for every $r \in p$. Set $T_r = h^{-1}(B_r)$; the clopen sets T_r are pairwise disjoint and $g(T_r) = T_{r+1}$ for all $r \in p$. For any permutation π of $\{0, \ldots, p-1\}$ set $g_\pi(t) = g^{\pi(r)-r}(t)$ whenever $t \in T_r$, $g_\pi(t) = t$ for all $t \in T$ outside $T_0 \cap \ldots \cap T_{p-1}$. It is routine to verify that $g_\pi \in \mathrm{Aut}(T)$; thus $\mathrm{Aut}(T)$ contains the symmetric group $\mathrm{Sym}(p)$ for any element $g \in \mathrm{Aut}(T)$ of sufficiently large prime order p. Therefore the variety V is not iso-universal. This completes the proof of Theorem 4.6.

REFERENCES

1. M. E. Adams, V. Koubek and J. Sichler, Homomorphisms and endomorphisms of distributive lattices, to appear in *Houston J. Math.*

2. M. E. Adams, V. Koubek and J. Sichler, Endomorphisms and homomorphisms of Heyting algebras, to appear in *Algebra Universalis.*

3. M. E. Adams and J. Sichler, Endomorphism monoids of distributive double p-algebras, *Glasgow Math. J.* **20** (1979), 81–86.

4. R. Beazer, The determination congruence on double p-algebras, *Algebra Universalis* **6** (1976), 121–129.

5. R. Beazer, Personal communication (1983).

6. B. Davey, Subdirectly irreducible distributive double p-algebras, *Algebra Universalis* **8** (1978), 73–88.

7. B. A. Davey and D. Duffus, Exponentiation and duality, *Ordered Sets*, NATO Advanced Study Institutes Series **83** (D. Reidel, 1982), 43–95.

8. Z. Hedrlín and A. Pultr, On rigid undirected graphs, *Canad. J. Math.* **18** (1966), 1237–1242.

9. Z. Hedrlín and A. Pultr, On full embeddings of categories of algebras, *Illinois J. Math.* **10** (1966), 392–406.

10. Z. Hedrlín and J. Sichler, Any boundable binding category contains a proper class of mutually disjoint copies of itself, *Algebra Universalis* **1** (1971), 97–103.

11. B. Jónsson, Algebras whose congruence lattices are distributive, *Math. Scand.* **21** (1967), 110–121.

12. T. Katriňák, The injective double Stone algebras, *Algebra Universalis* **4** (1974), 259–267.

13. L. Nachbin, *Topology and Order* (Van Nostrand, 1965).

14. H. A. Priestley, Representation of distributive lattices by means of ordered Stone spaces, *Bull. London Math. Soc.* **2** (1970), 186–190.

15. H. A. Priestley, Ordered topological spaces and the representation of distributive lattices, *Proc. London Math. Soc.* (3) **24** (1972), 507–530.

16. H. A. Priestley, The construction of spaces dual to pseudocomplemented distributive lattices, *Quart. J. Math. Oxford Ser.* (2) **26** (1975), 215–228.

17. H. A. Priestley, Ordered sets and duality for distributive lattices, *Proc. Conf. on Ordered Sets and their Applications, Lyon* (1982), to appear.

18. A. Pultr and V. Trnková, *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories* (North-Holland, 1980).

19. J. Varlet, A regular variety of type (2, 2, 1, 1, 0, 0), *Algebra Universalis* **2** (1972), 218–223.

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