

POTENTIAL THEORETIC PROPERTIES OF THE GRADIENT OF A CONVEX FUNCTION ON A FUNCTIONAL SPACE

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Introduction

In the previous paper [11], introducing the notions of potentials and of capacity associated with a convex function Φ given on a regular functional space $\mathfrak{X}(X; \xi)$, we discussed potential theoretic properties of the gradient $\nabla\Phi$ which were originally introduced and studied by Calvert [5] for a class of nonlinear monotone operators in Sobolev spaces. For example:

- (i) The modulus contraction operates.
- (ii) The principle of lower envelope holds.
- (iii) The domination principle holds.
- (iv) The contraction T_k onto the real interval $[0, k]$ ($k > 0$) operates.
- (v) The strong principle of lower envelope holds.
- (vi) The complete maximum principle holds.

In this paper we shall investigate relations among the properties mentioned above. For this purpose, we shall consider an operator A from a subset of $L^2(X; \xi) \cap \mathfrak{X}(X; \xi)$ into $L^2(X; \xi)$ associated with $\nabla\Phi$ and its resolvent $R_\lambda = (I + \lambda A)^{-1}$, $\lambda > 0$.

One aim is to show (in Theorem A) that each of properties (i) ~ (iii) is equivalent to:

- (vii) for any $\lambda > 0$, R_λ is order-preserving in $L^2(X; \xi)$. Another aim is to show (in Theorem B) that if (i) is satisfied, then each of properties (iv) ~ (vi) holds if and only if

- (viii) $R_\lambda(f + T_k g) \leq R_\lambda f + k$ holds for any $\lambda > 0$ and any $f, g \in L^2(X; \xi)$.

These assertions are nonlinear analogues of results in the Dirichlet space (cf. Deny [7; Théorèmes 1 et 2] and Itô [8; Theorems 3 and 5]). The crucial step in the proofs is to deduce both (i) and (iv) from (vii)

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and (viii). The key tool is the fact that if $f \in L^2(X; \xi)$ and if

$$\left\{ \int_X \lambda^{-1}(f - R_\lambda f) f d\xi; \lambda > 0 \right\}$$

is bounded, then $f \in \mathfrak{X}(X; \xi)$. This fact is a nonlinear extension of a result in [7]. At the end of this paper we shall consider the nonlinear contraction semigroup $S = \{S(t); t > 0\}$ on $L^2(X; \xi)$ generated by $-A$ and show (in Theorem C) that each of properties (i) ~ (iii) and (vii) is necessary and sufficient for S to be order-preserving in $L^2(X; \xi)$.

§ 1. Preliminaries

Let X be a locally compact Hausdorff space with a countable base and ξ be a positive (Radon) measure on X . Let $\mathfrak{X} = \mathfrak{X}(X; \xi)$ be a real reflexive Banach space whose elements are real-valued locally ξ -summable functions defined ξ -a.e. on X . We denote by \mathfrak{X}^* the dual space of \mathfrak{X} , by $\|u\|$ (resp. $\|u^*\|$) the norm of $u \in \mathfrak{X}$ (resp. $u^* \in \mathfrak{X}^*$) and by $\langle u^*, u \rangle$ the value of $u^* \in \mathfrak{X}^*$ at $u \in \mathfrak{X}$. By “ \xrightarrow{s} ” (resp. “ \xrightarrow{w} ”) we mean the strong (resp. weak) convergence. For functions $u, v \in L^1_{loc}(X; \xi)$, we write $u \vee v$, $u \wedge v$, u^+ and u^- for $\max(u, v)$, $\min(u, v)$, $\max(u, 0)$ and $-\min(u, 0)$, respectively, and for a ξ -measurable set S in X we write simply “ $u \geq v$ (resp. $u = v$) on S ” for “ $u \geq v$ (resp. $u = v$) ξ -a.e. on S ”. Especially we write “ $u \geq v$ (resp. $u = v$)” for “ $u \geq v$ (resp. $u = v$) on X ”.

Throughout this paper, let $1 < p < \infty$ and Φ be a strictly convex function on \mathfrak{X} such that

$$(1.1) \quad \begin{cases} \Phi(0) = 0, \\ \Phi(u) \geq C \|u\|^p, \quad \text{for any } u \in \mathfrak{X}, \end{cases}$$

where C is a positive constant. Suppose that Φ is bounded on each bounded subset of \mathfrak{X} and is everywhere differentiable in the sense of Gâteaux, that is, there is an operator $G: \mathfrak{X} \rightarrow \mathfrak{X}^*$ such that for any $u, v \in \mathfrak{X}$,

$$\langle Gu, v \rangle = \lim_{t \downarrow 0} \frac{\Phi(u + tv) - \Phi(u)}{t}.$$

This operator G is called the gradient of Φ and denoted by $\nabla\Phi$. We shall use the following properties of Φ and $\nabla\Phi$ without proof:

- (Φ_1) Φ is weakly sequentially lower semicontinuous in \mathfrak{X} .
- (Φ_2) Let $u \in \mathfrak{X}$ and $u^* \in \mathfrak{X}^*$. Then $u^* = \nabla\Phi(u)$ if and only if

$$\langle u^*, v - u \rangle \leq \Phi(v) - \Phi(u) \quad \text{for any } v \in \mathfrak{X} .$$

(Φ_1) $\nabla\Phi$ is strictly monotone, i.e.,

$$\langle \nabla\Phi(u) - \nabla\Phi(v), u - v \rangle > 0 \quad \text{for any } u, v \in \mathfrak{X}, u \neq v .$$

(Φ_4) $\nabla\Phi$ is bounded, i.e., it maps bounded sets in \mathfrak{X} to bounded sets in \mathfrak{X}^* .

(Φ_5) $\nabla\Phi$ is demicontinuous, i.e., if $u_n \xrightarrow{s} u$ in \mathfrak{X} as $n \rightarrow \infty$, then $\nabla\Phi(u_n) \xrightarrow{w} \nabla\Phi(u)$ in \mathfrak{X}^* as $n \rightarrow \infty$.

(Φ_6) For each $u \in \mathfrak{X}$, $\langle \nabla\Phi(v), v - u \rangle / \|v\| \rightarrow \infty$ as $\|v\| \rightarrow \infty$.

(Φ_7) For any $u, v \in \mathfrak{X}$,

$$\Phi(u) - \Phi(v) = \int_0^1 \langle \nabla\Phi(v + t(u - v)), u - v \rangle dt .$$

Remark. $\nabla\Phi$ is one to one and onto. For a proof, see Browder [2; Theorem 3].

LEMMA 1.1. Let $\{u_n\}$ be a sequence in \mathfrak{X} which converges weakly to u in \mathfrak{X} . If $\limsup_{n \rightarrow \infty} \langle \nabla\Phi(u_n), u_n - u \rangle \leq 0$, then $\nabla\Phi(u_n) \xrightarrow{w} \nabla\Phi(u)$ in \mathfrak{X}^* and $\Phi(u_n) \rightarrow \Phi(u)$ as $n \rightarrow \infty$.

Proof. From (Φ_2) and our assumption it follows that

$$0 \geq \limsup_{n \rightarrow \infty} \langle \nabla\Phi(u_n), u_n - u \rangle \geq \limsup_{n \rightarrow \infty} \Phi(u_n) - \Phi(u) .$$

On account of (Φ_1) we obtain

$$(1.2) \quad \lim_{n \rightarrow \infty} \Phi(u_n) = \Phi(u) .$$

Next, by (Φ_3), $\liminf_{n \rightarrow \infty} \langle \nabla\Phi(u_n), u_n - u \rangle \geq \lim_{n \rightarrow \infty} \langle \nabla\Phi(u), u_n - u \rangle = 0$. Hence

$$(1.3) \quad \lim_{n \rightarrow \infty} \langle \nabla\Phi(u_n), u_n - u \rangle = 0 .$$

The sequence $\{\nabla\Phi(u_n)\}$ is weakly relatively compact in \mathfrak{X}^* , since it is bounded in \mathfrak{X}^* on account of (Φ_4). Now, let $\{u_{n_j}\}$ be any subsequence of $\{u_n\}$ such that $\nabla\Phi(u_{n_j}) \xrightarrow{w} u^*$ in \mathfrak{X}^* as $j \rightarrow \infty$ for some $u^* \in \mathfrak{X}^*$. Then, using (Φ_2), (1.2) and (1.3), we see that for any $v \in \mathfrak{X}$

$$\begin{aligned}\langle u^*, v - u \rangle &= \lim_{j \rightarrow \infty} \langle \nabla \Phi(u_{n_j}), v - u_{n_j} \rangle \\ &\leq \lim_{j \rightarrow \infty} \{ \Phi(v) - \Phi(u_{n_j}) \} \\ &= \Phi(v) - \Phi(u),\end{aligned}$$

which implies that $u^* = \nabla \Phi(u)$, and simultaneously that $\nabla \Phi(u_{n_j}) \xrightarrow{w} \nabla \Phi(u)$ in \mathfrak{X}^* as $n \rightarrow \infty$. q.e.d.

DEFINITION 1.1. (cf. [7], [8]) $\mathfrak{X} = \mathfrak{X}(X; \xi)$ is called a functional space if the following axiom is satisfied:

AXIOM (a) For each compact subset K of X , there is a constant $M(K) > 0$ such that

$$\int_K |u| d\xi \leq M(K) \|u\| \quad \text{for all } u \in \mathfrak{X}.$$

Henceforth let \mathfrak{X} be a functional space satisfying the following axiom:

AXIOM (b') $L^2 \cap \mathfrak{X}$ is dense both in L^2 and in \mathfrak{X} , where $L^2 = L^2(X; \xi)$.

We consider an operator A from a subset of L^2 into L^2 associated with $\nabla \Phi$ which is defined as follows: If v is a function in $L^2 \cap \mathfrak{X}$ and f is a function in L^2 such that

$$(1.4) \quad \langle \nabla \Phi(v), w \rangle = \int_X w f d\xi \quad \text{for any } w \in L^2 \cap \mathfrak{X},$$

then we put $Av = f$. By Axiom (b'), such an f is uniquely determined by v . Thus we can define an operator $A: D_0 \rightarrow L^2$, where $D_0 = \{v \in L^2 \cap \mathfrak{X}; \text{ there is a function } f \in L^2 \text{ such that (1.4) holds}\}$. By (Φ_3) , A is strictly monotone, i.e.,

$$(1.5) \quad (Au - Av, u - v) > 0 \quad \text{for any } u, v \in D_0, u \neq v,$$

where (\cdot, \cdot) denotes the inner product in L^2 , i.e., $(v, w) = \int_X v w d\xi$ for any $v, w \in L^2$.

§ 2. The resolvent of A

In order to show that the resolvent of A exists, we prove

LEMMA 2.1. *Given $\lambda > 0$ and $f \in L^2$, we find a unique function $u \in L^2 \cap \mathfrak{X}$ such that*

$$(2.1) \quad 2\lambda\Phi(u) + \|u - f\|_2^2 = \inf \{2\lambda\Phi(v) + \|v - f\|_2^2; v \in L^2 \cap \mathfrak{X}\},$$

where by $\|v\|_2$ we mean the norm of v in L^2 .

Proof. Set $F(v) = 2\lambda\Phi(v) + \|v - f\|_2^2$ for $v \in L^2 \cap \mathfrak{X}$ and $\alpha = \inf \{F(v); v \in L^2 \cap \mathfrak{X}\}$. Then, clearly, $0 \leq \alpha < \infty$. We find a sequence $\{v_n\} \subset L^2 \cap \mathfrak{X}$ such that $F(v_n) \downarrow \alpha$ as $n \rightarrow \infty$. Since $\{v_n\}$ is bounded in \mathfrak{X} by (1.1) as well as L^2 , it is weakly relatively compact both in \mathfrak{X} and in L^2 and hence, by Axiom (a), there exists a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that $v_{n_j} \xrightarrow{w} u$ both in \mathfrak{X} and in L^2 for some $u \in L^2 \cap \mathfrak{X}$. Noting (Φ_1) and the weak lower semicontinuity of the functional: $v \rightarrow \|v - f\|_2^2$ relative to the topology of L^2 , we have

$$\Phi(u) \leq \liminf_{j \rightarrow \infty} \Phi(v_{n_j})$$

and

$$\|u - f\|_2^2 \leq \liminf_{j \rightarrow \infty} \|v_{n_j} - f\|_2^2,$$

so that $F(u) \leq \lim_{j \rightarrow \infty} F(v_{n_j}) = \alpha$. This implies that $F(u) = \alpha$. The uniqueness of such a u follows from the strict convexity of Φ . q.e.d.

For any $\lambda > 0$, we denote by R_λ the operator from L^2 into L^2 which assigns the function u satisfying (2.1) to each $f \in L^2$.

LEMMA 2.2. *Let λ be any positive number. Then:*

- (i) $\Phi(R_\lambda f) \leq \Phi(f)$ for any $f \in L^2 \cap \mathfrak{X}$.
- (ii) $\langle \nabla \Phi(R_\lambda f), v \rangle = (f - R_\lambda f, v) / \lambda$ for any $f \in L^2$ and $v \in L^2 \cap \mathfrak{X}$.
- (iii) $R_\lambda f \in D_0$ and $AR_\lambda f = (f - R_\lambda f) / \lambda$ for any $f \in L^2$.
- (iv) $R(I + \lambda A)$ (the range of $I + \lambda A$) = L^2 and $R_\lambda = (I + \lambda A)^{-1}$.
- (v) $\|R_\lambda f - R_\lambda g\|_2 \leq \|f - g\|_2$ for any $f, g \in L^2$, especially $\|R_\lambda f\|_2 \leq \|f\|_2$ for any $f \in L^2$.

Proof. (i) is clear. For any $f \in L^2$, $v \in L^2 \cap \mathfrak{X}$ and $t > 0$, setting $u = R_\lambda f$, we observe that

$$2\lambda\Phi(u + tv) + \|u + tv - f\|_2^2 \geq 2\lambda\Phi(u) + \|u - f\|_2^2,$$

that is

$$\frac{1}{t} \{\Phi(u + tv) - \Phi(u)\} \geq \frac{1}{2\lambda} \frac{1}{t} \{\|u - f\|_2^2 - \|u + tv - f\|_2^2\}.$$

Letting $t \downarrow 0$ in this inequality, we obtain

$$\langle \nabla \Phi(u), v \rangle \geq \frac{1}{\lambda}(f - u, v).$$

Hence we have (ii) and at the same time (iii). Besides, we see that

$$f = R_\lambda f + \lambda A R_\lambda f = (I + \lambda A)R_\lambda f \quad \text{for any } f \in L^2.$$

From this we infer (iv), since $(I + \lambda A)$ is one to one by (1.5). Finally we shall show (v). Let f and g be any functions in L^2 , and set $u = R_\lambda f$ and $v = R_\lambda g$. Then, as was seen above, $f = u + \lambda Au$ and $g = v + \lambda Av$. From (1.5) it follows that

$$\begin{aligned} \|f - g\|_2 \|u - v\|_2 &\geq (f - g, u - v) \\ &= (u + \lambda Au - v - \lambda Av, u - v) \\ &= \|u - v\|_2^2 + \lambda(Au - Av, u - v) \\ &\geq \|u - v\|_2^2, \end{aligned}$$

so we have (v).

q.e.d.

The fact (iv) of Lemma 2.2 says that R_λ is the (nonlinear) resolvent (at λ) of A for each $\lambda > 0$.

LEMMA 2.3 (cf. [9; Lemma 4.3]). *If $v \in D_0$, then $\|AR_\lambda v\|_2 \leq \|Av\|_2$ for any $\lambda > 0$.*

Proof. We have $v = R_\lambda(I + \lambda A)v = R_\lambda(v + \lambda Av)$, since $R_\lambda = (I + \lambda A)^{-1}$ by (iv) of Lemma 2.2. From (iii) and (v) of Lemma 2.2 it follows that

$$\|AR_\lambda v\|_2 = \frac{1}{\lambda}\|v - R_\lambda v\|_2 = \frac{1}{\lambda}\|R_\lambda(v + \lambda Av) - R_\lambda v\|_2 \leq \|Av\|_2. \quad \text{q.e.d.}$$

LEMMA 2.4. (1) *For any $f \in L^2$, $R_\lambda f \xrightarrow{s} f$ in L^2 as $\lambda \downarrow 0$.*

(2) *For any $f \in L^2 \cap \mathfrak{X}$, $R_\lambda f \xrightarrow{w} f$ in \mathfrak{X} as $\lambda \downarrow 0$.*

Proof. First, let f be any function in $L^2 \cap \mathfrak{X}$. We observe from (1.1) and (i), (v) of Lemma 2.2 that

$$C\|R_\lambda f\|^p \leq \Phi(R_\lambda f) \leq \Phi(f) \quad \text{for any } \lambda > 0$$

and

$$(2.2) \quad \|R_\lambda f\|_2 \leq \|f\|_2 \quad \text{for any } \lambda > 0.$$

Therefore $\{R_\lambda f; \lambda > 0\}$ is bounded in \mathfrak{X} as well as in L^2 . From (ii) of Lemma 2.2 and (Φ_λ) we derive that for each $v \in L^2 \cap \mathfrak{X}$

$$(f - R_\lambda f, v) = \lambda \langle \nabla \Phi(R_\lambda f), v \rangle \rightarrow 0 \quad \text{as } \lambda \downarrow 0 .$$

This fact and the boundedness of $\{R_\lambda f; \lambda > 0\}$ in L^2 imply that

$$(2.3) \quad R_\lambda f \xrightarrow{w} f \quad \text{in } L^2 \text{ as } \lambda \downarrow 0$$

and $\liminf_{\lambda \downarrow 0} \|R_\lambda f\|_2 \geq \|f\|_2$. On the other hand, we have by (2.2) $\limsup_{\lambda \downarrow 0} \|R_\lambda f\|_2 \leq \|f\|_2$. Hence, $\lim_{\lambda \downarrow 0} \|R_\lambda f\|_2 = \|f\|_2$. This together with (2.3) implies that

$$(2.4) \quad R_\lambda f \xrightarrow{s} f \quad \text{in } L^2 \text{ as } \lambda \downarrow 0 .$$

Now, let $\{\lambda_j\}$ be any sequence tending to 0 such that $\{R_{\lambda_j} f\}$ is weakly convergent in \mathfrak{X} . Then, denoting the weak limit by g , we see from (2.4) and Axiom (a) that $g = f$. This shows that $R_\lambda f \xrightarrow{w} f$ in \mathfrak{X} as $\lambda \downarrow 0$. Finally we can obtain (1) by using Axiom (b') and (v) of Lemma 2.2 and noting the fact that $R_\lambda f \xrightarrow{s} f$ in L^2 as $\lambda \downarrow 0$ for $f \in L^2 \cap \mathfrak{X}$.

q.e.d.

Remark. By the above lemma, D_0 is dense in L^2 . Also, we can prove that D_0 is weakly sequentially dense in \mathfrak{X} .

LEMMA 2.5. *If $f \in L^2 \cap \mathfrak{X}$, then $\nabla \Phi(R_\lambda f) \xrightarrow{w} \nabla \Phi(f)$ in \mathfrak{X}^* and $\Phi(R_\lambda f) \rightarrow \Phi(f)$ as $\lambda \downarrow 0$.*

Proof. Let $f \in L^2 \cap \mathfrak{X}$. Then we see from (ii) of Lemma 2.2 that

$$\limsup_{\lambda \downarrow 0} \langle \nabla \Phi(R_\lambda f), R_\lambda f - f \rangle = \limsup_{\lambda \downarrow 0} - \frac{1}{\lambda} \|f - R_\lambda f\|_2^2 \leq 0 .$$

Applying Lemma 1.1, we obtain the lemma.

q.e.d.

Following Deny (cf. [7; Théorème 2]), we define

$$H_\lambda(f) = \frac{1}{\lambda} (f - R_\lambda f, f) , \quad f \in L^2 .$$

We note that the following relations hold:

$$(2.5) \quad H_\lambda(f) \geq \frac{1}{\lambda} (f - R_\lambda f, R_\lambda f) = \langle \nabla \Phi(R_\lambda f), R_\lambda f \rangle \geq \Phi(R_\lambda f) .$$

LEMMA 2.6. *If $f \in L^2$ and if $\{H_\lambda(f); \lambda > 0\}$ is bounded, then $f \in \mathfrak{X}$.*

Proof. Let $f \in L^2$ and assume that $\{H_\lambda(f); \lambda > 0\}$ is bounded. Then

we see from (2.5) and (1.1) that $\{R_\lambda f; \lambda > 0\}$ is bounded in \mathfrak{X} . Hence there is a sequence $\{\lambda_j\}$ tending to 0 such that $R_{\lambda_j} f \xrightarrow{w} g$ in \mathfrak{X} for some $g \in \mathfrak{X}$. By (1) of Lemma 2.4 and Axiom (a), we have $g = f$, so that $f \in \mathfrak{X}$. q.e.d.

§ 3. Potential theoretic properties and their equivalence

In this section, we state potential theoretic properties of Φ , A and R_λ and our main results about their equivalence.

DEFINITION 3.1 (cf. [5], [11]). Let k be a positive number or ∞ . We say that the contraction T_k operates in \mathfrak{X} with respect to Φ if the following two conditions are satisfied:

- (C_k) $T_k v \equiv v^+ \wedge k \in \mathfrak{X}$ for all $v \in \mathfrak{X}$, where $T_k v = v^+$ if $k = \infty$.
- (ΦC_k) For any $u, v \in \mathfrak{X}$,

$$\Phi(u + T_k(v - u)) + \Phi(v - T_k(v - u)) \leq \Phi(u) + \Phi(v).$$

In particular, we say that the modulus contraction operates in \mathfrak{X} with respect to Φ , when T_∞ operates in \mathfrak{X} with respect to Φ .

Remark. It was shown in [11; Proposition 2.1] that under (C_k), condition (ΦC_k) is equivalent to the following:

$$(\Phi C_k)' \quad \langle \nabla \Phi(u + T_k v) - \nabla \Phi(u), v - T_k v \rangle \geq 0 \quad \text{for any } u, v \in \mathfrak{X}.$$

In particular, (ΦC_∞)' is of the form:

$$\langle \nabla \Phi(u + v^+) - \nabla \Phi(u), v^- \rangle \leq 0 \quad \text{for any } u, v \in \mathfrak{X}.$$

DEFINITION 3.2 (cf. [5], [11]). We say that the principle of lower envelope with respect to Φ holds if (C_∞) and the following are satisfied:

- (ΦL) If u and v are functions in D_0 , then

$$\langle \nabla \Phi(u \wedge v), w \rangle \geq \langle \nabla \Phi(u \wedge v), w \rangle$$

for any non-negative function $w \in L^2 \cap \mathfrak{X}$.

DEFINITION 3.3 (cf. [5], [11]). We say that the domination principle with respect to Φ holds, if the following is satisfied:

- (ΦD) If u and v are functions in D_0 and if there is $f \in L^2$ such that $Au \geq f$, $Av \geq f$ and $(Au - f, (u - v)^+) = 0$, then $u \leq v$.

DEFINITION 3.4 (cf. [5], [11]). We say that R_λ , $\lambda > 0$, is order-preserving in L^2 , if the following holds:

- (ΦO) $R_\lambda f \leq R_\lambda g$ for any $f, g \in L^2$ such that $f \leq g$.

Now, we are in a position to state one of main theorems.

THEOREM A. *The following statements are equivalent to each other:*

- (a₁) *The modulus contraction operates in \mathfrak{X} with respect to Φ .*
- (a₂) *The principle of lower envelope with respect to Φ holds.*
- (a₃) *The domination principle with respect to Φ holds.*
- (a₄) *For any $\lambda > 0$, the resolvent R_λ is order-preserving in L^2 .*

Next, to state another main theorem we give some definitions. Throughout the remainder of this section, let k be a positive number.

DEFINITION 3.5 (cf. [5], [11]). We say that the strong principle of lower envelope with respect to Φ and k holds, if the following two conditions are satisfied:

- (C_k)^{*} $u \wedge (v + k) \in \mathfrak{X}$ for any $u, v \in \mathfrak{X}$.
- (ΦSL_k) If u and v are functions in D_0 , then

$$\langle \nabla \Phi(u \wedge (v + k)), w \rangle \geq (Au \wedge Av, w)$$

for any non-negative function $w \in L^2 \cap \mathfrak{X}$.

It should be noticed that under (C_∞), conditions (C_k)^{*} and (C_k) are equivalent.

DEFINITION 3.6 (cf. [5], [11]). We say that the complete maximum principle with respect to Φ and k holds, if the following is satisfied:

(ΦCM_k) If u and v are functions in D_0 and if there is $f \in L^2$ such that $Au \geq f$, $Av \geq f$ and $(Au - f, (u - v - k)^+) = 0$, then $u \leq v + k$.

The second main theorem is stated as follows:

THEOREM B. *Assume that the modulus contraction operates in \mathfrak{X} with respect to Φ . Then the following statements are equivalent to each other:*

- (b₁) *The contraction T_k operates in \mathfrak{X} with respect to Φ .*
- (b₂) *The strong principle of lower envelope with respect to Φ and k holds.*
- (b₃) *The complete maximum principle with respect to Φ and k holds.*
- (b₄) *For any $\lambda > 0$ and any $f, g \in L^2$*

$$R_\lambda(f + T_k g) \leq R_\lambda f + k .$$

§ 4. Proofs of Theorems A and B

Before proving the theorems, we recall an existence theorem for

nonlinear variational inequalities. Let \mathfrak{Y} be a real reflexive Banach space and P be a (nonlinear) semicontinuous monotone¹⁾ operator from \mathfrak{Y} into \mathfrak{Y}^* (the dual space of \mathfrak{Y}). Let Ψ be a lower semicontinuous convex function on \mathfrak{Y} with values in $(-\infty, \infty]$.

THEOREM (cf. [3; Theorem 3], [10; Theorem 4.1]). *Let \mathfrak{R} be a non-empty closed convex subset of \mathfrak{Y} and assume that for some $w \in \mathfrak{R}$ with $\Psi(w) < \infty$,*

$$\frac{\langle Pv, v - w \rangle_{\mathfrak{Y}} + \Psi(v)}{\|v\|_{\mathfrak{Y}}} \rightarrow \infty \quad \text{as } \|v\|_{\mathfrak{Y}} \rightarrow \infty, \quad v \in \mathfrak{R},$$

where we denote by $\langle \cdot, \cdot \rangle_{\mathfrak{Y}}$ the natural pairing between \mathfrak{Y}^* and \mathfrak{Y} and by $\|\cdot\|_{\mathfrak{Y}}$ the norm in \mathfrak{Y} . Then, there is $u \in \mathfrak{R}$ such that

$$\langle Pu, u - v \rangle_{\mathfrak{Y}} \leq \Psi(v) - \Psi(u) \quad \text{for all } v \in \mathfrak{R}.$$

Moreover, if P is strictly monotone, then such a u is unique.

LEMMA 4.1. *The function Ψ on L^2 defined by*

$$\Psi(v) = \begin{cases} \Phi(v), & \text{if } v \in L^2 \cap \mathfrak{X}, \\ \infty, & \text{otherwise,} \end{cases}$$

is lower semicontinuous on L^2 .

Proof. Let $\{v_n\}$ be a sequence in L^2 which converges to v_0 in L^2 , and assume that $\alpha \equiv \liminf_{n \rightarrow \infty} \Psi(v_n) < \infty$. Then, by (1.1) there is a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that $v_{n_j} \xrightarrow{w} v$ in \mathfrak{X} for some $v \in \mathfrak{X}$ and $\Phi(v_{n_j}) \rightarrow \alpha$ as $j \rightarrow \infty$. By Axiom (a), $v_0 = v \in \mathfrak{X}$. Hence, from (Φ_1) it follows that $\alpha \geq \Phi(v_0) = \Psi(v_0)$. Thus Ψ is lower semicontinuous on L^2 .
q.e.d.

Proof of $(a_1) \rightarrow (a_2)$: Let $u \in D_0$ and $v \in D_0$, and set $f = Au$ and $g = Av$. Define an operator $P: L^2 \rightarrow L^2$ by $Pw = w - u \wedge v - f \wedge g$ and let Ψ be the same function as in Lemma 4.1. Obviously, P is a demicontinuous monotone operator from L^2 into L^2 and Ψ is a convex function on L^2 with values in $[0, \infty]$ such that $\{(Pw, w - u) + \Psi(w)\} / \|w\|_2 \rightarrow \infty$ as $\|w\|_2 \rightarrow \infty$. By Lemma 4.1, Ψ is lower semicontinuous on L^2 . Furthermore, setting $\mathfrak{R} = \{w \in L^2; w \geq u \wedge v\}$, we see that \mathfrak{R} is closed and convex in L^2 , $u \in \mathfrak{R}$ and $\Psi(u) = \Phi(u) < \infty$. By virtue of the existence theorem

1) An operator P from \mathfrak{Y} into \mathfrak{Y}^* is called monotone, if for any $u, v \in \mathfrak{Y}$, $\langle Pu - Pv, u - v \rangle_{\mathfrak{Y}} \geq 0$.

mentioned above, there is $u_0 \in \mathfrak{R}$ such that $(Pu_0, u_0 - w) \leq \Psi(w) - \Psi(u_0)$ for all $w \in \mathfrak{R}$. Clearly, $u_0 \in \mathfrak{R} \cap \mathfrak{X}$, so we have $(Pu_0, u_0 - w) \leq \Phi(w) - \Phi(u_0)$ for all $w \in \mathfrak{R} \cap \mathfrak{X}$. Since $u_0 + t(w - u_0) \in \mathfrak{R} \cap \mathfrak{X}$ for any $w \in \mathfrak{R} \cap \mathfrak{X}$ and $1 > t > 0$, we see that

$$(Pu_0, u_0 - w) \leq \frac{1}{t} \{ \Phi(u_0 + t(w - u_0)) - \Phi(u_0) \} .$$

As $t \downarrow 0$, we have

$$(4.1) \quad (Pu_0, u_0 - w) \leq \langle \nabla \Phi(u_0), w - u_0 \rangle \quad \text{for any } w \in \mathfrak{R} \cap \mathfrak{X} .$$

By (a₁), $u_0 \wedge u \in \mathfrak{R} \cap \mathfrak{X}$. Hence, taking $w = u_0 \wedge u$ in (4.1)

$$(4.2) \quad (Pu_0, u_0 - u \wedge u_0) \leq \langle \nabla \Phi(u_0), u \wedge u_0 - u_0 \rangle .$$

Since $f + Pu = u - u \wedge v + f - f \wedge g \geq 0$, it follows from $(\Phi C_\infty)'$ that $\langle \nabla \Phi(u \wedge u_0), u_0 - u \wedge u_0 \rangle = \langle \nabla \Phi(u - (u - u_0)^+), (u - u_0)^- \rangle \geq \langle \nabla \Phi(u), (u - u_0)^- \rangle = (f, u_0 - u \wedge u_0) \geq (-Pu, u_0 - u \wedge u_0)$. Hence, by (4.2), $\langle \nabla \Phi(u_0) - \nabla \Phi(u \wedge u_0), u_0 - u \wedge u_0 \rangle \leq -(Pu_0 - Pu, (u_0 - u)^+) \leq 0$, so we have by (Φ_3) $u_0 = u_0 \wedge u$, that is, $u_0 \leq u$. In a similar manner, we have $u_0 \leq v$. Therefore $u_0 \leq u \wedge v$, while $u_0 \geq u \wedge v$ because $u_0 \in \mathfrak{R}$. Consequently, $u_0 = u \wedge v$ and hence (4.1) yields (ΦL) . q.e.d.

Proof of $\{(a_1), (b_1)\} \rightarrow (b_2)$: Let u and v be any functions in D_0 , and set $f = Au$ and $g = Av$. Define an operator P from L^2 into L^2 by $Pw = w - u \wedge (v + k) - f \wedge g$ and denote by \mathfrak{R} the set of all $w \in L^2$ such that $w \geq u \wedge (v + k)$. Then, in the same manner as in the proof of $(a_1) \rightarrow (a_2)$, we can find $u_0 \in \mathfrak{R} \cap \mathfrak{X}$ such that $u_0 \leq u$ and

$$(4.3) \quad (Pu_0, u_0 - w) \leq \langle \nabla \Phi(u_0), w - u_0 \rangle \quad \text{for any } w \in \mathfrak{R} \cap \mathfrak{X} .$$

Moreover, $u_0 = u_0 \wedge (v + k)$ holds. In fact, by using $(\Phi C_k)'$ we observe that

$$\begin{aligned} & \langle \nabla \Phi(u_0 \wedge (v + k)), u_0 - u_0 \wedge (v + k) \rangle \\ &= \langle \nabla \Phi(u_0 \wedge v + T_k((u_0 - v)^+)), (u_0 - v)^+ - T_k((u_0 - v)^+) \rangle \\ &\geq \langle \nabla \Phi(u_0 \wedge v), (u_0 - v)^+ - T_k((u_0 - v)^+) \rangle \\ &= \langle \nabla \Phi(v - (v - u_0)^+), (u_0 - v - k)^+ \rangle . \end{aligned}$$

Now, putting $z = (v - u_0)^+ - (u_0 - v - k)^+$, we see that $w^+ = (v - u_0)^+$ and $w^- = (u_0 - v - k)^+$, so by $(\Phi C_\infty)'$ the right hand side of the above inequalities is

$$\geq \langle \nabla\Phi(v - (v - u_0)^+ + w^+), w^- \rangle = (g, u_0 - u_0 \wedge (v + k)) .$$

From this and (4.3) it follows that

$$\begin{aligned} & \langle \nabla\Phi(u_0) - \nabla\Phi(u_0 \wedge (v + k)), u_0 - u_0 \wedge (v + k) \rangle \\ & \leq -(Pu_0 + g, u_0 - u_0 \wedge (v + k)) \leq 0 . \end{aligned}$$

By (Φ_3) , $u_0 = u_0 \wedge (v + k)$. Hence we see that $u_0 = u \wedge (v + k)$ and obtain (ΦSL_k) from (4.3). q.e.d.

Proof of $(b_2) \rightarrow (b_3)$: Let u and v be any functions in D_0 and f be a function in L^2 such that $Au \geq f$, $Av \geq f$ and $(Au - f, (u - v - k)^+) = 0$. Then it follows from (ΦSL_k) that

$$\langle \nabla\Phi(u \wedge (v + k)), w \rangle \geq (f, w) \quad \text{for any non-negative } w \in L^2 \cap \mathfrak{X} .$$

Hence

$$\begin{aligned} & \langle \nabla\Phi(u) - \nabla\Phi(u \wedge (v + k)), u - u \wedge (v + k) \rangle \\ & = \langle \nabla\Phi(u) - \nabla\Phi(u \wedge (v + k)), (u - v - k)^+ \rangle \\ & \leq (Au - f, (u - v - k)^+) \\ & = 0 . \end{aligned}$$

From this and (Φ_3) we obtain $u = u \wedge (v + k)$, that is, $u \leq v + k$.

Proof of $(a_2) \rightarrow (a_3)$: By taking 0 instead of k in the above proof, we have the proof of $(a_2) \rightarrow (a_3)$.

Proof of $(b_3) \rightarrow (b_4)$: Let λ be any positive number and f and g be any functions in L^2 . Set $u = R_\lambda(f + T_k g)$ and $v = R_\lambda f$. Then we see by (iii) of Lemma 2.2 that $Au \geq \lambda^{-1}\{f - (u - T_k g) \vee v\}$ and $Av \geq \lambda^{-1}\{f - (u - T_k g) \vee v\}$. Moreover,

$$\left(Au - \frac{f - (u - T_k g) \vee v}{\lambda}, (u - v - k)^+ \right) = 0 .$$

Hence, we obtain from (ΦCM_k) that $u \leq v + k$, i.e., $R_\lambda(f + T_k g) \leq R_\lambda f + k$. q.e.d.

Proof of $(a_3) \rightarrow (a_4)$: In the above proof, replace k by 0 and $T_k g$ by $-g^+$. Then, by (ΦD) we have $R_\lambda(f - g^+) \leq R_\lambda f$. Since this inequality holds for any $f, g \in L^2$, R_λ is order-preserving in L^2 .

Remark. The proofs of $(a_2) \rightarrow (a_3) \rightarrow (a_4)$ and of $(b_2) \rightarrow (b_3) \rightarrow (b_4)$ given above are essentially due to Calvert [5; §2].

In order to prove the assertions $(a_i) \rightarrow (a_1)$ and $\{(a_i), (b_i)\} \rightarrow (b_1)$, we prepare some lemmas. In the rest of this section, let k be a positive number.

LEMMA 4.2. (i) *If (a_4) and (b_4) are satisfied, then $T_k f \in \mathfrak{X}$ for any $f \in D_0$.*

(ii) *If (a_4) is satisfied, then $f^+ \in \mathfrak{X}$ for any $f \in D_0$.*

Proof. Assume (a_4) and (b_4) . Let $f \in D_0$ and set $g = T_k f$. Then

$$\begin{aligned} H_\lambda(f - g) &= \frac{1}{\lambda}((f - g) - R_\lambda(f - g), f - g) \\ &= \frac{1}{\lambda} \sum_{i=1}^3 \int_{X_i} \{(f - g) - R_\lambda(f - g)\}(f - g) d\xi, \end{aligned}$$

where $X_1 = \{x \in X; f(x) < 0\}$, $X_2 = \{x \in X; 0 \leq f(x) \leq k\}$ and $X_3 = \{x \in X; f(x) > k\}$. By (a_4) and (b_4) we have

$$R_\lambda(f - g) \leq R_\lambda f \leq R_\lambda(f - g) + k,$$

so that $(f - g) - R_\lambda(f - g) \geq f - R_\lambda f$ on X_1 and $\leq f - R_\lambda f$ on X_3 . Moreover, $f - g = f < 0$ on X_1 , $f - g = 0$ on X_2 and $0 < f - g < f$ on X_3 . Hence we have $H_\lambda(f - g) \leq \lambda^{-1} \|f - R_\lambda f\|_2 \|f\|_2 \leq \|Af\|_2 \|f\|_2$ because of (iii) of Lemma 2.2 and Lemma 2.3, so from Lemma 2.6 it follows that $f - g \in \mathfrak{X}$, i.e., $g \in \mathfrak{X}$. Thus (i) is obtained. (ii) is similarly proved. q.e.d.

LEMMA 4.3. (i) *If (a_4) and (b_4) are satisfied, then*

$$(4.4) \quad \langle \nabla \Phi(u + T_k v) - \nabla \Phi(u), v - T_k v \rangle \geq 0 \quad \text{for any } u \in \mathfrak{X} \text{ and } v \in D_0.$$

(ii) *If (a_4) is satisfied, then*

$$\langle \nabla \Phi(u + v^+) - \nabla \Phi(u), v^- \rangle \leq 0 \quad \text{for any } u \in \mathfrak{X} \text{ and } v \in D_0.$$

Proof. Assume (a_4) and (b_4) . According to Axiom (b') and property (Φ_b) , it is sufficient to show the inequality in case $u \in L^2 \cap \mathfrak{X}$ and $v \in D_0$. By Lemma 4.2 we see that $T_k v \in L^2 \cap \mathfrak{X}$. Furthermore we have

$$\begin{aligned} &\langle \nabla \Phi(R_\lambda(u + T_k v)) - \nabla \Phi(R_\lambda u), v - T_k v \rangle \\ &= \frac{1}{\lambda}((u + T_k v) - R_\lambda(u + T_k v) - (u - R_\lambda u), v - T_k v) \\ &= \frac{1}{\lambda}(T_k v + R_\lambda u - R_\lambda(u + T_k v), v - T_k v). \end{aligned}$$

Now, $T_k v + R_\lambda u - R_\lambda(u + T_k v) \geq 0$ on $X_1 = \{x \in X; v(x) \geq k\}$ by (b_4) and ≤ 0 on $X_2 = \{x \in X; v(x) \leq 0\}$ by (a_4) . Moreover, $v - T_k v \geq 0$ on X_1 , ≤ 0 on X_2 and $= 0$ on $\{x \in X; 0 < v(x) < k\}$, so the right hand side of the above equalities is non-negative. Hence, by Lemma 2.5,

$$\begin{aligned} & \langle \nabla \Phi(u + T_k v) - \nabla \Phi(u), v - T_k v \rangle \\ &= \lim_{\lambda \downarrow 0} \langle \nabla \Phi(R_\lambda(u + T_k v)) - \nabla \Phi(R_\lambda u), u - T_k v \rangle \\ &\geq 0. \end{aligned}$$

The assertion (ii) is similarly proved. q.e.d.

LEMMA 4.4. (i) *If (a_4) and (b_4) are satisfied, then*

$$(4.5) \quad \Phi(u) + \Phi(u + v) \geq \Phi(u + T_k v) + \Phi(u + v - T_k v)$$

for any $u \in \mathfrak{X}$ and $v \in D_0$.

(ii) *If (a_4) is satisfied, then*

$$\Phi(u) + \Phi(u + v) \geq \Phi(u + v^+) + \Phi(u - v^-)$$

for any $u \in \mathfrak{X}$ and $v \in D_0$.

Proof. We shall show (i). By using property (Φ_7) and (i) of Lemma 4.3 we have

$$\begin{aligned} & \Phi(u + v) - \Phi(u + T_k v) \\ &= \int_0^1 \langle \nabla \Phi(u + T_k v + t(v - T_k v)), v - T_k v \rangle dt \\ &\geq \int_0^1 \langle \nabla \Phi(u + t(v - T_k v)), v - T_k v \rangle dt \\ &= \Phi(u + v - T_k v) - \Phi(u) \end{aligned}$$

for any $u \in \mathfrak{X}$ and $v \in D_0$. Similarly from (Φ_7) and (ii) of Lemma 4.3 we obtain (ii). q.e.d.

Proof of $\{a_4\}, (b_4) \rightarrow (b_1)$: Let v be any function in $L^2 \cap \mathfrak{X}$. Then $R_\lambda v \in D_0$, $\lambda > 0$, because of (iii) of Lemma 2.2. Taking 0 and $R_\lambda v$ for u and v in (4.5) respectively and using (i) of Lemma 2.2, we have

$$\Phi(T_k(R_\lambda v)) \leq \Phi(R_\lambda v) \leq \Phi(v),$$

so by (1.1), $\{T_k(R_\lambda v); \lambda > 0\}$ is bounded in \mathfrak{X} and hence it is weakly relatively compact in \mathfrak{X} . Now, let $\{T_k(R_{\lambda_n} v)\}$ be any sequence weakly convergent in \mathfrak{X} such that $\lambda_n \downarrow 0$ as $n \rightarrow \infty$, and denote by g the weak limit.

Then, since $T_k(R_{\lambda_n}v) \xrightarrow{s} T_kv$ in L^2 as $n \rightarrow \infty$ by (i) of Lemma 2.4, it follows that $g = T_kv$. This shows that $T_kv \in \mathfrak{X}$. Thus we have seen that $T_kv \in \mathfrak{X}$ for any $v \in L^2 \cap \mathfrak{X}$. Moreover, just as Lemma 4.3, we can prove that

$$\langle \nabla\Phi(u + T_kv) - \nabla\Phi(u), v - T_kv \rangle \geq 0 \quad \text{for any } u \in \mathfrak{X} \text{ and } v \in L^2 \cap \mathfrak{X} .$$

From this, by the same calculation as in the proof of Lemma 4.4, we deduce that

$$(4.6) \quad \begin{aligned} \Phi(u + T_kv) + \Phi(u + v - T_kv) &\leq \Phi(u) + \Phi(u + v) \\ &\text{for any } u \in \mathfrak{X} \text{ and } v \in L^2 \cap \mathfrak{X} . \end{aligned}$$

Again by the limit process, we see from (4.6) that $T_kv \in \mathfrak{X}$ for any $v \in \mathfrak{X}$, i.e., (C_k) holds, and simultaneously that

$$\Phi(u + T_kv) + \Phi(u + v - T_kv) \leq \Phi(u) + \Phi(u + v)$$

for any $u, v \in \mathfrak{X}$. Clearly, this inequality is equivalent to (ΦC_k) . q.e.d.

Proof of $(a_i) \rightarrow (a_1)$: We can prove the assertion just as $\{(a_i), (b_i)\} \rightarrow (b_1)$.

§ 6. The nonlinear contraction semigroup generated by $-A$

In this section we discuss the nonlinear contraction semigroup on L^2 generated by $-A$. In view of the generation theorem for contraction semigroups due to Kōmura [12; Theorem 4], there is a unique contraction semigroup $S = \{S(t); t \geq 0\}$ on L^2 whose infinitesimal generator is $-A$. Here we mean by a contraction semigroup $S = \{S(t); t \geq 0\}$ on L^2 a one-parameter family of operators $S(t), t \geq 0$, from L^2 into L^2 with the following properties:

- (s₁) $S(t)S(s) = S(t + s) \quad \text{for } t, s \geq 0.$
- (s₂) $S(0) = I.$
- (s₃) $\|S(t)v - S(t)w\|_2 \leq \|v - w\|_2 \quad \text{for } t \geq 0 \text{ and } v, w \in L^2.$
- (s₄) For each $v \in L^2, t \rightarrow S(t)v$ is continuous on $[0, \infty)$.

The contraction semigroup $S = \{S(t); t \geq 0\}$ on L^2 generated by $-A$ is in fact given by

$$S(t)v = \lim_{n \rightarrow \infty} R_{t/n}^n v \quad \text{for } t \geq 0 \text{ and } v \in L^2$$

(see Crandall-Liggett [6; Theorem I]).

THEOREM C. *The following statements $(c_1) \sim (c_3)$ and $(a_1) \sim (a_4)$ (in Theorem A) are equivalent to each other:*

(c_1) *A is T-accretive (or $-A$ is dispersive) (cf. [4], [13]), i.e.,*

$$(Au - Av, (u - v)^+) \geq 0 \quad \text{for any } u, v \in D_0.$$

(c_2) *For any $t, s \in [0, \infty)$, $s \leq t$, and any $u, v \in L^2$,*

$$\|(S(t)u - S(t)v)^+\|_2 \leq \|(S(s)u - S(s)v)^+\|_2.$$

(c_3) *For any $\lambda > 0$ and any $u, v \in L^2$,*

$$\|(R_\lambda u - R_\lambda v)^+\|_2 \leq \|(u - v)^+\|_2.$$

Proof. From (a_1) we see that

(c_1) Condition (C_∞) is satisfied and $\nabla\Phi$ is T -monotone (cf. [1]), i.e., for any $u, v \in \mathfrak{X}$,

$$\langle \nabla\Phi(u) - \nabla\Phi(v), (u - v)^+ \rangle \geq 0.$$

In fact, from $(\Phi C_\infty)'$ and the monotonicity of $\nabla\Phi$ we derive that

$$\begin{aligned} & \langle \nabla\Phi(u) - \nabla\Phi(v), (u - v)^+ \rangle \\ &= \langle \nabla\Phi(u) - \nabla\Phi(v), (v - u)^- \rangle \\ &\geq \langle \nabla\Phi(u + (v - u)^+) - \nabla\Phi(v), (v - u)^- \rangle \\ &\geq \langle \nabla\Phi(u \vee v) - \nabla\Phi(v), u \vee v - v \rangle \\ &\geq 0 \end{aligned}$$

for any $u, v \in \mathfrak{X}$ (cf. [5; Proposition 1.2]). The assertions $(c_1) \rightarrow (c_2)$ and $(c_3) \rightarrow (a_4)$ are trivial, and $(c_1) \rightarrow (c_2)$ and $(c_2) \rightarrow (c_3)$ are known (cf. [13]).

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