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Curvature of K-contact Semi-Riemannian Manifolds

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Abstract. In this paper we characterize K-contact semi-Riemannian manifolds and Sasakian semi-Riemannian manifolds in terms of curvature. Moreover, we show that any conformally flat K-contact semi-Riemannian manifold is Sasakian and of constant sectional curvature $\kappa = \varepsilon$, where $\varepsilon = \pm 1$ denotes the causal character of the Reeb vector field. Finally, we give some results about the curvature of a K-contact Lorentzian manifold.

1 Introduction

Contact Riemannian manifolds, in particular K-contact and Sasakian manifolds, have been intensively studied. The recent monographs [2, 5] give a wide and detailed overview of the results obtained in this framework. Contact semi-Riemannian structures (η, g), also called contact pseudo-metric structures, where η is a contact 1-form and g a semi-Riemannian metric associated to it, are a natural generalization of contact Riemannian structures also called contact metric structures. Contact Lorentzian structures are particular contact semi-Riemannian structures. The relevance of contact semi-Riemannian structures for physics was pointed out in [1, 9]. Contact structures equipped with semi-Riemannian metrics were first introduced and studied by Takahashi [16], who focused on the Sasakian case. However, in the semi-Riemannian case, even for Sasakian and K-contact manifolds, there are few results. Recently, in [6] (see also [7, 8]) we introduced a systematic study of contact structures with semi-Riemannian associated metrics. In this paper we continue this study, turning our attention to the case of K-contact semi-Riemannian manifolds, emphasizing analogies and differences with respect to the Riemannian case. The paper is organized in the following way. In Section 2 we report some basic information for contact pseudo-metric manifolds. In Section 3 we characterize K-contact and Sasakian, semi-Riemannian manifolds in terms of curvature (see Theorems 3.1, 3.3). Note that, in the Riemannian case, Theorem 3.1(i) holds in a stronger form (*cf.* Remark 3.2). Then, in Section 4 we show that any conformally flat K-contact semi-Riemannian manifold is Sasakian and of constant sectional curvature $\kappa = \varepsilon$, where $\varepsilon = \pm 1$ denotes the causal character of the Reeb vector field. Section 5 contains some results about the curvature of a K-contact Lorentzian manifold. In particular, a simply connected η-Einstein Lorentzian-Sasaki manifold of dimension $2n + 1 > 3$, with

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scalar curvature $r_L < 2n$, admits a transverse homothety whose resulting Lorentzian-Sasaki manifold $(M, \widetilde{g}_L)$ is Einstein, and thus it is a spin manifold. In dimension three, the Lie groups $SU(2)$, $\widetilde{SL}(2,R)$, and a special non-unimodular Lie group, are the only simply connected manifolds that admit a Lorentzian-Sasaki structure with constant scalar curvature $r_L \neq 2$. In particular, the unimodular Lie group $\widetilde{SL}(2,R)$ and a special non-unimodular Lie group are the only simply connected three-manifolds that admit a left invariant Lorentzian-Sasaki structure of constant sectional curvature $\kappa = -1$.

2 Preliminaries on Contact Semi-Riemannian Manifolds

In this section, we collect some basic facts about contact semi-Riemannian manifolds [6]. All manifolds are assumed to be connected and smooth. A $(2n+1)$-dimensional manifold M is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. Given η, there exists a unique vector field ξ, called the *characteristic vector field* or the *Reeb vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. Furthermore, a semi Riemannian metric g is said to be an *associated metric* if there exists a tensor φ of type $(1,1)$ such that

$$\eta = \varepsilon g(\xi, \cdot), \quad d\eta(\cdot, \cdot) = g(\cdot, \varphi \cdot), \quad \varphi^2 = -I + \eta \otimes \xi,$$

where $\varepsilon = \pm 1$, and so $g(\xi, \xi) = \varepsilon$ (the light-like case cannot occur for the Reeb vector field). In particular, the signature of an associated metric is either $(2p+1, 2n-2p)$ or $(2p, 2n-2p-1)$, according to whether ξ is space-like or time-like. Then (η, g, ξ, φ), or (η, g), is called a *contact semi Riemannian structure*, or *contact pseudo metric structure*, and $(M, \eta, g, \xi, \varphi)$ a *contact semi-Riemannian manifold* or a *contact pseudo metric manifold*. We denote by ∇ the Levi-Civita connection and by R the corresponding Riemann curvature tensor given by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

Moreover, we denote by Ric the Ricci tensor of type $(0,2)$, by Q the corresponding endomorphism field and by r the scalar curvature. The tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$, where $\mathcal{L}$ denotes the Lie derivative, is symmetric and satisfies

$$\nabla \xi = -\varepsilon\varphi - \varphi h, \quad \nabla_\xi \varphi = 0, \quad h\varphi = -\varphi h, \quad h\xi = 0. \tag{2.1}$$

If $\{E_1, \dots, E_{2n+1}\}$ is an arbitrary local pseudo-orthonormal basis on M and $\varepsilon_i = g(E_i, E_i)$, then

$$\operatorname{tr} \nabla\varphi = \sum_{i=1}^{2n+1} \varepsilon_i (\nabla_{E_i}\varphi) E_i = 2n\xi, \tag{2.2}$$

$$\operatorname{Ric}(\xi, \xi) = 2n - \operatorname{tr} h^2. \tag{2.3}$$

A contact semi-Riemannian manifold is said to be *η-Einstein* if the Ricci operator Q is of the form $Q = aI + b\eta \otimes \xi$, where a, b are smooth functions. A contact semi-Riemannian manifold is said to be a *K-contact manifold* if ξ is a Killing vector field,

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or equivalently, $h = 0$. Moreover, a contact semi-Riemannian structure (ξ, η, φ, g) is said to be *Sasakian* if it is *normal*, that is $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$. This last condition is equivalent to

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \varepsilon\eta(Y)X. \tag{2.4}$$

Any Sasakian manifold is K-contact and the converse also holds when $n = 1$, that is, for three-dimensional spaces. It is worthwhile to remark here a difference between the Riemannian case and the general semi-Riemannian one. In fact, in both cases, $\operatorname{tr} h^2 = 0$ implies $\operatorname{Ric}(\xi, \xi) = 2n$. Moreover, it is well known that K-contact *Riemannian* manifolds are characterized by the condition $\operatorname{Ric}(\xi, \xi) = 2n$, since it implies $\operatorname{tr} h^2 = 0$, and so $h = 0$, because h is diagonalizable. On the other hand, there exist contact pseudo-metric manifolds for which $\operatorname{tr} h^2 = 0$ but $h \neq 0$, as we showed in [6] (see also [7, Example 1.1]). For a contact semi-Riemannian manifold (M, η, g), $h^2 = 0$ does not imply $h = 0$. We refer to [6–8] for more information about contact pseudo metric geometry.

3 K-contact and Sasakian Semi-Riemannian Manifolds

Theorem 3.1 *Let $(M, \eta, g, \xi, \varphi)$ be a K-contact semi-Riemannian manifold. Then*

(i) *ξ is an eigenvector of the Ricci operator Q: $Q\xi = 2n\varepsilon\xi$;*

(ii) *M is Sasakian if and only if the curvature tensor R satisfies*

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X. \tag{3.1}$$

Proof (i) Since ξ is a Killing vector field, then it is affine and hence satisfies

$$R(X, \xi)Y = -\nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y}\xi;$$

moreover, by (2.1), $\nabla\xi = -\varepsilon\varphi$. Then

$$R(X, \xi)Y = \varepsilon\nabla_X \varphi Y - \varepsilon\varphi\nabla_X Y = \varepsilon(\nabla_X \varphi)Y. \tag{3.2}$$

Consequently, if E_i is a local pseudo-orthonormal basis, we have

$$Q\xi = \sum_{i=1}^{2n+1} \varepsilon_i R(E_i, \xi)E_i = \varepsilon \sum_{i=1}^{2n+1} \varepsilon_i (\nabla_{E_i}\varphi)E_i = \varepsilon \operatorname{tr} \nabla\varphi.$$

Since, by (2.2), $\operatorname{tr} \nabla\varphi = 2n\xi$, we get $Q\xi = 2n\varepsilon\xi$.

(ii) If M is Sasakian, by (2.4), we have

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \varepsilon\eta(Y)X. \tag{3.3}$$

Moreover, M is K-contact and thus holds (3.2). Using (3.2) and (3.3) we get (3.1). Conversely, if (3.1) holds, we have $R(X, \xi)\xi = \varphi^2 X$. On the other hand, ξ is Killing, that is, $\nabla\xi = -\varepsilon\varphi$. Thus holds (3.2). Consequently, using (3.1) and (3.2), we obtain

$$\begin{aligned}
\varepsilon g((\nabla_X \varphi)Y, Z) &= g(R(X, \xi)Y, Z) = -g(R(Y, Z)\xi, X) \\
&= -g(\eta(Y)Z - \eta(Z)Y, X) \\
&= g(-\eta(Y)X, Z) - \varepsilon g(X, Y)g(\xi, Z).
\end{aligned}$$

Therefore, we get (2.4) and thus M is Sasakian. ∎

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Remark 3.2 A contact semi-Riemannian manifold $(M, \eta, g, \xi, \varphi)$ is K-contact if and only if the tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ vanishes. In the Riemannian case, Theorem 3.1(i) holds in a stronger form; that is, M is K-contact if and only if $Q\xi = 2n\xi$ (*cf.* [2, Theorem 7.1 and Proposition 7.2]). In fact $Q\xi = 2n\xi$ implies $\mathrm{tr} h^2 = 0$, and so $h = 0$, because h is diagonalizable. When M is semi-Riemannian, the condition $Q\xi = 2n\varepsilon\xi$ implies, by using (2.3), $\mathrm{tr}h^2 = 0$, but as we showed in [6] (see also [7, Example 1.1]) in general $\mathrm{tr}h^2 = 0$ does not imply $h \neq 0$. In the Riemannian case, Theorem 3.1(ii) holds in the same form (*cf.* [2, Proposition 7.6]).

The following is a characterization of K-contact semi-Riemannian manifolds in the class of all semi-Riemannian manifolds. In the Riemannian case, the corresponding result was given in [11].

Theorem 3.3 *A semi-Riemannian manifold (M, g) is K-contact if and only if M admits a Killing vector field ξ, with $g(\xi, \xi) = \varepsilon$, such that the sectional curvature of all nondegenerate plane sections containing ξ equals ε.*

Proof Let p be a point of M. We recall that a plane section $\mathrm{span}(X_p, Y_p)$ is nondegenerate if $\mathcal{A}(X_p, Y_p) := g(X_p, X_p)g(Y_p, Y_p) - g(X_p, Y_p)^2 \neq 0$. Suppose first that (ξ, φ, η, g) is a K-contact structure on M. For a contact semi-Riemannian manifold, by (2.1), one gets

$$R(\,\cdot\,, \xi)\xi = -\varphi\nabla_\xi h + \varphi^2 + h^2. \tag{3.4}$$

Since ξ is Killing, *i.e.*, $h = 0$, for a nondegenerate plane section $\mathrm{span}(\xi_p, X_p)$, $g(\xi_p, X_p) = 0$, from (3.4) we have

$$K(\xi_p, X_p) = -\frac{g(R(X_p, \xi_p)\xi_p, X_p)}{\varepsilon g(X_p, X_p)} = -\frac{g(\varphi^2 X_p, X_p)}{\varepsilon g(X_p, X_p)} = \varepsilon.$$

Conversely, suppose that ξ is a Killing vector field with $g(\xi, \xi) = \varepsilon = \pm 1$, and define η and φ by

$$\eta = \varepsilon g(\xi, \cdot\,), \quad \varphi = -\varepsilon\nabla\xi.$$

Since $g(\xi, \xi) = \varepsilon$, the nondegenerate plane sections containing ξ are nondegenerate for any vector field $X \in \mathrm{Ker}\,\eta_p$, which is either space-like or time-like. Let p be a point of M. Then

$$\varepsilon = K(\xi, X_p) = -\frac{g(R(X_p, \xi_p)\xi_p, X_p)}{\varepsilon g(X_p, X_p)}, \quad \text{that is} \quad g(R(X_p, \xi_p)\xi_p + X_p, X_p) = 0,$$

for any $X_p \in \mathrm{Ker}\,\eta_p$, with X_p either space-like or time-like. Now, if $Y_p \in \mathrm{Ker}\,\eta$ is a null vector, that is, $\mathrm{span}(\xi_p, Y_p)$ is degenerate, by [14, Lemma 40, p. 78], the vector Y_p is limit of nonnull vectors X_p of $\mathrm{Ker}\,\eta_p$. Since $g(R(X_p, \xi_p)\xi_p + X_p, X_p)$ is a continuous function of X_p, we get

$$g(R(X_p, \xi_p)\xi_p + X_p, X_p) = 0, \quad \text{for any} \quad X_p \in \mathrm{Ker}\,\eta_p.$$

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Then, since the endomorphism $S(X_p) := R(X_p, \xi_p)\xi_p + X_p$ is self-adjoint, we have

$$R(X_p, \xi_p)\xi_p = -X_p, \quad \text{for any} \quad X_p \in \operatorname{Ker} \eta_p \quad \text{and} \quad p \in M. \tag{3.5}$$

Moreover, since ξ is Killing with $g(\xi, \xi) = \text{const}$, we have

$$\varphi\xi = -\varepsilon\nabla_\xi \xi = 0, \quad g(\varphi X, Y) = -g(X, \varphi Y)$$

and

$$R(X, \xi)\xi = -\nabla_X \nabla_\xi \xi + \nabla_{\nabla_X \xi}\xi = \nabla_{\nabla_X \xi}\xi = \varphi^2 X. \tag{3.6}$$

So from (3.5) and (3.6), we get $\varphi^2 X = -X$ for any $X \in \operatorname{Ker} \eta$. This gives $\varphi^2 X = -X + \eta(X)\xi$ for arbitrary X. Moreover,

$$\begin{aligned} 2\varepsilon(\mathrm{d}\eta)(X, Y) &= Xg(\xi, Y) - Yg(\xi, X) - g(\xi, [X, Y]) = g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) \\ &= -\varepsilon g(\varphi X, Y) + \varepsilon g(X, \varphi Y) \\ &= 2\varepsilon g(X, \varphi Y). \end{aligned}$$

This implies that η is a contact 1-form, ξ the associated Reeb vector field, and g an associated metric. Since ξ is Killing, the structure (η, g, ξ, φ) is K-contact. ■

4 Conformally Flat K-contact Semi-Riemannian Manifolds

Generalizing a result of Okumura [13], Tanno [17] proved that a conformally flat K-contact Riemannian manifold is of constant sectional curvature +1. In this section, we show the corresponding result in the semi-Riemannian case.

Theorem 4.1 *Let $M = (M, \eta, g, \xi, \varphi)$ be a conformally flat K-contact semi-Riemannian manifold. Then M is Sasakian and of constant sectional curvature $\kappa = \varepsilon = g(\xi, \xi)$.*

Proof We first consider M of dimension $2n + 1 > 3$. We recall that a semi-Riemannian $(2n + 1)$-manifold, $n > 1$, is conformally flat if and only if

$$\begin{aligned} (2n-1)R(X, Y)Z &= g(Z, X)QY + g(QZ, X)Y - g(Z, Y)QX - g(QY, Z)X \\ &\quad - \frac{r}{2n}\big(g(Z, X)Y - g(Z, Y)X\big). \end{aligned} \tag{4.1}$$

In particular, for $Z = \xi$, we have

$$\begin{aligned} (2n-1)R(X, Y)\xi &= g(\xi, X)QY + g(Q\xi, X)Y - g(\xi, Y)QX - g(QY, \xi)X \\ &\quad - \frac{\varepsilon r}{2n}\big(\eta(X)Y - \eta(Y)X\big). \end{aligned} \tag{4.2}$$

On the other hand, by Theorem 3.1, for a K-contact manifold we have $Q\xi = 2n\varepsilon\xi$, and hence (4.2) implies

$$2n(2n-1)R(X, \xi)\xi = 2n\big(4n\eta(X)\xi - \varepsilon QX - 2nX\big) - \varepsilon r\big(\eta(X)\xi - X\big). \tag{4.3}$$

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But, in the K-contact case, $R(X,\xi)\xi = \varphi^2 X = -X + \eta(X)\xi$. Then (4.3) implies

$$QX = \frac{r - 2n\varepsilon}{2n} X + \frac{2n(2n+1)\varepsilon - r}{2n} \eta(X)\xi. \tag{4.4}$$

From (4.2) and (4.4) we get $R(X,Y)\xi = \eta(X)Y - \eta(Y)X$. Then since ξ is Killing, by Theorem 3.1, M is Sasakian.

Next, we consider the $*$-*scalar curvature* r^* of a contact pseudo-metric manifold (M,η,g) by contracting the curvature tensor by φ insteaded of by the metric. Precisely,

$$r^* = \operatorname{tr}\operatorname{Ric}^* = \sum_{i,j=1}^{2n+1} \varepsilon_j \varepsilon_i g\big(R(E_j,E_i)\varphi E_j, \varphi E_i\big)$$

where $\{E_1,\dots,E_{2n+1}\}$ is a pseudo-orthonormal basis. Then we get

$$r^* - r + 4n^2\varepsilon = \varepsilon \operatorname{tr} h^2 + \frac{1}{2}\big(\|\nabla\varphi\|^2 - 4n\varepsilon\big) \tag{4.5}$$

(see [6, Lemma 4.6]). By using (4.1), a direct calculation gives

$$r^* = \sum_{i,j=1}^{2n+1} \varepsilon_j \varepsilon_i g\big(R(E_j,E_i)\varphi E_j, \varphi E_i\big) = \frac{r - 4n\varepsilon + 2\varepsilon \operatorname{tr} h^2}{2n-1}. \tag{4.6}$$

From (4.5) and (4.6), one gets

$$4(n-1)\big(-r + 2n(2n+1)\varepsilon\big) = 2\varepsilon(2n-3)\operatorname{tr} h^2 + (2n-1)(\|\nabla\varphi\|^2 - 4n\varepsilon). \tag{4.7}$$

Since M is Sasakian, $h = 0$, and by (2.4) we easily find $(\|\nabla\varphi\|^2 - 4n\varepsilon) = 0$. Then (4.7) and $n > 1$ give $r = 2n(2n+1)\varepsilon$, and by (4.4) we get $QX = 2n\varepsilon X$. Thus M is a conformally flat, Einstein semi-Riemannian manifold. Then formula (4.1), $QX = 2n\varepsilon X$, and $r = 2n(2n+1)\varepsilon$ give

$$R(X,Y)Z = \varepsilon\big(g(Z,X)Y - g(Z,Y)X\big),$$

namely M has constant sectional curvature $\kappa = \varepsilon$.

Now, let (M,η,g) be a three-dimensional conformally flat K-contact semi-Riemannian manifold. In this case a pseudo-orthonormal φ-basis $\{\xi, E, \varphi E\}$ of $\operatorname{Ker}\eta$, satisfies $g(\varphi E,\varphi E) = g(E,E) = \pm g(\xi,\xi) = \pm\varepsilon$. Moreover, in dimension three, any K-contact semi-Riemannian manifold is automatically Sasakian and η-Einstein (see Remark 5.2), thus

$$\operatorname{Ric} = \alpha g + \beta \eta\otimes\eta, \quad \text{where} \quad \alpha = \Big(\frac{r}{2} - \varepsilon\Big) \quad \text{and} \quad \beta = \Big(3 - \varepsilon\frac{r}{2}\Big). \tag{4.8}$$

Since ξ is Killing, it leaves Ric invariant, that is $\mathcal{L}_\xi \operatorname{Ric} = 0$. This and (4.8) imply

$$\big(\nabla_\xi \operatorname{Ric}\big)(E,\varphi E) = 0. \tag{4.9}$$

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Recall that a semi-Riemannian 3-manifold is conformally flat if and only if

$$(4.10)\quad \big(\nabla_X \operatorname{Ric}\big)(Y,Z) - \big(\nabla_Y \operatorname{Ric}\big)(X,Z) = (1/4)\big(g(Y,Z)X(r) - g(X,Z)Y(r)\big)$$

From (4.10) and (4.8), we have

$$\begin{aligned}\big(\nabla_\xi \operatorname{Ric}\big)(E,\varphi E) = \big(\nabla_E \operatorname{Ric}\big)(\xi,\varphi E) &= -\operatorname{Ric}(\nabla_E\xi,\varphi E) - \operatorname{Ric}(\xi,\nabla_E\varphi E)\\ &= \varepsilon \operatorname{Ric}(\varphi E,\varphi E) - \operatorname{Ric}(\xi,\nabla_E\varphi E)\\ &= \pm\,\varepsilon^2\alpha - \alpha g(\xi,\nabla_E\varphi E) - \beta\eta(\xi)\eta(\nabla_E\varphi E)\\ &= \pm\,\alpha \mp\,\alpha \mp\,\beta\varepsilon = \mp\,\beta\varepsilon.\end{aligned}$$

Therefore, (4.9) gives $\beta = 0$; that is, M is Einstein with $r = 6\varepsilon$, namely M has constant sectional curvature $\kappa = \varepsilon$. ∎

Corollary 4.2 *Any conformally flat K-contact Lorentzian manifold is Lorentzian-Sasaki and of constant sectional curvature $\kappa = \varepsilon = g(\xi,\xi)$.*

Besides, as a consequence of Theorems 4.1 and 3.3 we get the following corollary.

Corollary 4.3 *Let (M,g) be a conformally flat semi-Riemannian manifold. If M admits a Killing vector field ξ with $g(\xi,\xi) = \varepsilon$, such that the sectional curvature of all nondegenerate plane sections containing ξ equals ε, then M admits a Sasakian semi-Riemannian structure (η,g) of constant sectional curvature $\kappa = \varepsilon$.*

Example 4.4 (Sasakian semi-Riemannian manifolds of constant curvature) Consider $(\mathbb{R}^{2n+2}_{2s},\widetilde{g})$ the pseudo-Euclidean space with the standard indefinite Käler metric. The *pseudosphere* and the *pseudohyperbolic space* are defined by

$$S^{2n+1}_{2s}(1) = \{x \in \mathbb{R}^{2n+2}_{2s} : \widetilde{g}(x,x) = 1\} \text{ and } \mathbb{H}^{2n+1}_{2s-1}(-1) = \{x \in \mathbb{R}^{2n+2}_{2s} : \widetilde{g}(x,x) = -1\}.$$

They are hyperquadrics of $\mathbb{R}^{2n+2}_{2s}$, both of dimension $(2n+1)$, of index $2s$ and $(2s-1)$, and of constant sectional curvature 1 and -1 respectively. Moreover, they have a canonical Sasakian semi-Riemannian structure, with characteristic vector field space-like and time-like respectively [16].

5 Some Remarks on Contact Lorentzian Manifolds

It is easy to see that a smooth manifold admits a Lorentzian metric if and only if it admits a nowhere vanishing vector field. So contact semi-Riemannian geometry is quite natural in the Lorentzian setting. Lorentzian Sasaki structures are related to the Kaehler structures by the following (*cf.* [1, p. 46]): M has a Lorentzian Sasakian structure (g_L,η) if and only if the cone $C(M) = (M\times\mathbb{R}, g_C = t^2 g_L - dt\otimes dt)$ has a (semi-Riemannian) Kaehler structure. In this section we give some results about the curvature of a contact Lorentzian manifold.

Let (M,η,g) be a contact semi-Riemannian manifold of dimension $2n+1$, with $g(\xi,\xi) = \varepsilon$. Then it is easy to check that for any real constant $t \neq 0$ the tensors

$$(5.1)\qquad \widetilde{\eta} = t\eta,\quad \widetilde{\xi} = \frac{1}{t}\xi,\quad \widetilde{\varphi} = \varphi,\quad \widetilde{g} = tg + \varepsilon t(t-1)\eta\otimes\eta$$

describe another contact semi-Riemannian structure on M, having the same contact distribution $\operatorname{Ker}\widetilde{\eta} = \operatorname{Ker}\eta$, called a $\mathcal{D}$-*homothetic deformation* (or a *transverse homothety*) of (φ, ξ, η, g). Clearly, (5.1) is the natural semi-Riemannian generalization of $\mathcal{D}$-homothetic deformations of a contact Riemannian structure, where one has $g(\xi, \xi) = 1$ and needs to assume $t > 0$ so that $\widetilde{g}$ is still Riemannian [18]. Notice that $\widetilde{g}(\widetilde{\xi}, X) = \varepsilon\widetilde{\eta}(X)$. In particular, $\widetilde{\varepsilon} = \widetilde{g}(\widetilde{\xi}, \widetilde{\xi}) = g(\xi, \xi) = \varepsilon$, that is, $\mathcal{D}$-homothetic deformation preserves the causal character of the Reeb vector field. For $t < 0$, if g is of signature $(2p+1, 2n-2p)$, then $\widetilde{g}$ is of signature $(2n-2p+1, 2p)$. The Ricci tensors, the scalar curvatures, and the sectional curvatures satisfy

$$\widetilde{\operatorname{Ric}} = \operatorname{Ric} - 2\varepsilon(t-1)g + 2(t-1)(nt+n+1)\eta\otimes\eta + \frac{t-1}{t} g\big(\varepsilon(\nabla_\xi h)\varphi + 2h, \cdot\big)\,, \tag{5.2}$$

$$\widetilde{r} = \frac{1}{t} r - \varepsilon\frac{t-1}{t^2}\operatorname{Ric}(\xi,\xi) - 2n\varepsilon\frac{(t-1)^2}{t^2}, \tag{5.3}$$

$$\widetilde{K}(\widetilde{\xi}, X) = \frac{1}{t^2}K(\xi, X) + \varepsilon\frac{t^2-1}{t^2} + 2\frac{t-1}{t^2}\frac{g(hX,X)}{g(X,X)}, \tag{5.4}$$

$$\widetilde{K}(X, \varphi X) = \frac{1}{t}K(X,\varphi X) - 3\varepsilon\frac{t-1}{t} - \varepsilon\frac{t-1}{t^2}\frac{g(hX,X)^2 + g(\varphi hX, X)^2}{g(X,X)^2}, \tag{5.5}$$

for all $X \in \operatorname{Ker}\eta = \operatorname{Ker}\widetilde{\eta}$, either space-like or time-like (see [6, Section 3]).

Recall that there is a canonical way to associate a contact Riemannian structure with a contact Lorentzian structure (and conversely). Let $(\varphi, \xi, \eta, g_L)$ be a contact Lorentzian structure on a smooth manifold M, where the Reeb vector field ξ is time-like. Then

$$g = g_L + 2\eta\otimes\eta$$

is a Riemannian metric, and is still compatible with the same contact structure (φ, ξ, η). Moreover, in such case $g(\xi,\xi) = -g_L(\xi,\xi) = +1$. Hence, (φ, ξ, η, g) is a contact Riemannian structure on M. We remark that $g_L = -g_{-1}$, where

$$g_{-1} = -g + 2\eta\otimes\eta$$

is obtained by the $\mathcal{D}$-homothetic deformation of g for $t = -1$. Consequently, the Levi-Civita connection and curvature of g_L can be easily deduced from the formulae valid for a general $\mathcal{D}$-homothetic deformation. Taking into account that in the Lorentzian case the tensor h is diagonalizable, for a unit vector field $X \in \operatorname{Ker}\eta$, $hX = \lambda X$, from (5.3)–(5.5) we have the following formulae (see also [6, Proposition 3.9]):

$$r_L = r + 4n + 2\operatorname{tr} h^2 \geq r + 4n\,,$$
$$K_L(\xi, X) = -K(\xi, X) + 4\lambda\,,$$
$$K_L(X, \varphi X) = K(X, \varphi X) + 2(3 - \lambda^2)\,.$$

So we obtain the following proposition.

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Proposition 5.1 *Let (M, η, g_L) be a contact Lorentzian manifold. If the eigenvalues of h are constant, then the scalar curvature, respectively the vertical sectional curvature and the holomorphic sectional curvature, of (M, η, g_L) is constant if and only if the corresponding curvature of (M, η, g) is constant. Moreover, $r_L = r + 4n$ if and only if (M, η, g_L) is K-contact Lorentzian.*

Since the operator $h_L = \frac{1}{2}\mathcal{L}_\xi\varphi = h$ does not depend on the metric, we have *(η, g_L) is K-contact if and only if (η, g)* is. Moreover, since $\widetilde{g} := g_{-1} = -g_L, \widetilde{\eta} = -\eta, \widetilde{\xi} = -\xi$, and $\widetilde{\varepsilon} = \varepsilon = 1$, we get

$$(\nabla^L_X\varphi)Y - \big(g_L(X,Y)\xi + \eta(Y)X\big) = (\widetilde{\nabla}_X\varphi)Y - \big(\widetilde{g}(X,Y)\widetilde{\xi} - \widetilde{\eta}(Y)X\big),$$

where ∇^L is the Levi-Civita connection of g_L. This formula, using (2.4), gives that *(η, g_L) is Sasakian if and only if (η, g) is* (see also [6, Theorem 3.1]).

Remark 5.2 The Ricci tensor of an arbitrary η-Einstein semi-Riemannian contact manifold is given by

$$\mathrm{Ric} = \alpha\, g + \beta\, \eta \otimes \eta,$$

where $\alpha = \big(\frac{r}{2n} + \varepsilon(\frac{\operatorname{tr} h^2}{2n} - 1)\big)$ and $\beta = -\big(\varepsilon\frac{r}{2n} + (2n+1)(\frac{\operatorname{tr} h^2}{2n} - 1)\big)$. In particular, the Ricci tensor of the η-Einstein K-contact structure (η, g) is given by

$$\mathrm{Ric} = \Big(\frac{r}{2n} - 1\Big) g + \Big(-\frac{r}{2n} + 2n + 1\Big)\eta \otimes \eta,$$

where the scalar curvature r is a constant when $n > 1$, and g is Einstein if and only if $r = 2n(2n+1)$. Then, from (5.2) and (5.3), the Ricci tensor of the corresponding Lorentzian K-contact structure (η, g_L) is given by

$$\mathrm{Ric}_L = \mathrm{Ric} + 4g - 4\eta \otimes \eta = \Big(\frac{r_L}{2n} + 1\Big) g_L + \Big(\frac{r_L}{2n} + 2n + 1\Big)\eta \otimes \eta, \tag{5.6}$$

where the scalar curvature $r_L = r + 4n$ is a constant when $n > 1$, and g_L is Einstein if and only if $r_L = -2n(2n+1)$. In dimension three, every K-contact structure (η, g) is automatically Sasakian and η-Einstein, and thus by (5.6) every K-contact Lorentzian structure (η, g_L) is also automatically Sasakian and η-Einstein. Moreover, for a K-contact Lorentzian 3-manifold, the scalar curvature r_L and the φ-sectional curvature H_L are related by $r_L = 2H_L - 4$.

A Lorentzian Sasakian manifold (M, g, η) is Einsteinian if and only if the cone $C(M)$ is Ricci-flat [1]. Moreover, geometries of this type are interesting because they provide examples of twistor spinors on Lorentzian manifolds (see, for example, [1, 4]). In particular, [1, Proposition 6.2] gives a twistorial characterization of Einstein Lorentzian-Sasaki manifolds. Now, we see as the η-Einstein Lorentzian-Sasaki structures are related to the Einstein Lorentzian-Sasaki structures. Let (η, g_L) be a

K-contact Lorentzian structure on M with ξ time-like and $\dim M = 2n+1 > 3$. For the new K-contact Lorentzian structure

$$\widetilde{\eta} = t\eta, \quad \widetilde{\xi} = \frac{1}{t}\xi, \quad \widetilde{\varphi} = \varphi, \quad \widetilde{g}_L = tg_L - t(t-1)\eta \otimes \eta, \; t > 0,$$

from (5.2) and (5.3) we have

$$\widetilde{\operatorname{Ric}}_L = \operatorname{Ric}_L + 2(t-1)g_L + 2(t-1)(nt+n+1)\eta \otimes \eta, \quad \widetilde{r}_L = \frac{r_L - 2n}{t} + 2n.$$

Then, if (η, g_L) is η-Einstein, the Ricci tensor of the new K-contact Lorentzian structure $(\widetilde{\eta}, \widetilde{g}_L)$ is given by

$$\begin{aligned}\widetilde{\operatorname{Ric}} &= \Big(\frac{r_L}{2n} + 2t - 1\Big) g_L + \Big(\frac{r_L}{2n} + 2n + 1 + 2(t-1)(nt+n+1)\Big)\eta \otimes \eta, \\ &= \Big(\frac{\widetilde{r}_L}{2n} + 1\Big)\widetilde{g}_L + \Big(\frac{\widetilde{r}_L}{2n} + 2n + 1\Big)\widetilde{\eta} \otimes \widetilde{\eta}.\end{aligned}$$

So for any $t > 0$ the K-contact Lorentzian structure $(\widetilde{\eta}, \widetilde{g}_L)$ is $\widetilde{\eta}$-Einstein. If the scalar curvature r_L of the η-Einstein K-contact Lorentzian manifold (η, g_L) satisfies $r_L < 2n$, then the K-contact Lorentzian structure $(\widetilde{\eta}, \widetilde{g})$ obtained in correspondence to

$$t = \frac{2n - r_L}{4n(n+1)} > 0.$$

is Einstein. If $r_L \geq 2n$, the contact Riemannian structure (η, g) that corresponds to the η-Einstein K-contact Lorentzian structure (g_L, η) is η-Einstein K-contact with scalar curvature $r \geq -2n$, and thus, when M is compact, by a result of Boyer and Galicki (*cf.* [5, p. 418]) the structure is Sasakian. Summing up, we get the following proposition.

Proposition 5.3 *Let (M, η, g_L) be a η-Einstein K-contact Lorentzian manifold of dimension $2n+1 > 3$. If the scalar curvature satisfies $r_L < 2n$, then there exists a transverse homothety whose resulting structure $(\widetilde{\eta}, \widetilde{g}_L)$ is Einstein K-contact Lorentzian structure. Moreover, if $r_L \geq 2n$, and M is compact, then the structure (η, g_L) is η-Einstein Lorentzian-Sasaki.*

The result of this proposition is peculiar to the Lorentzian case. From our Proposition 5.3 and [1, Proposition 6.2], we get the following theorem.

Theorem 5.4 *Let (M, η, g_L, ξ) be a simply connected η-Einstein Lorentzian-Sasaki manifold of dimension $2n+1 > 3$. If the scalar curvature satisfies $r_L < 2n$, then there exists a transverse homothety whose resulting Lorentzian manifold $(M, \widetilde{g}_L)$ is a spin manifold. Moreover, there exists a twistor spinor φ that is an imaginary Killing spinor such that the associated vector field V_φ (the Dirac current) is $\widetilde{\xi}$.*

We note that any connected sum of $S^2 \times S^3$ admits a Einstein Lorentzian-Sasaki structure [10]. In [8, p. 19] we proved that if a compact contact Lorentzian manifold $(M, \eta, \xi, g, \varphi)$ is a contact Ricci soliton, then it is a Einstein Lorentzian-Sasaki manifold. Now, we give the following

Example 5.5 Consider a simply connected bounded domain Ω in $\mathbb{C}^n$, equipped with the Kaehler structure (G, J) of constant holomorphic sectional curvature $\kappa < -3$. Let ω be the Kaehler form; such form is closed and thus $\omega = \mathrm{d}\vartheta$. Let $\pi\colon M = \Omega \times \mathbb{R} \to \Omega$ the natural projection, and t the coordinate on $\mathbb{R}$. We construct a Lorentzian-Sasaki structure on M like the Riemannian case (*cf.* [2, Ch.7]). We define the tensor

$$\eta = \pi^*\vartheta + \mathrm{d}t, \quad \xi = \partial/\partial t, \quad g_L = \pi^* G - \eta \otimes \eta.$$

Moreover, we define the tensor φ such that to be the horizontal lift of the complex structure J and zero in the vertical direction. Then $(\eta, g_L, \varphi, \xi)$ is a η-Einstein Lorentzian-Sasaki structure with ξ time-like. The scalar curvature is given by

$$r_L = \big(n(2n+1)(\kappa+3) + n(\kappa+7)\big)/2.$$

Since $r_L - 2n = n(n+1)(\kappa+3) < 0$, for $t = -\frac{\kappa+3}{4}$ the resulting structure $(\widetilde{\eta}, \widetilde{g}_L)$ is Einstein Lorentzian-Sasaki.

In the 3-dimensional case, Proposition 5.3 does not hold. However, a Lorentzian K-contact 3-manifold (M, η, g_L) is automatically Sasakian and η-Einstein. If, in addition, we assume that the scalar curvature is constant, then the corresponding K-contact Riemannian manifold (M, η, g) is a locally φ-symmetric space, and so it is locally homogeneous (see [3]). Equivalently, a 3-dimensional Lorentzian Sasakian space with constant scalar curvature is locally homogeneous. Then from the classification of 3-dimensional homogeneous Lorentzian contact manifolds given in [6] (which is a consequence of [15, Theorem 3.1]), we deduce the following proposition.

Proposition 5.6 *A simply connected Lorentzian-Sasaki three-manifold with constant scalar curvature, is a Lie group G equipped with a left-invariant contact Lorentzian-Sasaki structure* $(\varphi, \xi, \eta, g_L)$*. More precisely, one of the following cases occurs. If G is unimodular, then it is*

(i) *the Heisenberg group* H^3 *when* $r_L = 2$*;*
(ii) *the* 3*-sphere group* $SU(2)$ *when* $r_L > 2$*;*
(iii) $\widetilde{SL}(2, R)$ *when* $r_L < 2$.

If G is non-unimodular, then its Lie algebra is given by

$$[e_1, e_2] = \alpha e_2 + 2\xi, \quad [e_1, \xi] = [e_2, \xi] = 0, \tag{5.7}$$

where α *is a constant* $\neq 0$*. In this case,* $r_L = -2\alpha^2 + 2 < 2$.

When $r_L < 2$, the K-contact Lorentzian structure $(\widetilde{\eta}, \widetilde{g})$ obtained in correspondence to $t = \frac{2-r_L}{8}$ is Einstein, and so of constant sectional curvature -1. Therefore, we get the following corollary, which does not have a Riemannian counterpart.

Corollary 5.7 *The unimodular Lie group* $\widetilde{SL}(2, R)$ *and the non-unimodular Lie group with Lie algebra defined by* (5.7) *are the only simply connected three-manifolds that admit a left invariant Lorentzian-Sasaki structure of constant sectional curvature* $\kappa = -1$.

In the paper [12], the authors considered the problem of classifying 3-dimensional complete Lorentzian manifold of constant sectional curvature.

Another consequence of Proposition 5.6 is the following corollary.

Corollary 5.8 *The Heisenberg group H^3 is the only simply connected three-manifold that admits a left invariant Lorentzian-Sasaki structure of constant scalar curvature* $r_L = 2$.

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