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THE APPROXIMATE JORDAN-HAHN DECOMPOSITION

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1. Introduction. Non-commutative measure theory embraces measure theory on σ-fields of subsets of a set, on projection lattices of von Neumann algebras or JBW-algebras and on hypergraphs alike [**20**], [**27**], [**33**], [**37**], [**39**], [**40**], [**41**]. Due to the unifying structure of an orthoalgebra concepts can easily be transferred from one branch to the other. Additional conceptual inpetus is obtained from the logico-probabilistic foundations of quantum mechanics (see [**6**], [**19**], [**21**]).

In the late seventies the author studied the Jordan-Hahn decomposition of measures on orthomodular posets and certain graphs. These investigations revealed an interesting geometrical aspect of this decomposition in that the Jordan-Hahn property of the convex set of probability charges on a finite orthomodular poset can be characterized in terms of the extreme points of the unit ball of the Banach space dual of the base normed space of Jordan charges. Subsequent investigations let into the study of stable faces of convex polyhedra [**35**] and attempts to generalize these results above into questions of a functional analytic nature [**9**], [**10**].

The present paper is concerned with a relaxed form of the Jordan-Hahn decomposition which is, as it turns out, in important cases the suitable kind of decomposition when considering not necessarily σ-additive charges. A convex set Δ of probability charges on an orthoalgebra L is said to have the *approximate Jordan-Hahn property* provided that for every linear combination μ of elements of Δ there exist elements ν, ξ in Δ and elements s, t in $\mathbf{R}_+$ such that (i) $\mu = s\nu - t\xi$ and (ii) to each $\epsilon > 0$ there exists an element p in L with $(s\nu)(p')$, $(t\xi)(p) \leqq \epsilon$. This is clearly an extension of the classical notion of an approximate Jordan-Hahn decomposition (see e.g. [**34**]) from the context of fields of sets to the non-commutative setting of orthoalgebras. For the special case of a σ-orthocomplete orthomodular poset, an even weaker version of a Jordan-Hahn decomposition has been considered in [**9**].

The main result of this paper consists in establishing in the context of orthoalgebras two necessary and sufficient conditions for a convex subset Δ of probability charges to possess the approximate Jordan-Hahn property. This is done in terms of a positive dominance property for charges in one case and in terms of the variation norm and an intrinsic norm (base norm) determined by Δ in the other. The usefulness of these criteria is demonstrated in the case of JBW-algebras, locally finite orthoalgebras and Boolean algebras.

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In Section 2 we present a summary of the basic facts on orthoalgebras following the work of Foulis and Randall [**17**], [**32**]. Further contributions to this field can be found in [**30**], [**31**] and [**42**]. In Section 3 we introduce the notion of a charge on an orthoalgebra and develop the appropriate functional analytic framework. Then Section 4 is devoted to the aforementioned characterization of the approximate Jordan-Hahn property. In Section 5 we relate local finiteness of an orthoalgebra L to the Jordan-Hahn property of a convex set Δ of probability charges and to reflexivity of the base normed space generated by Δ. A corollary of the main result in this paragraph states that unitality together with the Jordan-Hahn property of the convex set of all probability charges on L entails local finiteness of L. Section 6 is concerned with various applications of the previous results. The collection of idempotents $U(A)$ of a JBW-Algebra A forms an orthoalgebra in a natural way. We prove that the collection of probability charges on $U(A)$ obtained by restricting states on A to $U(A)$ enjoys the approximate Jordan-Hahn property. As a second application we exhibit a large class of locally finite orthoalgebras for which the convex set of all probability charges has the approximate Jordan-Hahn property. Finally, we specialize our results to Boolean algebras to obtain a new result in that classical context.

2. Prerequisites. Let L be a set and $\perp$ a binary relation on L. If (p,q) is an element of $\perp$, we write $p \perp q$ and call the pair *orthogonal*. Let $\oplus$ be a map from $\perp$ into L. For an element (p,q) in $L \times L$ we write $p \oplus q$ to assert that $p \perp q$ and to denote the image of (p,q) under the map $\oplus$.

A quintuple $(L, \perp, \oplus, 0, 1)$, where $L, \perp$ and $\oplus$ are as above and 0, 1 are two distinct distinguished elements of L, is said to be an *orthoalgebra* if the following holds true, for elements p, q and r in L,

(i) If $p \perp q$ and $p \oplus q \perp r$ then $q \perp r$, $p \perp (q \oplus r)$ and $(p \oplus q) \oplus r = p \oplus (q \oplus r)$,
(ii) if $p \perp q$ then $q \perp p$ and $p \oplus q = q \oplus p$,
(iii) to each element p in L there exists a unique element q in L such that $p \perp q$ and $p \oplus q$ equals 1,
(iv) if $p \perp p$ then $p = 0$.

For an element p in L we denote with p' the unique element in L which satisfies condition (iii).

Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra. One easily verifies that, for every element p in L,

$$\text{(v)} \quad 1' = 0, \quad \text{(vi)} \quad p'' = p \text{ and } \quad \text{(vii)} \quad p \oplus 0 = p.$$

Moreover, for elements p, q and r in L,

(viii) if $p \perp q$ then $p \oplus (p \oplus q)' = q'$,
(ix) if $p \oplus q = p \oplus r$ then $q = r$.

For elements p and q in L, we write $p \leqq q$ to mean that there exists an element r in L such that $p \perp r$ and $p \oplus r$ is equal to q. The binary relation $\leqq$ is an ordering relation on L with respect to which 0 is the least and 1 is the greatest element. Also, the mapping $p \longrightarrow p'$ is an orthocomplementation on the poset $(L, \leqq)$. The orthocomplemented poset $(L, \leqq, ')$ is referred to as the orthocomplemented poset *associated* with the orthoalgebra $(L, \perp, \oplus, 0, 1)$. Then, for elements p and q in L,

(x) $p \perp q$ if and only if $p \leqq q'$,

(xi) if $p \leqq q$ then $q = p \oplus (p \oplus q')'$

holds true. Furthermore, if $p \perp q$ then $p \oplus q$ is a minimal upper bound of the set $\{p, q\}$ in the poset $(L, \leqq)$.

We now turn our attention to examples of orthoalgebras.

Let $(P, \leqq, ')$ be an orthocomplemented poset with 0 as the least and 1 as the greatest element. For a subset M of P we write $\vee M$, resp. $\wedge M$ to denote the supremum, resp. the infimum, of M provided it exists. Also, we write $p \vee q$, resp. $p \wedge q$, to denote $\vee\{p, q\}$, resp. $\wedge\{p, q\}$. An orthocomplemented poset $(P, \leqq, ')$ is said to be an *orthomodular poset* [**16**], [**37**] provided that, for elements p, q in P,

(i) if $p \leqq q'$ then $p \vee q$ exists,

(ii) if $p \leqq q'$ and $p \vee q = 1$ then $p = q'$.

Let $(P, \leqq, ')$ be an orthomodular poset of cardinality greater than one. We define a binary relation $\perp$ on P by

$$\perp := \{(p, q) \in P \times P : p \leqq q'\}$$

and a map $\oplus : \perp \longrightarrow P$ by

$$p \oplus q := p \vee q, \quad (p, q) \in \perp.$$

It follows that $(P, \perp, \oplus, 0, 1)$ is an orthoalgebra and that the associated orthocomplemented poset is ortho-order isomorphic to $(P, \leqq, ')$ under the identity map of P.

A *Boolean algebra* is a sixtuple $(B, \sqcup, \sqcap, \dagger, 0, 1)$ where B is a set, $\sqcup$, $\sqcap$ are binary operations, $\dagger$ is a unary operation and 0, 1 are two distinct distinguished elements of B such that the conditions $L1 - L10$ of [**5**] are satisfied. We define a binary relation $\perp$ on B by

$$\perp := \{(p, q) \in B \times B : p \sqcap q = 0\}$$

and a map $\oplus : \perp \longrightarrow B$ by

$$p \oplus q := p \sqcup q, \quad (p, q) \in \perp.$$

Then $(B, \perp, \oplus, 0, 1)$ is an orthoalgebra, the orthoalgebra *corresponding* to the Boolean algebra B, and the associated orthocomplemented poset $(B, \leqq, ')$ is a distributive orthocomplemented lattice, in fact

$$p \wedge q = p \sqcap q, \quad p \vee q = p \sqcup q \text{ and } p' = q^{\dagger},$$

for elements p, q in B.

Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra. A non-empty subset P of L is said to be a *suborthoalgebra* provided that the following holds true:

(i) if $p \in P$ then $p' \in P$,

(ii) if $p, q \in P$ and $p \perp q$ then $p \oplus q \in P$.

Clearly, a suborthoalgebra contains the elements 0 and 1 and, in the induced structure, is an orthoalgebra in its own right.

An orthoalgebra is said to be *orthomodular* provided that the associated orthocomplemented poset is an orthomodular poset.

Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra. Let p, q be elements of L with $p \perp q$. Then the subset $\{0, p, q, p \oplus q, p', q', (p \oplus q)', 1\}$ is an orthomodular suborthoalgebra. A subset M of L is called *orthogonal* if $p, q \in M$ with $p \neq q$ implies that $p \perp q$. Notice that the empty subset and singleton subsets of L are orthogonal. A subset M of L is said to be *jointly orthogonal* if it is an orthogonal subset contained in an orthomodular suborthoalgebra. Also notice that an orthogonal subset of cardinality less than or equal to 2 is jointly orthogonal. An orthoalgebra is said to be *locally finite* if every jointly orthogonal subset is finite.

Let p, q be elements in an orthoalgebra L with $q \leqq p$. We define the *difference* between p and q, denoted by $p - q$, to be the element $(p' \oplus q)'$. A subset D of L is called a *difference set* if it is empty or if there exists a finite, strictly isotone sequence $(p_i)_{i=0}^n$, $1 \leqq n$, in $(L, \leqq)$ such that

$$D = \{p_i - p_{i-1} : i = 1, 2, \ldots, n\}.$$

We say that the sequence $(p_i)_{i=0}^n$ *yields* the difference set D. Notice that n is the cardinality of D. A difference set is orthogonal and a subset of a difference set is a difference set. Also, an orthogonal set of non-zero elements of cardinality at most 2 is a difference set. Let $(p_i)_{i=0}^m$ and $(q_i)_{i=0}^n$, $1 \leqq m, n$, be strictly isotone sequences in $(L, \leqq)$ both of which yield the difference set D. Then $p_m - p_0$ and $q_n - q_0$ coincide (see Theorem 2.12 [**42**]). We now define a map $\bigoplus$ from the collection of difference sets of L into L as follows: if D is a non-empty difference set and yielded by the strictly isotone sequence $(p_i)_{i=0}^n$ then

$$\bigoplus D := p_n - p_0;$$

if D is empty, we set

$$\bigoplus D := 0.$$

Notice that if p is a non-zero element then it is equal to $\bigoplus\{p\}$ and that for every orthogonal pair (p, q) of non-zero elements $\bigoplus\{p, q\}$ coincides with $p \oplus q$. We now present two results on difference sets which are essential for the following.

THEOREM 2.1. *Let* $(L, \perp, \oplus, 0, 1)$ *be an orthoalgebra and let* M *be an orthogonal subset of* L. *Then* M *is jointly orthogonal if and only if every finite subset of* $M \backslash \{0\}$ *is a difference set.*

Proof. See Theorem 2.16 [**42**].

THEOREM 2.2. *Let* $(L, \perp, \oplus, 0, 1)$ *be an orthoalgebra and let* D *be a difference set of* L *of cardinality* $n > 0$. *Let* $i \in \{1, 2, \dots, n\} \longrightarrow p_i \in D$ *be any enumeration of* D. *Then*

$$p_1 \perp p_2, (p_1 \oplus p_2) \perp p_3, \quad ((p_1 \oplus p_2) \oplus p_3) \perp p_4, \dots$$

$$(\dots(((p_1 \oplus p_2) \oplus p_3) \oplus p_4) \oplus \dots \oplus p_{n-1}) \perp p_n.$$

Moreover, the sequence $(q_i)_{i=0}^n$ *defined by*

$$q_0 := 0, \quad q_i := (\dots((p_1 \oplus p_2) \oplus p_3) \dots \oplus p_{i-1}) \oplus p_i$$

for $1 \leqq i \leqq n$, *is strictly isotone and yields the difference set* D.

Proof. See Corollary 2.17[**42**].

For details and proofs the reader is referred to [**17**], [**30**], [**31**], [**32**] and [**42**].

3. Charges on orthoalgebras. Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra. We consider the vector space $\mathbf{R}^L$ in the product topology τ, a locally convex Hausdorff topology. An element μ in $\mathbf{R}^L$ is said to be a *charge* on L if

$$\mu(p \oplus q) = \mu(p) + \mu(q)$$

for all orthogonal pairs (p, q) in L. It follows that a charge μ on L evaluates to zero on the element 0 of L. Also, if D is a non-empty difference set of L then, by Theorem 2.2,

$$\mu(\bigoplus D) = \sum_{p \in D} \mu(p)$$

for any charge μ on L. A charge is said to be *positive* if it is positive as a functional on L. Trivially, the zero element of $\mathbf{R}^L$ is a positive charge on L. Similarly, a charge is said to be *bounded* if it is bounded as a functional on L.

LEMMA 3.1. *Let* $(L, \perp, \oplus, 0, 1)$ *be an orthoalgebra and let* $(L, \leqq, ')$ *be its associated orthocomplemented poset. Then a positive charge on* L *is bounded and is an isotone functional on the poset* $(L, \leqq)$.

Proof. Let μ be a positive charge on L and let p, q be elements in L with $p \leqq q$. Then there exists an element r in L such that $p \oplus r$ coincides with q. Therefore

$$\mu(p) \leqq \mu(p) + \mu(r) = \mu(p \oplus r) = \mu(q).$$

Then, for all elements p in L,

$$0 = \mu(0) \leqq \mu(p) \leqq \mu(1).$$

We denote with $W(L)$ the subspace of $\mathbf{R}^L$ consisting of all bounded charges on L and with $J_+(L)$ the subset of $\mathbf{R}^L$ of positive charges on L. Notice that $J_+(L)$ is a cone in $\mathbf{R}^L$ (i.e., (i) $J_+(L)+J_+(L) \subseteq J_+(L)$, (ii) $\mathbf{R}_+J_+(L) \subseteq J_+(L)$ and (iii) $J_+(L) \cap -J_+(L) = \{0\}$). The subspace $J_+(L) - J_+(L)$ of $\mathbf{R}^L$ is denoted by $J(L)$; an element of $J(L)$ is called a *Jordan charge*. By Lemma 3.1, $J(L)$ is a subset of $W(L)$.

Let μ be an element of $W(L)$, we define elements μ^+, μ^- and $|\mu|$ of $\mathbf{R}^L$ as follows:

$$\mu^+(p) := \sup_{q \leqq p} \mu(q), \ \ \mu^-(p) := -\inf_{q \leqq p} \mu(q) \text{ and } |\mu| := \mu^+ + \mu^-$$

for an element p of L. Notice that μ^- is equal to $(-\mu)^+$. It follows from Section 2 that μ^+ is a super-additive bounded positive functional on L. We define functionals $\mu \longrightarrow \|\mu\|_v$ and $\mu \longrightarrow \|\mu\|_s$ on $W(L)$ by

$$\|\mu\|_v := |\mu|(1) \text{ and } \|\mu\|_s := \sup_{p \in L} |\mu(p)|,$$

called the *variation norm* and the *sup-norm*, respectively.

A charge is said to be a *probability charge* if it is an element of $J_+(L)$ and evaluates to one on the element 1 of L. We denote with $\Omega(L)$ the τ-compact convex set of probability charges on L. Notice that the set $\Omega(L)$ is a base of the cone $J_+(L)$ (i.e., $\Omega(L)$ is a convex set, every non-zero element of $J_+(L)$ is a positive multiple of an element of $\Omega(L)$ and this representation is unique).

On occasion, we restrict our attention to a subset of $\Omega(L)$ rather than considering all probability charges. This justifies the treatment in the generality to follow. Let Δ be a convex subset of $\Omega(L)$. We denote by $J(\Delta)$ the linear hull of Δ and we write $J_+(\Delta)$ for the positive hull of Δ. Then $J_+(\Delta)$ is a generating cone in $J(\Delta)$, Δ is a base of $J_+(\Delta)$ and the absolutely convex hull acon(Δ) of Δ is absorbing in $J(\Delta)$. Clearly, $J(L)$ and $J_+(L)$ coincide with $J(\Omega(L))$ and $J_+(\Omega(L))$, respectively.

Let $J(\Delta)'$ be the algebraic dual space of $J(\Delta)$. With each element p of L we associate an element $e_\Delta(p)$ in $J(\Delta)'$ by

$$e_\Delta(p)(\mu) := \mu(p) \quad \text{for } \mu \in J(\Delta).$$

Clearly, the set $\{e_\Delta(p) : p \in L\}$, denoted by $P(\Delta)$, is total on $J(\Delta)$ (i.e., if $e_\Delta(p)(\mu) = 0$ for all elements p in L then $\mu = 0$).

For every convex subset Δ of $\Omega(L)$ the pair $(J(\Delta), \Delta)$ is a base normed space [**39**], i.e., $J(\Delta)$ is a real vector space, Δ is a base of a generating cone in $J(\Delta)$ and the Minkowski functional [**29**] over the absorbing absolutely convex set aconΔ, denoted by $\|\cdot\|_\Delta$, is a norm, the *base norm on* $J(\Delta)$.

Notice that

$$\|\mu\|_\Delta = \inf\{s+t : \mu = s\nu + t\xi, \ s,t \in \mathbf{R}_+, \nu, \xi \in \Delta\}$$

provided that Δ is not empty. Let $J(\Delta)_1$ be the unit ball of the normed vector space $(J(\Delta), \|\cdot\|_\Delta)$ and let $J(\Delta)_1^0$ denote its (norm-) interior then $J(\Delta)_1^0 \subseteq \text{acon}\Delta \subseteq J(\Delta)_1$. Also, if Δ is non-empty then the absolutely convex hull aconΔ of Δ coincides with convex hull con$(\Delta \cup -\Delta)$ of the set $\Delta \cup -\Delta$. Conditions on Δ which ensure that the corresponding base norm is a complete norm are discussed in [**41**].

We now follow the general theory of base normed spaces and order unit normed spaces [**1**], [**4**], [**15**], [**38**]. With $J(\Delta)^*$ we denote the $\|\cdot\|_\Delta$-continuous members of $J(\Delta)'$. An element f in $J(\Delta)'$ is $\|\cdot\|_\Delta$-continuous if and only if the functional f is bounded on the set Δ. It follows that $P(\Delta)$ is a subset of $J(\Delta)^*$. If we order the vector space $J(\Delta)^*$ as follows, for elements f, g in $J(\Delta)^*$,

$$f \leqq g :\Leftrightarrow f(\mu) \leqq g(\mu) \quad \text{for all } \mu \in \Delta,$$

then the triple $(J(\Delta)^*, \leqq, e_\Delta(1))$ becomes an order unit normed space, i.e., an Archimedean ordered real vector space with order unit $e_\Delta(1)$. Clearly the functional

$$f \longrightarrow \|f\|_\Delta := \sup\{f(\mu) : \mu \in J(\Delta)_1\}$$

defined on $J(\Delta)^*$ is a complete norm on $J(\Delta)^*$. Also

$$\|f\|_\Delta := \sup\{|f(\mu)| : \mu \in \Delta\} = \inf\{t \geqq 0 : f \in t[-e_\Delta(1), e_\Delta(1)]\},$$

where, for elements g, h of a partially ordered set, $[g, h]$ denotes the order-interval from g to h. Moreover, the set $[-e_\Delta(1), e_\Delta(1)]$ is the unit ball of the Banach space $(J(\Delta)^*, \|\cdot\|_\Delta)$.

The relationship between variation norm, sup-norm and base norm is given in the following lemma.

LEMMA 3.2. *Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra. Let τ be the product topology on the vector space $\mathbf{R}^L$ and let $W(L)$ be the subspace of $\mathbf{R}^L$ of bounded charges on L. Then the variation norm $\mu \longrightarrow \|\mu\|_v$ and the sup-norm $\mu \longrightarrow \|\mu\|_s$ are equivalent norms on $W(L)$. Moreover, for all elements μ in $W(L)$,*

$$\|\mu\|_v = \sup_{p \in L}(\mu(p) - \mu(p')) = 2\|\mu\|_s - |\mu(1)|$$

and the topology $\tau \mid W(L)$ is coarser than the topology determined by the variation norm (or sup-norm).

Let $\Omega(L)$ be the convex set of probability charges on L. Let Δ_1, Δ_2 be convex subsets of $\Omega(L)$ with $\Delta_1 \subseteq \Delta_2$. Let $J(\Delta_1)$ and $J(\Delta_2)$ denote the linear hull of Δ_1 and Δ_2 in $W(L)$, respectively. Then the variation norm, the sup-norm, the base norm on $J(\Delta_1)$ $\mu \longrightarrow \|\mu\|_{\Delta_1}$ and the base norm on $J(\Delta_2)$ $\mu \longrightarrow \|\mu\|_{\Delta_2}$ satisfy

$$\|\mu\|_s \leqq \|\mu\|_v \leqq \|\mu\|_{\Delta_2} \leqq \|\mu\|_{\Delta_1}$$

for elements μ in $J(\Delta_1)$.

Proof. The functional $\nu \longrightarrow \sup_{p\in L} |\nu(p)|$ is a complete norm on the subspace V of $\mathbf{R}^L$ of bounded functionals on L and the topology determined by this norm is finer than the topology $\tau \mid V$. Since the subspace of $\mathbf{R}^L$ of all charges on L is τ-closed it follows that $W(L)$ is $\tau \mid V$-closed and therefore $(W(L), \|\cdot\|_s)$ is a complete normed vector space.

Let μ be an element of $W(L)$. Then

$$\begin{aligned}\|\mu\|_v &= |\mu|(1) = \mu^+(1) + (-\mu)^+(1) = \sup_{p\in L} \mu(p) + \sup_{p\in L}(-\mu)(p') \\ &= 2\sup_{p\in L} \mu(p) - \mu(1) = \sup_{p\in L}(\mu(p) - \mu(p')).\end{aligned}$$

Hence

$$\|\mu\|_v = \|-\mu\|_v \quad \text{and} \quad \mu^+(1) = (\|\mu\|_v + \mu(1))/2.$$

Then,

$$\|\mu\|_s = \max\{\mu^+(1), \mu^-(1)\} = (\|\mu\|_v + |\mu(1)|)/2.$$

It follows that $\|\mu\|_v \leqq 2\|\mu\|_s$ and since $|\mu(1)| \leqq \|\mu\|_s$ we conclude that $\|\mu\|_s \leqq \|\mu\|_v$. This proves that the variation norm is indeed a norm and as such it is equivalent to the sup-norm.

Let μ be an element in $J(\Delta)$. Since $P(\Delta)$ is a subset of $[0, e_\Delta(1)]$ it follows that

$$\|\mu\|_v = \sup_{p\in L}(e_\Delta(p) - e_\Delta(p'))(\mu) \leqq \sup_{f\in[-e_\Delta(1), e_\Delta(1)]} f(\mu) = \|\mu\|_\Delta.$$

If $\Delta_1 \subseteq \Delta_2$ then acon$\Delta_1 \subseteq$ aconΔ_2 and the remaining assertion follows.

A subset Δ of $\Omega(L)$ which coincides with the intersection of $\Omega(L)$ and the affine hull affΔ of Δ is referred to as a *section* of $\Omega(L)$. A section of $\Omega(L)$ is necessarily convex. Notice that a face of $\Omega(L)$ is a section of $\Omega(L)$.

LEMMA 3.3. *Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra and let $W(L)$ be the vector space of bounded charges on L equipped with the variation norm $\mu \longrightarrow \|\mu\|_v$ and the sup-norm $\mu \longrightarrow \|\mu\|_s$. Let Δ be a convex subset of the convex set $\Omega(L)$ of probability charges on L and let $J(\Delta)$ be the linear hull of Δ in $W(L)$. Then the following conditions are pairwise equivalent:*

(i) *The set Δ is a section of $\Omega(L)$.*

(ii) *There exists a subspace V of $W(L)$ such that Δ equals $\Omega(L) \cap V$.*

(iii) *The set Δ coincides with the set $\{\mu \in J(\Delta) : \mu(1) = 1$ and $\mu(p) \geqq 0$ for all $p \in L\}$.*

(iv) *The set Δ coincides with the set $\{\mu \in J(\Delta) : \|\mu\|_s = 1 = \mu(1)\}$.*

(v) *The set Δ coincides with the set $\{\mu \in J(\Delta) : \|\mu\|_v = 1 = \mu(1)\}$.*

Proof. (i) $\Rightarrow$ (ii): Let Δ be a section of $\Omega(L)$ and let μ be an element of $\Omega(L) \cap \text{lin}\Delta$. Then there exist elements μ_i in Δ and elements t_i in $\mathbf{R}$, $i = 1, 2, \ldots, n$, such that μ equals $\sum_{i=1}^{n} t_i\mu_i$. Then

$$1 = \mu(1) = \sum_{i=1}^{n} t_i\mu_i(1) = \sum_{i=1}^{n} t_i$$

and therefore μ belongs to the set $\text{aff}\Delta \cap \Omega(L)$, hence μ is an element in Δ. This proves that Δ coincides with the set $\Omega(L) \cap \text{lin}\Delta$.

(ii) $\Rightarrow$ (iii): Let V be a subspace of $W(L)$ such that Δ equals $\Omega(L) \cap V$. Then Δ is a subset of V, hence $J(\Delta)$ is a linear subspace of V and we conclude that $\Delta \subseteq \Omega(L) \cap J(\Delta) \subseteq \Omega(L) \cap V$. Therefore Δ is equal to $\Omega(L) \cap J(\Delta)$. Now, if μ is a positive charge in $J(\Delta)$ such that $\mu(1)$ equals 1 then μ belongs to $\Omega(L) \cap J(\Delta)$. This proves the claim.

(iii) $\Rightarrow$ (iv): Clearly, Δ is a subset of the set $\{\mu \in J(\Delta) : \|\mu\|_s = 1 = \mu(1)\}$. Now let μ be an element of $J(\Delta)$ such that $\|\mu\|_s$ and $\mu(1)$ equals 1. Then for all elements p in L,

$$1 \geqq \mu(p') = \mu(1) - \mu(p) = 1 - \mu(p),$$

therefore $\mu(p)$ is greater than zero. It follows that μ is an element of Δ.

(iv) $\Leftrightarrow$ (v): This is a consequence of Lemma 3.2.

(iv) $\Rightarrow$ (i): Let μ be an element of the set $\text{aff}\Delta \cap \Omega(L)$. Then $\mu(1)$ equals one and μ is a positive charge. Therefore μ is of sup-norm one. Since $\text{aff}\Delta$ is a subset of $J(\Delta)$ it follows that μ belongs to Δ.

4. Approximate Jordan-Hahn property. Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra and let $\Omega(L)$ be the convex set of probability charges on L. For a convex subset Δ of $\Omega(L)$ we denote with $J(\Delta)$ the linear hull and with $J(\Delta)_+$ the positive hull of Δ in the vector space $W(L)$ of bounded charges on L.

A convex subset Δ of $\Omega(L)$ is said to have the *Jordan-Hahn property* [**36**], [**45**] if for each element μ in $J(\Delta)$ there exist elements ν, ξ in $J(\Delta)_+$ and an element p in L such that

$$\mu = \nu - \xi \quad \text{and} \quad \nu(p') = 0 = \xi(p).$$

Trivially, for any orthoalgebra L, the empty subset and singleton subsets of $\Omega(L)$ have the Jordan-Hahn property.

We now relax this condition in the following manner: a convex subset Δ of $\Omega(L)$ is said to have the *approximate Jordan-Hahn property* if for each element μ in $J(\Delta)$ there exist elements ν, ξ in $J(\Delta)_+$ satisfying

(i) $\mu = \nu - \xi$ and (ii) to every $\epsilon > 0$ there exists an element p in L with $\nu(p')$, $\xi(p) \leqq \epsilon$.

Notice that the two properties coalesce for finite orthoalgebras.

LEMMA 4.1. *Let* $(L, \perp, \oplus, 0, 1)$ *be an orthoalgebra and let* Δ *be a non-empty convex subset of the convex set* $\Omega(L)$ *of probability charges on* L. *Let* $J(\Delta)$ *and* $J_+(\Delta)$ *be the linear hull and the positive hull of* Δ, *respectively.*

Let μ *be an element in* $J(\Delta)$ *and let* ν *be an element in* $J_+(\Delta)$. *If* $\mu(p) \leqq \nu(p) \leqq 1$ *for all elements* p *in* L *then there exists an element* ξ *in* Δ *such that* $\mu(p) \leqq \xi(p)$ *for all elements* p *in* L.

Proof. If $\nu(1)$ equals zero then ν is the zero element and every element in Δ satisfies the condition. If $\nu(1)$ is different from zero then $\nu/\nu(1)$ is an element in Δ since, in this case, $J_+(\Delta)$ coincides with $\mathbf{R}_+\Delta$. The assertion follows immediately.

LEMMA 4.2. *Same preliminaries as in Lemma* 4.1. *Let* $W(L)$ *be the vector space of bounded charges on* L *equipped with the sup-norm* $\mu \longrightarrow \|\mu\|_s$ *and let* $\mu \longrightarrow \|\mu\|_\Delta$ *be the base norm on* $J(\Delta)$.

If Δ *is such that for each element* μ *in* $J(\Delta)$ *with* $\|\mu\|_s \leqq 1$ *there exists an element* ν *in* $J_+(\Delta)$ *with* $\mu(p) \leqq \nu(p) \leqq 1$ *for all elements* p *in* L *then* Δ *is a section of* $\Omega(L)$. *Moreover,* $\|\eta\|_\Delta$ *equals* $2\|\eta\|_s$ *for all elements* η *in* $J(\Delta)$ *for which* $\eta(1)$ *vanishes.*

Proof. We may assume that Δ is not empty. Let μ be an element of $J(\Delta)$ such that $\|\mu\|_s$ and $\mu(1)$ equal one. Since $|\mu(p)| \leqq 1$ for all elements p in L there exists, by Lemma 4.1, an element ξ in Δ such that $\mu(p) \leqq \xi(p)$ for all elements p in L. Then

$$\begin{aligned} 1 &= \mu(1) = \mu(p \oplus p') = \mu(p) + \mu(p') \\ &\leqq \xi(p) + \xi(p') = \xi(p \oplus p') = \xi(1) = 1. \end{aligned}$$

Therefore $\mu(p)$ equals $\xi(p)$ for all elements p in L, which shows that μ is an element in Δ. It follows from Lemma 3.3 that Δ is a section of $\Omega(L)$.

Suppose now that η is an element in $J(\Delta)$ such that $\eta(1)$ equals 0 and $\|\eta\|_\Delta$ equals 1. Then for every $\epsilon > 0$ there exist elements s, t in $\mathbf{R}_+$ and elements κ, λ in Δ such that η equals $s\kappa - t\lambda$ and $1 \leqq s + t \leqq 1 + \epsilon$. Since $\eta(1)$ vanishes, we conclude that s equals t and, hence, $1/2 \leqq s \leqq (1+\epsilon)/2$. Then, for all elements p in L,

$$-(1+\epsilon)/2 \leqq -s\lambda(p) \leqq \eta(p) \leqq s\kappa(p) \leqq (1+\epsilon)/2$$

and therefore $\|\eta\|_s \leqq 1/2$. On the other hand, there exists, by Lemma 4.1, an element ω in Δ such that

$$(\eta/\|\eta\|_s)(p) \leqq \omega(p) \text{ for all elements } p \text{ in } L.$$

Then $(\omega - \eta/\|\eta\|_s)(p) \geqq 0$ for all elements p in L and

$$(\omega - \eta/\|\eta\|_s)(1) = \omega(1) = 1.$$

Since Δ is a section of $\Omega(L)$ and $\omega - \eta/\|\eta\|_s$ is an element of $J(\Delta)$ it follows, by Lemma 3.3, that this element belongs to Δ. Then

$$1/\|\eta\|_s = \|\eta/\|\eta\|_s\|_\Delta \leqq \|\omega\|_\Delta + \|\omega - \eta/\|\eta\|_s\|_\Delta = 2.$$

We now proceed to the main result of this Section.

THEOREM 4.3. *Let* $(L, \perp, \oplus, 0, 1)$ *be an orthoalgebra and let* $W(L)$ *be the vector space of bounded charges on L equipped with the variation norm. Let* Δ *be a convex subset of the convex set* $\Omega(L)$ *of probability charges on L. Let* $J(\Delta)$ *and* $J_+(\Delta)$ *be the linear hull and the positive hull of* Δ, *respectively. Then the following conditions are pairwise equivalent:*

(i) *For each element* μ *in* $J(\Delta)$ *with* $\mu(p) \leqq 1$ *for all elements p in L there exists an element* ν *in* $J_+(\Delta)$ *such that* $\mu(q) \leqq \nu(q) \leqq 1$ *for all elements q in L.*

(ii) *For each element* μ *in* $J(\Delta)$ *with* $-1 \leqq \mu(p) \leqq 1$ *for all elements p in L there exists an element* ν *in* $J_+(\Delta)$ *such that* $\mu(q) \leqq \nu(q) \leqq 1$ *for all elements q in L.*

(iii) *The base norm on* $J(\Delta)$ *and the the variation norm coincide on* $J(\Delta)$ *and the absolutely convex hull of* Δ *is closed in the base norm.*

(iv) *The subset* Δ *has the approximate Jordan-Hahn property.*

If either is the case then Δ *is a section.*

Proof. We may assume that Δ is not empty.

(i) $\Rightarrow$ (ii): This is obvious.

(ii) $\Rightarrow$ (iii): It follows form Lemma 4.2 that Δ is a section. Let μ be an element of $J(\Delta)$ of sup-norm one. If $\mu(1)$ is equal to zero then, by Lemma 4.2 and Lemma 3.2, $\|\mu\|_\Delta$ and $\|\mu\|_v$ are equal. This also holds true when $\mu(1)$ equals ± 1, by Lemma 3.3. We now consider the case where $\mu(1)$ is contained in the interval (0, 1). By Lemma 4.1, there exists an element ν in Δ such that $0 \leqq (\nu - \mu)(p)$ for all elements p in L. Since Δ is a section there exists, by Lemma 3.3, an element ξ in Δ and a scalar $t \geqq 0$ such that $t\xi$ equals $\nu - \mu$. Then

$$t = t\xi(1) = 1 - \mu(1) \in (0, 1)$$

and one verifies that

$$\mu = (1 - t)\nu + t(\nu - \xi).$$

Since μ is of sup-norm one there exists, by the Theorem of Bourbaki-Alaoglu [**29**], a $\| \cdot \|_s$-continuous linear functional f of norm one on the normed vector space $(J(\Delta), \| \cdot \|_s)$ such that

$$1 = f(\mu) = (1 - t)f(\nu) + tf(\nu - \xi).$$

Since the elements ν and $\nu - \xi$ are of sup-norm less than or equal to one we conclude that

$$1 = f(\nu) = f(\nu - \xi).$$

It follows that the element $\nu - \xi$ is of sup-norm one and since $(\nu - \xi)(1)$ equals zero we infer from Lemma 4.2 that $\|(\nu - \xi)/2\|_\Delta$ equals one. Again, by the Theorem of Bourbaki-Alaoglu, there exists a $\|\cdot\|_\Delta$-continuous linear functional g of norm one on the normed vector space $(J(\Delta), \|\cdot\|_\Delta)$ such that $(1/2)g(\nu) - (1/2)g(\xi)$ equals one. Since the elements ν and ξ belong to Δ we conclude that both $g(\nu)$ and $-g(\xi)$ are equal to one. Notice that

$$(1+t)^{-1}\mu = (1+t)^{-1}\nu - (1+t)^{-1}t\xi \in \mathrm{acon}\Delta \quad \text{and}$$
$$g((1+t)^{-1}\mu) = 1.$$

Then $\|(1+t)^{-1}\mu\|_\Delta$ equals one and therefore

$$\|\mu\|_\Delta = 2 - \mu(1).$$

Suppose now that $\mu(1)$ is an element of the interval $(-1, 0)$. Then

$$\|\mu\|_\Delta = \|-\mu\|_\Delta = 2 - (-\mu)(1) = 2 + \mu(1).$$

It follows that

$$\|\mu\|_\Delta = 2\|\mu\|_s - |\mu(1)|$$

for all elements μ in $J(\Delta)$ and we conclude, by Lemma 3.2, that both the base norm on $J(\Delta)$ and the variation norm agree on $J(\Delta)$.

Next we prove that the absolutely convex hull aconΔ of Δ coincides with the unit ball of the normed vector space $(J(\Delta), \|\cdot\|_\Delta)$. Since the set acon$\Delta$ is circled it suffices to show that each element μ in $J(\Delta)$ of base norm one belongs to that set. Let μ be such an element in $J(\Delta)$. Then, by Lemma 3.2, $-1 \leqq \mu(p) \leqq 1$ for all elements p in L. Assume first that $\mu(1)$ is an element of the interval $[0, 1]$. Let ν denote the element $2(1 + \mu(1))^{-1}\mu$. Then, by the previously established relationship between the sup-norm and the base norm on $J(\Delta)$, we obtain

$$\|\nu\|_s = 2(1+\mu(1))^{-1}\|\mu\|_s = 2(1+\mu(1))^{-1}2^{-1}(1+\mu(1)) = 1.$$

By hypothesis, Lemma 4.1 and Lemma 3.3, there exist elements κ, λ in Δ such that ν equals $\kappa - (1 - \nu(1))\lambda$. It follows that

$$(2-\nu(1))^{-1}\kappa - (1-\nu(1))(2-\nu(1))^{-1}\lambda = (2-\nu(1))^{-1}\nu$$
$$= 2(1+\mu(1))^{-1}(2-\nu(1))^{-1}\mu$$
$$= 2(1+\mu(1))^{-1}(2-2\mu(1)(1+\mu(1))^{-1})^{-1}\mu = \mu$$

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and therefore μ is an element of aconΔ.

If $\mu(1)$ belongs to the interval $[-1, 0]$ then μ is an element of aconΔ since the latter is circled.

(iii) $\Rightarrow$ (iv): Let λ be a non-zero element of $J(\Delta)$ and let $\epsilon > 0$. Since the set aconΔ coincides with the unit ball of $(J(\Delta), \|\cdot\|_\Delta)$ there exist elements ν, ξ in Δ and a scalar t in the interval [0, 1] such that

$$\mu := \lambda/\|\lambda\|_\Delta = t\nu - (1-t)\xi.$$

Then the elements $\|\lambda\|_\Delta t\nu$, $\|\lambda\|_\Delta(1-t)\xi$ belong to $J_+(\Delta)$. By hypothesis and Lemma 3.2, $\sup_{p\in L}(\mu(p)-\mu(p'))$ equals one and therefore there exists an element p in L such that

$$1-(\mu(p)-\mu(p')) \leqq 2\|\lambda\|_\Delta^{-1}\epsilon.$$

Then

$$\begin{aligned} 2\|\lambda\|_\Delta^{-1}\epsilon &\geqq 1-(2\mu(p)-\mu(1)) = 1-2\mu(p)+2t-1 \\ &= 2(t-\mu(p)) = 2((1-t)\xi(p)-t\nu(p)+t) \\ &= 2(t\nu(p')+(1-t)\xi(p)). \end{aligned}$$

This shows that

$$\|\lambda\|_\Delta t\nu(p'), \|\lambda\|_\Delta(1-t)\xi(p) \leqq \epsilon.$$

(iv)$\Rightarrow$(i): Let μ be an element of $J(\Delta)$ and suppose that $\mu(p) \leqq 1$ for all elements p in L. There exist elements ν, ξ in Δ and elements s, t in $\mathbf{R}_+$ and for every $\epsilon > 0$ there exists an element p in L such that

$$\mu = s\nu - t\xi \quad \text{and} \quad s\nu(p'), t\xi(p) \leqq \epsilon.$$

Then

$$\begin{aligned} 1 \geqq \mu(p) &= s\nu(p) - t\xi(p) = s\nu(1) - s\nu(p') - t\xi(p) \\ &= s\nu(1) - (s\nu(p') + t\xi(p)) \geqq s - 2\epsilon. \end{aligned}$$

Therefore s belongs to the interval [0, 1] which implies that $\mu(p) \leqq s\nu(p) \leqq \nu(p) \leqq 1$ for all elements p in L.

COROLLARY 4.4. *Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra and let $W(L)$ be the vector space of bounded charges on L equipped with the variation norm. Let $J(L)$ be the real vector space of Jordan charges on L. Let $J_+(L)$ be the cone of positive charges and let $\Omega(L)$ be the convex set of probability charges on L. Then the following conditions are pairwise equivalent:*

(i) *For each element μ in $W(L)$ with $\mu(p) \leqq 1$ for all elements p in L there exists an element ν in $J_+(L)$ such that $\mu(q) \leqq \nu(q) \leqq 1$ for all elements q in L.*

(ii) *For each element μ in $W(L)$ with $-1 \leqq \mu(p) \leqq 1$ for all elements p in L there exists an element ν in $J_+(L)$ such that $\mu(q) \leqq \nu(q) \leqq 1$ for all elements q in L.*

(iii) *Every bounded charge is Jordan, i.e., $J(L)$ exhausts $W(L)$, and the base norm on $J(L)$ coincides with the variation norm.*

(iv) *Every bounded charge is Jordan and $\Omega(L)$ has the approximate Jordan-Hahn property.*

Proof. If condition (ii) is met then every bounded charge is Jordan. To see this, let μ be a non-zero element in $W(L)$. Then

$$|(\mu/\|\mu\|_s)(p)| \leqq 1$$

for all elements p in L. Let ν be an element in $J_+(L)$ with

$$|(\mu/\|\mu\|_s)(p)| \leqq \nu(p) \leqq 1$$

for all elements p in L. Then

$$\xi := \nu - \mu/\|\mu\|_s \in J_+(L) \quad \text{and} \quad \mu = \|\mu\|_s\nu - \|\mu\|_s\xi.$$

Since $\Omega(L)$ is τ-compact it follows that acon$\Omega(L)$ is τ-compact as well. Now, the topology $\tau \mid J(L)$ is coarser than the topology determined by the base norm $\mu \longrightarrow \|\mu\|_{\Omega(L)}$ on $J(L)$, by Lemma 3.2. Therefore acon$\Omega(L)$ is $\|\cdot\|_{\Omega(L)}$-closed.
With these two observations, the assertion now follows easily from Theorem 4.3.

5. Supplementary results. Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra. A subset Δ of the convex set $\Omega(L)$ of probability charges on L is said to be *unital* if for each non-zero element p in L there exists an element in Δ which evaluates to one on p.

THEOREM 5.1. *Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra and let $W(L)$ be the vector space of bounded charges on L. Let Δ be a unital convex subset of the convex set $\Omega(L)$ of probability charges on L and let $J(\Delta)$ be the linear hull of Δ equipped with the base norm $\mu \longrightarrow \|\mu\|_\Delta$.*

If the Banach space completion of the normed vector space $(J(\Delta), \|\cdot\|_\Delta)$ is reflexive then L is locally finite.

Proof. Suppose that there exists an infinite jointly orthogonal subset M of L and let $i \in \mathbf{N} \longrightarrow p_i \in M\backslash\{0\}$ be an injection. Let l^∞ be the Banach space of bounded real sequences $r = (r_i)_{i\in\mathbf{N}}$ with norm

$$\|r\|_\infty = \sup_{i\in\mathbf{N}} |r_i|.$$

Let r be an element in l^∞ and define a sequence $(f_n)_{n\in\mathbf{N}}$ by

$$f_n = \sum_{i=1}^{n} r_i e_\Delta(p_i).$$

Since $\{p_i : i = 1, 2, \ldots, n\}$ is a difference set, by Theorem 2.1, we obtain, for each element μ in Δ,

$$|f_n(\mu)| \leqq \sum_{i=1}^{n} |r_i| \mu(p_i) \leqq \max_{i=1,2,\ldots,n} |r_i| \sum_{k=1}^{n} \mu(p_k)$$
$$\leqq \max_{i=1,2,\ldots,n} |r_i| \mu(\oplus\{p_i : i = 1, 2, \ldots, n\}) \leqq \max_{i=1,2,\ldots,n} |r_i|.$$

Therefore, for all natural numbers n, $\|f_n\|_\Delta \leqq \|r\|_\infty$. By $\sigma(J(\Delta)^*, J(\Delta))$-compactness of the unit ball $[-e_\Delta(1), e_\Delta(1)]$ of $J(\Delta)^*$ there exists a subnet $(f_{n'})_{n'}$ in $\|r\|_\infty[-e_\Delta(1), e_\Delta(1)]$ which converges to an element f in this set in the $\sigma(J(\Delta)^*, J(\Delta))$-topology. On the other hand, for every element μ in Δ, the real sequence $(\sum_{i=1}^{n} \mu(p_i))_{n \in \mathbf{N}}$ is isotone and is contained in the interval $[0, 1]$. Since the sequence $(r_i)_{i \in \mathbf{N}}$ is bounded, we conclude that $(f_n(\mu))_{n \in \mathbf{N}}$ converges in $\mathbf{R}$. The subnet $(f_{n'}(\mu))_{n'}$ of the sequence $(f_n(\mu))_{n \in \mathbf{N}}$ converges to $f(\mu)$ and therefore $(f_n(\mu))_{n \in \mathbf{N}}$ converges to $f(\mu)$ as well. This proves that $(f_n)_{n \in \mathbf{N}}$ converges to f in the $\sigma(J(\Delta)^*, J(\Delta))$-topology.

For each natural number j we select a charge μ_j in Δ such that $\mu_j(p_j)$ equals 1. Then the sequence $(f_n(\mu_j))_{n \in \mathbf{N}}$ converges to r_j, for all natural numbers j. Hence r_j equals $f(\mu_j)$ and we conclude that

$$\|r\|_\infty = \sup_{j \in \mathbf{N}} |f(\mu_j)| \leqq \sup_{\mu \in \Delta} |f(\mu)| \leqq \|r\|_\infty.$$

To this end we have shown that the linear map $\psi : l^\infty \longrightarrow J(\Delta)^*$ defined by

$$\psi(r) := \sigma(J(\Delta)^*, J(\Delta)) - \lim_n \sum_{i=1}^{n} r_i e_\Delta(p_i)$$

is isometric. This proves that $J(\Delta)^*$ contains a non-reflexive closed sub-space. Hence $J(\Delta)^*$ is not reflexive and therefore the Banach space completion of $J(\Delta)$ is not reflexive.

The following result is a variation of Theorem 4.1 [**39**]. Notice that $\Omega(L)$ as well as the empty subset and singleton subsets of $\Omega(L)$ are τ-closed subsets of $\mathbf{R}^L$.

THEOREM 5.2. *Let* $(L, \perp, \oplus, 0, 1)$ *be an orthoalgebra and let* τ *be the product topology on* $\mathbf{R}^L$. *Let* Δ *be a convex subset of the convex set* $\Omega(L)$ *of probability charges on L and let* $J(\Delta)$ *be the linear hull of* Δ *equipped with the base norm* $\mu \longrightarrow \|\mu\|_\Delta$.

If Δ *is* τ*-closed and has the Jordan-Hahn property then* $(J(\Delta), \|\cdot\|_\Delta)$ *is a reflexive Banach space.*

Proof. If Δ is τ-closed then aconΔ is τ-compact and therefore coincides with the unit ball of the normed vector space $(J(\Delta), \|\cdot\|_\Delta)$; also, this space is complete

[**1**], [**41**]. Let $(J(\Delta)^*, \|\cdot\|_\Delta)$ be the Banach space dual of $(J(\Delta), \|\cdot\|_\Delta)$. Then the subset $J(\Delta)_*$ of $J(\Delta)^*$ defined by

$$J(\Delta)_* := \{f \in J(\Delta)' : f \mid \text{acon}\Delta \text{ is } (\tau \mid \text{acon}\Delta) - \text{continuous}\}$$

is a $\|\cdot\|_\Delta$-closed subspace. By the theorem of Dixmier-Ng (see e.g. [**24**]), the Banach space dual $(J(\Delta)_*)^*$ of $J(\Delta)_*$ is linearly isometric to $J(\Delta)$. Notice that $\{e_\Delta(p) - e_\Delta(p') : p \in L\}$ is a subset of the unit ball of $J(\Delta)_*$.

Let μ be an element of $J(\Delta)$. Then there exist elements μ, ξ in Δ, scalars s, t in $\mathbf{R}_+$ and an element p in L such that the real numbers $s\nu(p')$, $t\xi(p)$ are zero and μ equals $s\nu - t\xi$. Then

$$\begin{aligned} s + t &\geqq \|\mu\|_\Delta \geqq (e_\Delta(p) - e_\Delta(p'))(\mu) \\ &= (e_\Delta(p) - e_\Delta(p'))(s\mu - t\xi) = s + t. \end{aligned}$$

To this end we have shown that every element of $(J(\Delta)_*)^* \approx J(\Delta)$ attains its supremum (=norm) on the unit ball of $J(\Delta)_*$. By the Theorem of James [**25**] (also see e.g. [**12**]), the unit ball of $J(\Delta)_*$ is $\sigma(J(\Delta)_*, (J(\Delta)_*)^*)$-compact, showing that $(J(\Delta)_*, \|\cdot\|_\Delta)$ and finally, $(J(\Delta), \|\cdot\|_\Delta)$ are reflexive Banach spaces.

COROLLARY 5.3. *Let $(L, \perp, \oplus, 0, 1)$ be an orthoalgebra and let τ be the product topology on $\mathbf{R}^L$. Let Δ be a τ-closed unital convex subset of the convex set $\Omega(L)$ of probability charges on L.*

If Δ has the Jordan-Hahn property then L is locally finite.

Proof. This follows from Theorem 5.1 and Theorem 5.2.

There are examples which show that the converse statements of Theorem 5.1 and Corollary 5.3 are false.

6. Applications

6.1 JBW-*Algebras.* A real algebra A, not necessarily associative, for which

$$a \circ b = b \circ a, \quad a \circ (b \circ a^2) = (a \circ b) \circ a^2,$$

holds true and which is also a Banach space with respect to a norm $a \longrightarrow \|a\|$ satisfying

$$\|a \circ b\| \leqq \|a\| \cdot \|b\|, \quad \|a^2\| = \|a\|^2 \quad \text{and} \quad \|a^2\| \leqq \|a^2 + b^2\|$$

is said to be a *JB-algebra.*

An element a in A is called *positive* if there exists an element b such that

$$a = b \circ b.$$

The set A_+ consisting of positive elements in A forms a generating cone in A. If A has a unit, denoted by 1, then the triple $(A, A_+, 1)$ is an order unit normed space and the order unit norm coincides with the norm of A. An *idempotent* is an element p in A satisfying

$$p \circ p = p;$$

$U(A)$ denotes the collection of idempotents in A. Trivially, the zero-element 0 and the unit 1 of the algebra A are idempotents.

A JB-algebra which is the Banach space dual of a, necessarily unique, Banach space is called a *JBW-algebra*. A JBW-algebra has a unit.

Let A be a JBW-algebra. We define a binary relation $\perp$ on the set $U(A)$ of idempotents in A by

$$\perp := \{(p,q) \in U(A) \times U(A) : p \circ q = 0\}$$

and a map $\oplus : \perp \longrightarrow U(A)$ by

$$p \oplus q := p + q \quad \text{for } (p,q) \in \perp.$$

Then the quintuple $(U(A), \perp, \oplus, 0, 1)$ is an orthomodular orthoalgebra. In fact, the associated orthocomplemented poset $(U(A), \leqq, ')$ is a complete orthomodular lattice with

$$p \leqq q \quad \text{if and only if} \quad q - p \in A_+,\ p' = 1 - p.$$

Let A be a JB-algebra and let A^* be its Banach space dual. A *state* on A is an element ϕ of A^* such that

$$\phi(1) = 1 \quad \text{and} \quad \phi(A_+) \subseteq \mathbf{R}_+;$$

$S(A)$ denotes the collection of states on A. Then the pair $(A^*, S(A))$ is a base normed space and the unit ball of A^* coincides with the absolutely convex hull $\operatorname{acon} S(A)$ of $S(A)$. For details the reader is referred to [**2**], [**3**], [**13**], [**14**], [**23**], [**44**].

Let A be a JBW-algebra. The restriction $R\phi$ of an element ϕ in A^* to $U(A)$ is a bounded charge on the orthoalgebra $(U(A), \perp, \oplus, 0, 1)$. An element ϕ in A^* is a state on A if and only if $R\phi$ is a probability charge, by the spectral theorem. By the same token, the linear map $R : A^* \longrightarrow W(U(A))$ is injective. We denote with Δ the image of $S(A)$ under the map R. Notice that Δ is a unital convex subset of $\Omega(U(A))$. It then follows that $(J(\Delta), \Delta)$ and $(A^*, S(A))$ are isomorphic as base normed spaces, hence $\|R\phi\|_\Delta$ equals $\|\phi\|$ for all elements ϕ in A^*. The question as to when Δ exhausts $\Omega(U(A))$ is treated in [**8**].

THEOREM 6.1. *Let A be a* JBW-*algebra and let $(U(A), \perp, \oplus, 0, 1)$ be the orthoalgebra of idempotents in A.*

Then the convex set of probability charges obtained by restricting the states on A to $U(A)$ has the approximate Jordan-Hahn property.

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Proof. It is an immediate consequence of the spectral theorem that the order-interval [0, 1] of the order unit normed space $(A, A_+, 1)$ coincides with the norm-closure of the convex hull of $U(A)$. The affine map $a \longrightarrow 2a - 1$ is a norm-homeomorphism of A and maps the order-interval [0, 1] bijectively onto the order-interval $[-1, 1]$. Since the set $[-1, 1]$ is the unit ball of A we conclude, by Lemma 3.2, that for every element ϕ in A^*

$$\begin{aligned} \|R\phi\|_\Delta = \|\phi\| &= \sup_{\|a\|=1} \phi(a) \\ &= \sup\{\phi(a) : a \in \mathrm{con}\{p - p' : p \in U(A)\}\} \\ &= \sup_{p \in U(A)} \phi(p - p') = \sup_{p \in U(A)} (R\phi(p) - R\phi(p')) = \|R\phi\|_v. \end{aligned}$$

An allusion to Theorem 4.3 and the previous remarks concludes the proof.

6.2 Locally finite orthoalgebras. A non-zero element p of an orthoalgebra $(L, \perp, \oplus, 0, 1)$ is said to be an *atom* in L if, for elements q, r in L,

$$p = q \oplus r \text{ implies that } q = 0 \text{ or } r = 0.$$

LEMMA 6.2. *Let $(L, \perp, \oplus, 0, 1)$ be a locally finite orthoalgebra. Then:*

(i) *For each non-zero element p in L there exists an atom q and an element r in L such that p equals $q \oplus r$.*

(ii) *For each non-zero element p in L there exists a difference set D consisting of atoms such that $p = \bigoplus D$.*

(iii) *A difference set consisting of atoms is maximal as such if and only if* 1 *equals $\bigoplus D$.*

Proof. (i): Suppose that there exists a non-zero element p in L for which the assertion is false. Then there exists a strictly antitone sequence $(p_i)_{i=1}^\infty$ with p_1 equals p. For each natural number n, the set

$$D_n := \{p_i - p_{i+1} : i = 1, 2, \dots, n\}$$

is a difference set, hence an orthogonal subset. Since $D_{n \in \mathbf{N}}$ is a strictly isotone sequence of subsets of L it follows that $\bigcup_{i=1}^\infty D_n$ is an orthogonal set and that each of its finite subsets is a difference set. Therefore, by Theorem 2.1, $\bigcup_{i=1}^\infty D_n$ is an infinite jointly orthogonal set; a contradiction.

(ii): Let p be a non-zero element of L. By (i) and by a similar argument as in the proof of (i), we find a finite sequence $(q_i)_{i=1}^n$, $n \geqq 1$, of atoms such that

$$p > p - q_1 > (p - q_1) - q_2 > \dots > (\dots((p - q_1) - q_2) \dots - q_n) = 0.$$

Clearly, this so defined strictly isotone sequence yields the difference set $\{q_1, q_2, \dots, q_n\}$. Then $\bigoplus\{q_1, q_2, \dots, q_n\} = p - 0 = p$.

(iii): Let D be a difference set consisting of atoms and suppose that $\bigoplus D < 1$. Then $(\bigoplus D)'$ is different from 0 and therefore, by (i), there exists an atom p such that $p \perp \bigoplus D$. By Theorem 2.2, $D \cup \{p\}$ is a difference set showing that D is not maximal.

Conversely, if D is a difference set consisting of atoms and not maximal as such then there exists a difference set E properly containing D. it follows, by Theorem 2.2, that $\bigoplus D < \bigoplus E \leqq 1$.

Let $(L, \perp, \oplus, 0, 1)$ be a locally finite orthoalgebra. We denote with $A(L)$ the collection of atoms in L and with $O(L)$ the collection of maximal difference sets consisting of atoms. Both sets are not empty and the pair $(A(L), O(L))$ is a hypergraph [**7**], the *atom-hypergraph* of L. Up to isomorphisms, a locally finite orthoalgebra is uniquely determined by its atom-hypergraph [**17**], [**30**], [**31**], [**32**]. From the results of the aforementioned papers an intrinsic characterization may be deduced, namely, of those hypergraphs which are atom-hypergraphs of locally finite orthoalgebras.

LEMMA 6.3. *Let $(L, \perp, \oplus, 0, 1)$ be a locally finite orthoalgebra with atom-hypergraph $(A(L), O(L))$. Then:*

(i) *For every charge μ on L and for all elements E, F in $O(L)$,*

$$\sum_{p\in E} \mu(p) = \sum_{q\in F} \mu(q).$$

(ii) *A functional $\omega : A(L) \longrightarrow \mathbf{R}$ such that*

$$\sum_{p\in E} \omega(p) = \sum_{q\in F} \omega(q) \quad \textit{for all} \quad E, F \in O(L)$$

admits a unique extension to a charge on L.

Proof. (i): This follows from 6.2(iii).

(ii): Let $\omega : A(L) \longrightarrow \mathbf{R}$ be a functional satisfying the condition. Let M, N be difference sets consisting of atoms such that $\bigoplus M$ and $\bigoplus N$ are equal. Let E be an element of $O(L)$ such that $N \subseteq E$. Then

$$1 = \bigoplus E = \bigoplus N \oplus \bigoplus (E \backslash N),$$

thus $\bigoplus M \perp \bigoplus (E\backslash N)$ and it follows that $M \cup (E\backslash N)$ is a difference set. Also $M \cap (E\backslash N)$ is empty and

$$\bigoplus (M \cup (E\backslash N)) = \bigoplus M \oplus \bigoplus (E\backslash N) = \bigoplus M \oplus (\bigoplus M)' = 1.$$

We conclude, by Lemma 6.2(iii), that

$$F := M \cup (E\backslash N) \in O(L).$$

Then in the case that N is different from E,

$$\sum_{p\in E}\omega(p)=\sum_{p\in F}\omega(p)=\sum_{p\in M}\omega(p)+\sum_{p\in E\setminus N}\omega(p)$$
$$=\sum_{q\in E}\omega(q)-\sum_{q\in N}\omega(p)+\sum_{p\in M}\omega(p).$$

It follows that $\sum_{q\in N}\omega(q)$ equals $\sum_{p\in M}\omega(p)$. We define an element μ_ω in $\mathbf{R}^L$ by

$$\mu_\omega(p)=\begin{cases}0 & \text{, if } p=0\\ \sum_{q\in N}\omega(q) & \text{, if } p\neq 0\end{cases}$$

where N is a difference set consisting of atoms such that $\bigoplus N$ equals p, and claim that μ_ω is a charge on L which extends ω. Let p,q be non-zero elements in L with $p\perp q$ and let M,N be difference sets consisting of atoms, such that p equals $\bigoplus M$ and q equals $\bigoplus N$. Then $M\cup N$ is a difference set and $\bigoplus(M\cup N)=\bigoplus M\oplus\bigoplus N=p\oplus q$. Then, since $M\cap N$ is empty,

$$\mu_\omega(p\oplus q)=\sum_{r\in M\cup N}\omega(r)=\sum_{r\in M}\omega(r)+\sum_{r\in N}\omega(r)$$
$$=\mu_\omega(p)+\mu_\omega(q).$$

Finally, $\mu_\omega(p)$ is equal to $\omega(p)$ for all elements p in $A(L)$ since p equals $\bigoplus\{p\}$. Uniqueness of the extension follows from Lemma 6.2(ii).

A locally finite orthoalgebra $(L,\perp,\oplus,0,1)$ is said to satisfy the *outer point condition* (see [**28**]) if for every element E in $O(L)$ there exists an element p such that, for all elements F in $O(L)$,

(1) $\quad p\in F \quad\Leftrightarrow\quad E=F.$

It is a consequence of the following theorem that the convex sets of all probability charges of the locally finite orthoalgebras J_{18}(Janowitz [**26**]) and D_{16}(Dilworth [**11**]) have the Jordan-Hahn property. This theorem generalizes a result by Schindler [**43**].

THEOREM 6.4. *Let $(L,\perp,\oplus,0,1)$ be a locally finite orthoalgebra. If L satisfies the outer point condition then every bounded charge on L is Jordan and the convex set $\Omega(L)$ of probability charges on L has the approximate Jordan-Hahn property.*

Proof. We show that condition (i) of Corollary 4.4 holds true. Let μ be a bounded charge on L such that $\mu(p)\leqq 1$ for all elements p in L. For each element E in $O(L)$ we define a scalar t_E by

$$t_E:=\sum_{p\in E}\mu^+(p);$$

then $0 \leqq t_E \leqq 1$, by hypothesis. Also for each element E in $O(L)$ we select an element p_E which satisfies condition (1) and define a functional $\omega : A(L) \longrightarrow \mathbf{R}$ as follows

$$\omega(p) := \begin{cases} 1 - t_E + \mu^+(p_E) & \text{, if } p = p_E \text{ for some } E \in O(L) \\ \mu^+(p) & \text{, otherwise.} \end{cases}$$

Then $\omega(p) \geqq 0$ and $\mu(p) \leqq \mu^+(p) \leqq \omega(p)$ for all elements p in $A(L)$. Then for every element F in $O(L)$ with more than one element

$$\begin{aligned} \sum_{p \in F} \omega(p) &= \omega(p_F) + \sum_{p \in F \setminus \{p_F\}} \omega(p) \\ &= 1 - t_F + \mu^+(p_F) + \sum_{p \in F \setminus \{p_F\}} \mu^+(p) \\ &= 1 - t_F + \sum_{p \in F} \mu^+(p) = 1 - t_F + t_F = 1. \end{aligned}$$

The same conclusion is reached when F is a singleton set. Let μ_ω be the charge on L which extends the functional ω (see Lemma 6.2) then μ_ω is a positive charge and $\mu(p) \leqq \mu_\omega(p) \leqq 1$ for all elements p in L.

Using Lemma 4.1 and Theorem 4.3 it is easily verified that the "Schindler Fork" Sch_{20} [**43**] and Greechie's G_{32} [**18**] are finite orthomodular orthoalgebras for which the convex sets of all probability charges do not have the Jordan-Hahn property.

6.3 *Boolean Algebras*. Let $(B, \sqcup, \sqcap, \dagger, 0, 1)$ be a Boolean algebra. An element μ in $\mathbf{R}^B$ is said to be a *charge*, resp. *bounded charge*, *positive charge*, *Jordan charge*, *probability charge*, on the Boolean algebra $(B, \sqcup, \sqcap, \dagger, 0, 1)$ if μ is a charge, resp. bounded charge, positive charge, Jordan charge, probability charge on the corresponding orthoalgebra $(B, \perp, \bigoplus, 0, 1)$ (see 2).

Let μ be a bounded charge on B. We claim that μ^+ is a subadditive functional on B, hence a positive charge on B: Let p, q be elements in B such that $p \perp q$. If r is an element of B with $r \leqq p \oplus q$ then r is equal to $(r \sqcap p) \oplus (r \sqcap q)$. Therefore

$$\mu(r) = \mu(r \sqcap p) + \mu(r \sqcap q) \leqq \mu^+(p) + \mu^+(q),$$

hence,

$$\mu^+(p \oplus q) \leqq \mu^+(p) + \mu^+(q)$$

The following classical result (see e.g. [**34**]) is obtained as a corollary to Theorem 4.3.

COROLLARY 6.5. *Let* $(B, \sqcup, \sqcap, \dagger, 0, 1)$ *be a Boolean algebra.*

Then every bounded charge on B is Jordan and the convex set $\Omega(B)$ of probability charges on B has the approximate Jordan-Hahn property.

Proof. Let μ be a bounded charge such that $\mu(p) \leqq 1$ for all elements in p in B. As observed above, μ^+ is a positive charge on B and, clearly, $\mu(p) \leqq \mu^+(p) \leqq 1$ for all elements p in B. The assertion follows from 4.4.

The next result is an immediate consequence of the above and Corollary 5.3.

COROLLARY 6.6. *Let $(B, \sqcup, \sqcap, \dagger, 0, 1)$ be a Boolean algebra and let $\Omega(B)$ be the convex set of probability charges on B.*

Then $\Omega(B)$ has the Jordan-Hahn property if and only if B is finite.

Proof. Let p be a non-zero element of B. Then there exists a two-valued Boolean homomorphism $\mu : B \longrightarrow \{0, 1\}$ such that $\mu(p)$ equals one (see [**22**]). Then clearly, μ is a bounded charge on B and therefore $\Omega(B)$ is unital. If $\Omega(B)$ has the Jordan-Hahn property then, by Corollary 5.3, B is locally finite. The set of atoms of the orthomodular orthoalgebra $(B, \perp, \oplus, 0, 1)$ is orthogonal, hence, jointly orthogonal and therefore finite. Finiteness of B now follows from Lemma 6.2. The converse follows from Corollary 6.5.

REFERENCES

1. E. M. Alfsen, *Compact convex sets and boundary integrals*, (Springer Verlag, Berlin, 1971).
2. E. M. Alfsen and F. W. Shultz, *Non-commutative spectral theory for affine function spaces on convex sets*, Memoirs of the American Mathematical Society *172* (1976).
3. E. M. Alfsen, F. W. Schultz and E. Størmer, *A Gelfand-Neumark theorem for Jordan algebras*, Advances in Mathematics *28* (1978).
4. L. Asimow and A. J. Ellis, *Convexity theory and its applications in functional analysis* (Academic Press, London, 1980).
5. G. Birkhoff, *Lattice theory*, AMS Colloquium Publications *25*, 3rd edition (American Mathematical Society, Providence, 1967).
6. E. G. Beltrametti and G. Cassinelli, *The logic of quantum mechanics*, Encyclopedia of Mathematics and its Applications *15* (Addison-Wesley, Reading, 1981).
7. C. Berge, *Graphs and hypergraphs* (North Holland, New York, 1973).
8. L. J. Bunce and J. D. M. Wright, *Quantum measures and states on Jordan algebras*, Communications in Mathematical Physics *98* (1985).
9. T. A. Cook, *The geometry of generalized quantum logic*, International Journal of Theoretical Physics *17* (1978).
10. T. A. Cook and G. T. Rüttimann, *Symmetries on quantum logics*, Reports on Mathematical Physics *21* (1985).
11. R. P. Dilworth, *On complemented lattices*, Tôhoku Mathematical Journal *47* (1940).
12. J. Diestel, *Geometry of Banach spaces – selected topics*, Lecture Notes in Mathematics *485* (Springer-Verlag, Berlin, 1975).
13. C. M. Edwards and G. T. Rüttimann, *On the facial structure of the unit balls in a GL-space and its dual*, Mathematical Proceedings of the Cambridge Philosophical Society *98* (1985).
14. ——— *On the facial structure of the unit ball of a GM-space*, Mathematische Zeitschrift *193* (1986).
15. A. J. Ellis, *The duality of partially ordered normed linear spaces*, Journal of the London Mathematical Society *39* (1964).
16. D. J. Foulis, *A note on orthomodular lattices*, Portugaliae Mathematica *21* (1962).

17. D. J. Foulis and C. H. Randall, *New definitions and results*, Ditto Notes, University of Massachusetts, Autumn (1979).
18. R. J. Greechie, *Orthomodular lattices admitting no states*, Journal of Combinatorial Theory *4* (1971).
19. S. P. Gudder, *Stochastic methods in quantum mechanics* (North Holland, New York, 1979).
20. S. P. Gudder, M. P. Kläy and G. T. Rüttimann, *States on hypergraphs*, Demonstratio Mathematica *19* (1986).
21. S. P. Gudder, *Quantum probability* (Academic Press, Boston, 1988).
22. P. R. Halmos, *Boolean algebras* (Van Nostrand Company, Princeton, 1963).
23. H. Hanche-Olson and E. Størmer, *Jordan operator algebras* (Pitman, Boston, 1984).
24. R. B. Holmes, *Geometrical functional analysis and its applications* (Springer-Verlag, Berlin, 1975).
25. R. James, *Weak compactness and reflexivity*, Israel Journal of Mathematics *2* (1964).
26. M. F. Janowitz, *Quantifiers on quasi-orthomodular lattices*, Doctoral Thesis, Wayne State University (1963).
27. G. Kalmbach, *Measures and Hilbert lattices* (World Scientific Publishing Company, Singapore, 1986).
28. M. P. Kläy, *Stochastic models and empirical systems, empirical logics and quantum logics, and states on hypergraphs*, Doctoral Thesis, University of Berne (1985).
29. G. Köthe, *Topological vector spaces 1* (Springer-Verlag, 1969).
30. P. F. Lock and G. M. Hardegree, *Connections among quantum logics. Part 1: Quantum propositional logics*, International Journal of Theoretical Physics *24* (1984).
31. ——— *Connections among quantum logics. Part 2: Quantum event logics*, International Journal of Theoretical Physics *24* (1984).
32. C. H. Randall and D. J. Foulis, *Operational statistics and tensor products*, In *Interpretations and foundations of quantum theory* (Bibliographisches Institut, Zürich, 1981).
33. ——— *Stochastic entities*, In *Recent developments in quantum logics* (Bibliographisches Institut, Zürich, 1985).
34. K. P. S. B. Rao and M. B. Rao, *Theory of charges* (Academic Press, New York, 1983).
35. G. T. Rüttimann, *Stable faces of a polytope*, Bulletin of the American Mathematical Society *82* (1976).
36. ——— *Jordan-Hahn decomposition of signed weights on finite orthogonality spaces*, Commentarii Mathematici Helvetici *52* (1977).
37. ——— *Non-commutative measure theory* (Habilitationsschrift, Universität Bern, 1980).
38. ——— *Lecture notes on base normed and order unit normed spaces* (University of Denver, 1981/84).
39. ——— *Facial sets of probability measures*, Probability and Mathematical Statistics *6* (1985).
40. G. T. Rüttimann and Chr. Schindler, *The Lebesgue decomposition of measures on orthomodular posets*, Quarterly Journal of Mathematics Oxford *37* (1986).
41. ——— *On σ-convex sets of probability measures*, Bulletin of the Polish Academy of Science, Mathematics, *35* (1987).
42. G. T. Rüttimann, *Orthoalgebras*. Preprint, University of Berne (1989).
43. Chr. Schindler, *Decomposition of measures on orthologics*, Doctoral Thesis, University of Berne (1986).
44. F. W. Schultz, *On normed Jordan algebras which are Banach space duals*, Journal of Functional Analysis *31* (1979).
45. N. Zierler, *On general measure theory*, Doctoral Thesis, Harvard University (1959).

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