CYCLES AND CONNECTIVITY IN GRAPHS

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1. Introduction. In this note, G will denote a finite undirected graph without multiple edges, and $V = V(G)$ will denote its vertex set. The largest integer n for which G is n-vertex connected is the *vertex-connectivity* of G and will be denoted by $\lambda = \lambda(G)$. One defines ζ to be the largest integer z not exceeding $|V|$ such that for any set $U \subset V$ with $|U| = z$, there is a cycle in G which contains U. The symbol $i(U)$ will denote the component index of U. As a standard reference for this and other terminology, the authors recommend O. Ore **(3)**.

The purpose of this note is to characterize those graphs G for which $\lambda = \zeta \geqslant 2$. Since it is known **(1**, Theorem 9**)** that

$$\text{if } \lambda \geqslant 2, \quad \text{then } \zeta \geqslant \lambda, \tag{1.1}$$

these graphs have a certain "marginal" character. The characterization is obtained in two parts: (1) for $\lambda \geqslant 3$, and (2) for $\lambda = 2$.

THEOREM 1. *Let G be a graph with $\lambda \geqslant 3$. A necessary and sufficient condition that $\zeta = \lambda$ is that there exist a set $S \subset V$ with $|S| = \lambda$ and $i(S) \geqslant \lambda + 1$.*

THEOREM 2. *Let G be a graph with $\lambda = 2$. A necessary and sufficient condition that $\zeta = 2$ is that there exist a set $S \subset V$ such that one of the following three (sets of) conditions holds:*

I. $|S| = 2$ *and* $i(S) \geqslant 3$.

II. (a) $S = \{s^1, s^2, s^3, s\}$.
(b) *Each set* $S^m = \{s^m, s\}$ *separates* G $(m = 1, 2, 3)$.
(c) *Each pair of elements of S is joined by an arc in G having no interior vertex in S.*

III. (a) $S = \{s_n^m : m = 1, 2, 3; n = 1, 2\}$.
(b) *Each set* $S^m = \{s_1^m, s_2^m\}$ *separates* G $(m = 1, 2, 3)$.
(c) *There is an arc in G joining s_n^m to s_q^p with no interior vertex in S if and only if $m = p$ or $n = q$.*

We shall say that G is of Type I, II, or III according as conditions I, II, or III are satisfied. The simplest representations of these three types are shown in Figure 1.

2. Preliminaries. If H is a subgraph of G, written $H \subset G$, then $V(H)$ denotes the vertex set of H. If $U \subset V$, then $G(U)$ denotes the section subgraph of G with vertex set U. Thus $i(U)$ is the number of components in $G(V - U)$.

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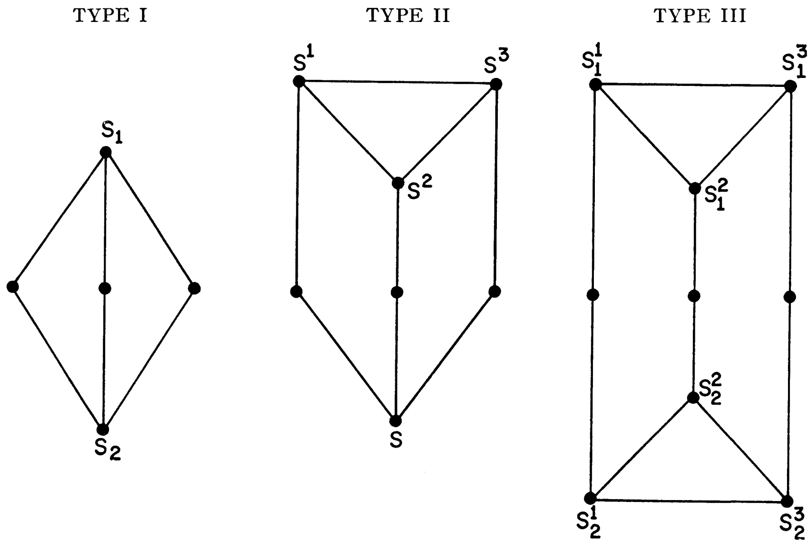

FIGURE 1

To say that $A = A[a, b]$ is an arc, shall mean that a and b are its terminal vertices. If $c, d \in V(A)$, then $A[c, d]$ denotes the subarc of A with terminal vertices c and d. The symbol $A(c, d)$ denotes the arc $A[c, d]$ with the edge incident to c deleted. Analogously we define $A[c, d)$ and $A(c, d)$. To any cycle Z in G, an orientation may be assigned. If $a, b \in V(Z)$, the arc traversed by moving along Z in the positive sense from a to b is denoted by $Z[a, b]$. Its complement in Z is naturally $Z[b, a]$. The above conventions hold for writing $Z(a, b]$, etc.

If $H \subset G$ where $|V(H)| \geqslant n$ and if $a \in V - V(H)$, then a family of arcs $\{A_i[a, b_i] : i = 1, \ldots, n\}$ is said to *radiate from a to H* if $A_i \cap H = \{b_i\}$ and $A_i \cap A_j = \{a\}$ for $i, j = 1, \ldots, n$ and $i \neq j$. Each A_i is said to *meet H* at b_i.

An immediate corollary to a special case of another result by G. A. Dirac (**2**, Theorem I) is stated without proof:

LEMMA 2.1. *Let H be a subgraph of G with* $|V(H)| \geqslant \lambda(G)$. *Let* $a \in V - V(H)$. *Then there is a family of* λ *arcs radiating from a to H.*

3. The sufficiency proofs.

For Theorem 1. Let $S = \{s_1, \ldots, s_\lambda\}$ be a subset of V with $i(S) \geqslant \lambda + 1$, and choose vertices $c_1, \ldots, c_{\lambda+1}$ from distinct components of $G(V - S)$. Any arc $A[c_i, c_j]$ must contain at least one member of S as an interior vertex. But a cycle $Z \subset G$ which contained $c_1, \ldots, c_{\lambda+1}$ would be the union of $\lambda + 1$ such arcs A having no interior vertices in common. Hence, $\zeta \leqslant \lambda$. By (1.1), $\zeta = \lambda$.

For Theorem 2. If G is of Type 1, the proof is precisely that of Theorem I for $\lambda = 2$.

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Let G be of Type II or Type III. Then there exists a component H_m of $G(V - S^m)$ for each $m = 1, 2, 3$ which contains no vertex of S. We assert that

$$H_m \cap H_p = \emptyset, \qquad m \neq p. \tag{3.1}$$

If G is of Type III, let $x \in V(H_m \cap H_p)$. Since $\lambda = 2$, $G(V - \{s_1^m\})$ is connected, so by condition (b) of the theorem, s_2^m is adjacent to some vertex of H_m. There exists, therefore, an arc $A[x, s_2^m]$ such that $A[x, s_2^m) \subset H_m$. Hence, A contains neither vertex of S^p. This is impossible since $x \in V(H_p)$ while $s_2^m \notin V(H_p)$. If G is of Type II, we replace s_1^m by s and replace s_2^m by s^m in the foregoing argument, and (3.1) follows.

Choose vertices $c_m \in V(H_m)$ $(m = 1, 2, 3)$. Thus $c_p \notin V(H_m)$ for $p \neq m$. Any cycle Z through c_1, c_2, c_3 would be the union of three arcs $A_1[c_1, c_2]$, $A_2[c_2, c_3]$, and $A_3[c_3, c_1]$ having no interior vertices in common. Since each arc A_m must begin in H_m and terminate in some H_p, A_m must contain at least one vertex of S^m and at least one vertex of S^p, and hence precisely one vertex of each.

If G is of Type II, then S has but four elements. Hence no such cycle Z exists, and $\zeta \leqslant 2$. If G is of Type III, suppose for definiteness that the vertex of S^1 lying on A_1 is s_2^1. By condition (c) of the theorem, $s_1^2 \notin V(A_1)$. Hence the vertex of S^2 on A_1 is s_2^2. Therefore, $s_1^2 \in V(A_2)$ and by the same argument, $s_1^3 \in V(A_2)$. This leaves s_2^3 and s_1^1 for the arc A_3. But by condition (c), the arc $A_3[s_2^3, s_1^1]$ would require an interior vertex in S. Hence, Z cannot exist, and again $\zeta \leqslant 2$. By (1.1), $\zeta = 2$.

4. The necessity proofs.

LEMMA 4.1. *Let $\lambda(G) \geqslant 2$ and suppose vertices $\{c_1, \ldots, c_\lambda\}$ lie on the cycle $Z \subset G$ but that the set $C = \{c_1, \ldots, c_\lambda, c_{\lambda+1}\}$ lies on no cycle of G. Then* (i) *any largest family F of arcs $\{R_i[c_{\lambda+1}, s_i]\}$ radiating from $c_{\lambda+1}$ to Z contains precisely λ arcs, and* (ii) *any two vertices s_i, s_j are separated in Z by the set $\{c_1, \ldots, c_\lambda\}$.*

Proof. By Lemma 2.1, the required family F of λ arcs exists. Suppose for some $i \neq j$ that $Z[s_i, s_j]$ contains no element of C. Then the cycle $Z[s_j, s_i] \cup R_i \cup R_j$ contains all of C. This proves (ii) and also demonstrates that F cannot have more than λ arcs.

LEMMA 4.2. *Let $\zeta = \lambda \geqslant 2$ and suppose that $C = \{c_1, \ldots, c_{\lambda+1}\}$ lies on no cycle in G. Then to each $c_i \in C$, there corresponds a set of λ vertices*

$$S^i = \{s_1^i, \ldots, s_\lambda^i\}$$

which separates c_i from $C - \{c_i\}$. Moreover, there is a cycle Z^i passing through $S^i \cup (C - \{c_i\})$.

Proof. By symmetry, we may let $i = \lambda + 1$. By (1.1), a cycle Z through $c_1, \ldots, c_\lambda$ exists. Let Z be oriented and let the elements of C be renumbered if necessary so that by proceeding around Z in the positive sense from c_1, one encounters in order $c_1, \ldots, c_\lambda$. Let the family of arcs $\{R_j[c_{\lambda+1}, s_j] : j = 1, \ldots, \lambda\}$ radiate from $c_{\lambda+1}$ to Z. By Lemma 4.1, the elements of $S = \{s_1, \ldots, s_\lambda\}$ may be renumbered so that s_j lies on $Z(c_j, c_{j+1})$ $(j = 0, 1, \ldots, \lambda - 1)$. (Throughout the proofs of the present lemma and of Theorem 1, whenever 0 or -1 appears as a subscript, it is to be read as λ or $\lambda - 1$, respectively. Thus $c_0 \equiv c_\lambda$, etc.)

Certain alterations in the cycle Z will now be made which do not alter the order in which Z passes through $c_1, \ldots, c_\lambda$. If for some $p = 0, \ldots, \lambda - 1$, there exists a vertex $t \neq s_p$ on the arc $Z[c_p, c_{p+1}]$ and an arc $Q[t, s']$ with the three properties: (i) $Q \cap Z = \{t\}$, (ii) $Q \cap R_p(c_{\lambda+1}, s_p) = \{s'\}$, and (iii) $Q \cap R_q = \emptyset$ when $q \neq p$, then, assuming for definiteness that t follows s_p on $Z[c_p, c_{p+1}]$, we alter $Z[c_p, c_{p+1}]$ by replacing $Z[s_p, t]$ by the arc $R_p[s_p, s'] \cup Q$. The arc $R_p[c_{\lambda+1}, s']$ is then considered to be all of R_p, and s' is renamed s_p. We repeat this operation as many times as it is possible, i.e., as long as for some $p = 0, \ldots, \lambda - 1$ there exist a vertex t and an arc Q as described above. Each time this operation is performed, the arc R_p, for some p, is shortened. Hence, after some finite number, possibly zero, of these alterations, the required t and Q will no longer exist. Thus Z contains in order

$$c_1, s_1, c_2, s_2, \ldots, c_\lambda, s_\lambda.$$

To show that S separates $c_{\lambda+1}$ from any other vertex in C, let A be an arc from $c_{\lambda+1}$ to another vertex in C and suppose $V(A) \cap S = \emptyset$. Proceeding along A from $c_{\lambda+1}$, let z be the first vertex of Z encountered and let r be the last vertex of $\bigcup_{i=1}^{\lambda} R_i$ encountered before z. For some $q = 0, \ldots, \lambda - 1$, z must lie on $Z[c_q, c_{q+1}]$. Then r cannot lie on $R_q(c_{\lambda+1}, s_q)$ or else r and $A[r, z]$ would correspond respectively to t and Q above, which is no longer possible. Suppose then that $r \in V(R_p)$ for some $p \neq q$ and, for definiteness, that z follows s_q on $Z[c_q, c_{q+1}]$. (We note that r could be $c_{\lambda+1}$). But then the cycle

$$Z[z, s_q] \cup R_q \cup R_p[c_{\lambda+1}, r] \cup A[r, z]$$

contains $c_1, \ldots, c_{\lambda+1}$, contrary to assumption. This proves the lemma.

Necessity proof for Theorem 1. We continue all of the notation of Lemma 4.2, in the light of which it remains to show merely that, given $i = 1, \ldots, \lambda$, then S separates c_i from the vertex set $\{c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_\lambda\}$. It will follow from this that $c_1, \ldots, c_{\lambda+1}$ necessarily lie in $\lambda + 1$ distinct components of $G(V - S)$.

Arbitrarily choose and then fix $i = 0, \ldots, \lambda - 1$, and define the cycle

$$Z^i = Z[s_i, s_{i-1}] \cup R_{i-1} \cup R_i.$$

(Clearly Z^i excludes c_i.) Consider a family of λ arcs $R_j{}^i[c_i, s_j{}^i]$ $(j = 1, \ldots, \lambda)$ radiating from c_i to Z^i. Such a family exists by Lemma 4.1. Moreover, the elements of $S^i = \{s_1{}^i, \ldots, s_\lambda{}^i\}$ can be named and Z^i can be oriented so that as one proceeds around Z^i in the positive sense from c_1, one encounters in order $c_1, s_1{}^i, \ldots, c_{i-1}, s_{i-1}{}^i, c_{\lambda+1}, s_i{}^i, c_{i+1}, s_{i+1}{}^i, \ldots, c_\lambda, s_\lambda{}^i$. In particular,

$$s_{i-1}{}^i \in V(Z^i(c_{i-1}, c_{\lambda+1})) = V(Z(c_{i-1}, s_{i-1}) \cup R_{i-1}), \tag{4.1}$$

$$s_i{}^i \in V(Z^i(c_{\lambda+1}, c_{i+1})) = V(R_i \cup Z(s_i, c_{i+1})). \tag{4.2}$$

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We remark that

$$R_j^i \cap R_k \subset \{s_k\} \qquad (j, k = 1, \ldots, \lambda), \tag{4.3}$$

for, if (4.3) were false, there would exist a vertex $x \in V(R_j^i \cap R_k)$, $x \neq s_k$, for some j, $k = 1, \ldots, \lambda$. Then the arc $A = R_k[c_{\lambda+1}, x] \cup R_j^i[x, c_i]$ would join $c_{\lambda+1}$ and c_i and contain no vertex of S. But S separates $c_{\lambda+1}$ and c_i by Lemma 4.2. Hence (4.1) and (4.2) can be strengthened to read

$$s_{i-1}^i \in V(Z(c_{i-1}, s_{i-1}]), \tag{4.4}$$

$$s_i^i \in V(Z[s_i, c_{i+1})). \tag{4.5}$$

It is asserted that

$$s_j^i = s_j \qquad (i, j = 0, \ldots, \lambda - 1; j \neq i - 1, i). \tag{4.6}$$

Suppose that (4.6) is false. Then some s_j^i $(j \neq i - 1, i)$ lies either on $Z(c_j, s_j)$ or on $Z(s_j, c_{j+1})$. In the first case, by (4.3) and (4.4),

$$R_j^i \cup Z[s_i, s_j^i] \cup R_i \cup R_j \cup Z[s_j, s_{i-1}^i] \cup R_{i-1}^i$$

is a cycle which contains C. In the second case,

$$R_j^i \cup Z[s_j^i, s_{i-1}] \cup R_{i-1} \cup R_j \cup Z[s_i^i, s_j] \cup R_i^i$$

is such a cycle, by (4.3) and (4.5).

It is in fact true that

$$s_j^i = s_j \qquad (i, j = 1, \ldots, \lambda). \tag{4.7}$$

By symmetry and in the light of (4.6), it suffices to prove that $s_i^i = s_i$ $(i = 1, \ldots, \lambda)$. If this were not true for some i, then proceeding along R_i^i toward c_i, let d be the first vertex of Z encountered after s_i^i. Clearly d lies on $Z(s_{i-1}, s_i)$. If d lies on $Z(s_{i-1}, c_i]$, then $R_i^i[d, s_i^i] \cup Z^i[s_i^i, s_i] \cup Z[d, s_i]$ is a cycle which contains C. Hence,

$$d \in V(R_i^i \cap Z(c_i, s_i)). \tag{4.8}$$

Since $\lambda \geqslant 3$, there is an integer k such that $c_i \neq c_{k-1}, c_k$. Hence by (4.6), $s_i^k = s_i$; that is, $R_i^k = R_i^k[c_k, s_i]$.

Case 1. Suppose that the condition

$$R_i^i[d, s_i^i] \cap R_j^k = \emptyset \tag{4.9}$$

holds for $j = i, k$. Then

$$R_i^i[d, s_i^i] \cup Z[s_i^i, s_{k-1}] \cup R_{k-1} \cup R_i \cup R_i^k \cup R_k^k \cup Z[s_k^k, d]$$

is a cycle, by (4.3) and (4.8), which contains C.

Case 2. Suppose (4.9) holds for $j = k$ but fails for $j = i$. Proceeding along R_i^i from s_i^i, let u be the first vertex of R_i^k encountered. Since u lies on $R_i^i(d, s_i^i)$, C is contained in the cycle

$$R_i \cup R_{k-1} \cup Z[s_i^i, s_{k-1}] \cup R_i^i[s_i^i, u] \cup R_i^k[u, c_k] \cup R_k^k \cup Z[s_k^k, s_i].$$

Case 3. Suppose (4.9) fails for $j = k$. Proceeding along R_i^i from s_i^i, let v be the first vertex of R_k^k encountered. Thus v lies on $R_i^i(d, s_i^i)$, and

$$R_i^k \cup R_k^k[c_k, v] \cup R_i^i[v, s_i^i] \cup Z^k[s_i^i, s_i]$$

is a cycle containing C.

We have now proved (4.7).

Now let B be any arc from a vertex c_i to any one of the vertices $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_\lambda$, where $i = 0, \ldots, \lambda - 1$. Suppose $V(B) \cap S = \emptyset$. Proceeding along B from c_i, let w be the first vertex of Z^i encountered and let e be the last vertex of $\bigcup\{R_j^i : j = 1, \ldots, \lambda\}$ encountered before w. Since S separates c_i and $c_{\lambda+1}$, w must lie on $Z(s_i, s_{i-1})$. In particular, w lies on $Z(c_j, c_{j+1})$ for some $j = 0, \ldots, \lambda - 1$. If e lies on $R_j^i(c_i, s_j)$, consider the family of λ arcs: $P_m = R_m^i$, $m \neq j$; $P_j = R_j^i[c_i, e] \cup B[e, w]$, which radiates from c_i to Z^i. By (4.7), each arc P_m meets Z^i at s_m. In particular, $w = s_j$. If, on the other hand, e lies on some R_p^i for $p \neq j$, assume for definiteness that $s_j \in V(Z(c_j, w))$. The cycle $Z^i[w, s_j] \cup R_j^i \cup R_p^i[c_i, e] \cup B[e, w]$ then contains C. Hence, S separates c_i from the other vertices in C, and the proof of Theorem 1 is complete.

Necessity proof for Theorem 2. Suppose there are vertices c_1, c_2, c_3 which lie on no cycle in G. Henceforth symbols i, j, k will be used to denote some arbitrary rearrangement of the integers 1, 2, 3. By Lemma 4.2, there corresponds to each c_i the pair $S^i = \{s_1^i, s_2^i\}$ which separates c_i from c_j, c_k. Let $S = S^i \cup S^j \cup S^k$. If Z is any cycle through c_i and c_j, then, regardless of its orientation, the arcs $Z(c_i, c_j)$ and $Z(c_j, c_i)$ will be called the *sides* of Z. Using this terminology, Lemma 4.1 states:

(4.10) *Any pair of arcs radiating from c_k to a cycle Z through c_i and c_j must meet Z on opposite sides of Z.*

We show next:

LEMMA 4.3. *If Z is a cycle through c_i, c_j, s_1^k, and s_2^k, then $S \subset V(Z)$, and if either side of Z is traversed from c_i to c_j, the vertices c_i, $s_{p_i}^i$, $s_{p_k}^k$, $s_{p_j}^j$, c_j are encountered in the given order, where $p_i, p_j, p_k \in \{1, 2\}$.*

Proof. Since S^i separates c_i from c_j, the vertices s_1^i, s_2^i belong one to each side of Z. Similarly, the vertices s_1^j, s_2^j lie on opposite sides of Z. Since S^k separates c_k from c_i and c_j, any pair of arcs radiating from c_k to Z must meet Z at s_1^k and s_2^k. By (4.10), s_1^k and s_2^k must then lie on opposite sides of Z. Thus, for either orientation of Z, there are $p_i, p_j, p_k \in \{1, 2\}$ such that

$$V(Z[c_i, c_j]) \cap S = \{s_{p_i}^i, s_{p_j}^j, s_{p_k}^k\},$$

and it remains only to determine their order. Unless $s_{p_i}^i \in V(Z(c_i, s_{p_k}^k])$, then $Z[c_i, s_{p_k}^k]$ together with some arc $P[c_k, s_{p_k}^k]$ joins c_i to c_k without passing through S^i. Similarly we conclude that $s_{p_j}^j$ lies on $Z[s_{p_k}^k, c_j)$. This proves the lemma.

COROLLARY. *Subscripts p_i, p_j, p_k may be assigned so that when Z is given some orientation, the vertices c_i, s_1^i, s_1^k, s_1^j, c_j, s_2^j, s_2^k, s_2^i are encountered in the given order as one proceeds around Z from c_i.*

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Pursuing the argument in the proof of the lemma, it is also immediate that

(4.11) *if* $s_p{}^i = s_p{}^j$, *then* $s_p{}^i = s_p{}^k = s_p{}^j$ $(p = 1, 2)$.

This result will be used to show that if the vertices of S are not all distinct, then G is of either Type I or Type II.

Suppose that $s_n{}^m = s_q{}^p$ for some $m \neq p$ or $n \neq q$. Since $\lambda = 2$, each set S^m contains two (distinct) vertices. By the Corollary, therefore, no generality is lost in assuming that $s_1{}^i = s_1{}^j$. Two cases now arise.

Case 1: $s_2{}^i = s_2{}^j$. By (4.11), $s_1{}^i = s_1{}^j = s_1{}^k \equiv s_1$ and $s_2{}^i = s_2{}^j = s_2{}^k \equiv s_2$. Since $\{s_1, s_2\} = S$ separates each c_i from c_j, c_k, $G(V - S)$ has at least three components, and G is of Type I.

Case 2: $s_2{}^i \neq s_2{}^j$. Again, $s_1{}^i = s_1{}^j = s_1{}^k \equiv s$; but $s_2{}^k \neq s_2{}^i, s_2{}^j$, for otherwise (4.11) with superscripts permuted implies $s_2{}^i = s_2{}^j$. By Lemma 4.2, each set $S^m = \{s, s_2{}^m\}$ separates G, and there is a cycle Z through s, $s_2{}^k$, c_i, and c_j. By Lemma 4.3, Z can be oriented to contain c_i, s, c_j, $s_2{}^j$, $s_2{}^k$, $s_2{}^i$ in the given order. Z contains arcs with no interior vertices in S that join each pair of vertices of S except the pairs s, $s_2{}^k$ and $s_2{}^j$, $s_2{}^i$. These pairs are joined by arcs free of interior vertices in S which are contained in any cycle Z' through s, $s_2{}^j$, c_i, and c_k. Thus G is of Type II.

It remains to prove that if the vertices of S are all distinct, then G is of Type III. It will first be shown that $s_1{}^i$ can be joined to precisely one of $s_1{}^m$, $s_2{}^m$ $(m = j, k)$ by an arc having no interior vertex in S. By symmetry, we assume that $m = j$.

There exists a cycle Z through $s_1{}^j$, $s_2{}^j$, c_i, c_k by Lemma 4.2, and by the Corollary to Lemma 4.3, we may suppose that Z is oriented so that, proceeding around Z from c_i, one encounters in order: c_i, $s_1{}^i$, $s_1{}^j$, $s_1{}^k$, c_k, $s_2{}^k$, $s_2{}^j$, $s_2{}^i$. Let $A = Z[s_1{}^i, s_1{}^j]$, $B = Z[s_2{}^j, s_2{}^i]$, $C = Z[s_1{}^j, s_1{}^k]$, and $D = Z[s_2{}^k, s_2{}^j]$. For any arc $E[s_1{}^j, s_2{}^j]$ through c_j, it is clear that $E \cap Z = S^j$; see Figure 2.

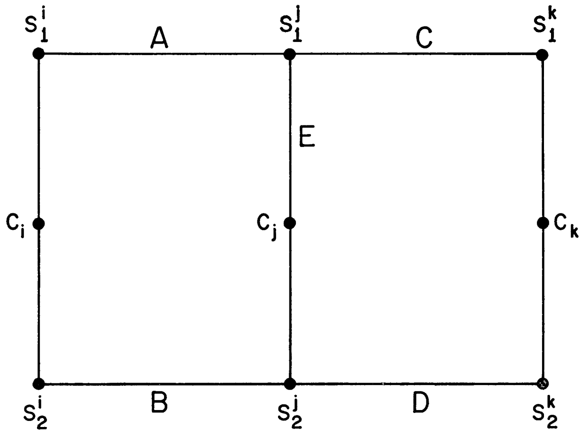

FIGURE 2

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Let $T[x, y]$ be an arc with $x \in V(A \cup C) - \{s_1^j\}$ and $y \in V(B \cup D)$ but not satisfying $x = s_1^p$, $y = s_2^p$, for any $p = 1, 2, 3$. Suppose also that $T \cap (Z \cup E) = \{x, y\}$. If there were an arc $T^*[s_1^i, s_2^j]$ in G free of interior vertices in S, it would necessarily contain a subarc $T[x, y]$ as described. Our procedure, therefore, is to prove that the existence of T leads inevitably to a contradiction. We first disallow

$$x \in V(A[s_1^i, s_1^j)) \qquad y \in V(D[s_2^k, s_2^j)), \tag{4.12}$$

since (4.12) would imply a cycle $E \cup Z[s_2^j, x] \cup T \cup Z[s_1^j, y]$ through c_i, c_j, and c_k. Symmetrically,

$$x \in V(C(s_1^j, s_1^k]), \qquad y \in V(B(s_2^j, s_2^i])$$

is impossible. There remain the two cases

$$x \in V(A[s_1^i, s_1^j)), \qquad y \in V(B(s_2^j, s_2^i])$$

and

$$x \in V(C(s_1^j, s_1^k]), \qquad y \in V(D[s_2^k, s_2^j)). \tag{4.13}$$

These cases are also symmetrical. We shall derive a contradiction from (4.13).

By Lemma 4.2 and the Corollary, there is a cycle Z' which may be oriented to pass through c_i, s_1^k, c_j, s_2^k in the order given. Let the first vertex of $A \cup B$ encountered when one proceeds along Z' from:

s_1^k in the reverse sense be a,
s_1^k in the forward sense be a',
s_2^k in the reverse sense be b',
s_2^k in the forward sense be b.

The existence of these vertices is assured by Lemma 4.3. Thus, on Z' one encounters in order: $c_i, a, s_1^k, a', c_j, b', s_2^k, b$. Let $A' = Z'[a, s_1^k]$, $B' = Z'[s_2^k, b]$, $C' = Z'[s_1^k, a']$, and $D' = Z'[b', s_2^k]$. Define the cycle $Y = Z[s_2^j, s_1^j] \cup E$.

We show that $(B' \cup D') \cap C = \emptyset$. Four pairs of arcs radiating from c_k to the cycle Y are obtained by choosing one arc from

$$A' \cup Z[s_1^k, c_k], \qquad C' \cup Z[s_1^k, c_k] \tag{4.14}$$

and one arc from the pair

$$Z[c_k, s_2^k] \cup B', \qquad Z[c_k, s_2^k] \cup D'. \tag{4.15}$$

By (4.10), the arc chosen from (4.14) must meet Y on the opposite side from the arc chosen from (4.15). Thus, a and a' lie on one side of Y while b and b' lie on the other side. Proceeding along C from s_1^k, let h be the first vertex (after s_1^k) encountered on $B' \cup D'$. For definiteness, say $h \in V(B')$; see

Figure 3 or Figure 4. The pair of arcs $Z[c_k, s_2^k] \cup D'$ and $Z[h, c_k] \cup B'[h, b]$ radiating from c_k meet Y on the same side, contrary to (4.10). Hence,

$$(B' \cup D') \cap C = \emptyset.$$

(In particular, $C[x, s_1^k] \cap (B' \cup D') = \emptyset$.) Since (4.12) is impossible, $(B' \cup D') \cap A = \emptyset$. Hence, $b, b' \in V(B)$.

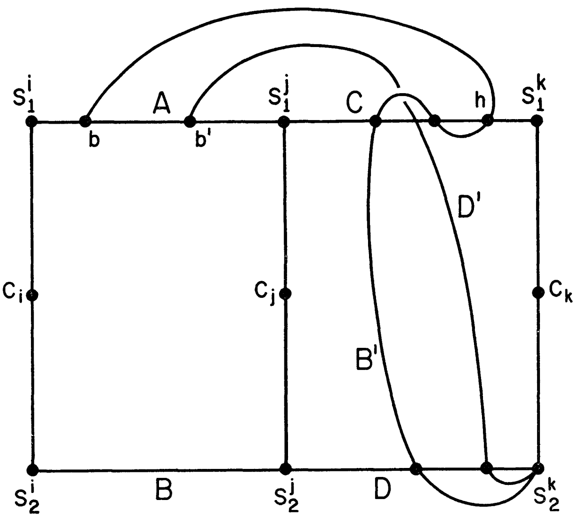

FIGURE 3

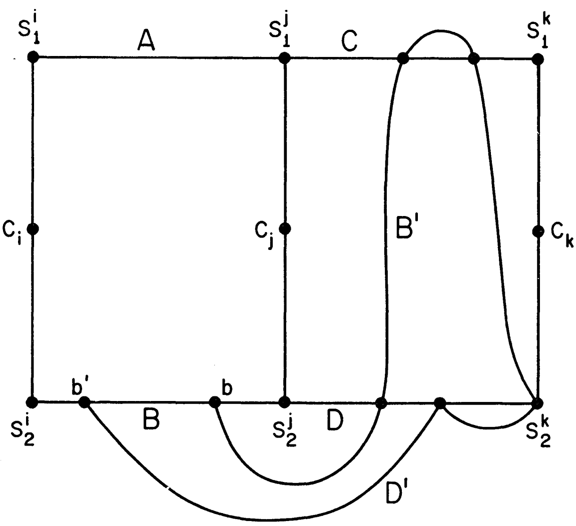

FIGURE 4

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We define a vertex y' as follows. If $T \cap (B' \cup D') \neq \emptyset$, then proceeding along T from x, let y' be the first vertex of $B' \cup D'$ encountered. If

$$T \cap (B' \cup D') = \emptyset,$$

then proceeding along D from y toward $s_2{}^j$, let y' be either the first vertex of $B' \cup D'$ encountered or $s_2{}^j$, whichever comes first, and extend T to include $D[y, y']$. If $y' \neq s_2{}^j$, assume for definiteness that $y' \in V(B')$. Then the two arcs $Z[c_k, s_2{}^k] \cup D'$ and $Z[x, c_k] \cup T[x, y'] \cup B'[y', b]$, radiating from c_k to Y, meet on the same side of Y, contrary to (4.10). If $y' = s_2{}^j$, this latter arc becomes $Z[x, c_k] \cup T[x, s_2{}^j]$. This concludes the proof that an arc $T^*[s_1{}^i, s_2{}^j]$ with no interior vertices in S cannot exist.

We complete the proof that condition III(c) is satisfied by noting that at least one of the arcs

$$A[s_1{}^i, a] \cup A', \qquad A[s_1{}^i, a'] \cup C'$$

joins $s_1{}^i$ to $s_1{}^k$ with no interior vertices in S, and that at least one of the arcs

$$B[s_2{}^i, b] \cup B', \qquad B[s_2{}^i, b'] \cup D'$$

joins $s_2{}^i$ to $s_2{}^k$ with no interior vertices in S. The other required connections are all subarcs of Z.

5. Concluding remarks. In the proof of Lemma 4.2, if for some $i = 0, \ldots, \lambda - 1$ there really were to exist a vertex t and an arc Q as described, then the vertex finally chosen as s_i would not be a vertex of the original cycle Z. Thus the arc $Z[c_i, c_{i+1}]$ of that cycle would contain no vertex in S, contrary to the lemma itself. It follows then that vertices t and arcs Q never did exist, that $s_1, \ldots, s_\lambda$ never had to be renamed, and that set S was uniquely determined by the original cycle Z. Moreover, Z was an arbitrary cycle excluding precisely one of the vertices $c_1, \ldots, c_{\lambda+1}$. The conclusion is that, *given a set* $\{c_1, \ldots, c_{\lambda+1}\}$ *of* $\lambda + 1$ *vertices in no cycle of* G, *there is a unique subset* $S \subset V$ *as described in Theorem* 1 *or Theorem* 2 *such that* $c_1, \ldots, c_{\lambda+1}$ *lie in distinct components of* $G(V - S)$. The same set S is determined by any set $\{c_1', \ldots, c_{\lambda+1}'\}$ where c'_i and c_i lie in the same component of $G(V - S)$.

In conclusion, we pose a question. Given a non-negative integer n, which graphs G have the property $\zeta = \lambda(G) + n$? For $n = 0$, the answer has been given in this paper. At the other extreme, when $n = |V| - \lambda$, is the problem of characterizing the Hamiltonian graphs of connectivity λ.

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