J. Austral. Math. Soc. 71 (2001), 259–277

RESTRICTED LAZARD ELIMINATION AND MODULAR LIE POWERS

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To Laci Kovács on his 65th birthday

(Received 22 January 2001; revised 23 May 2001)

Communicated by R. A. Bryce

Abstract

We exhibit a variation of the Lazard Elimination Theorem for free restricted Lie algebras, and apply it to two problems about finite group actions on free Lie algebras over fields of positive characteristic.

2000 *Mathematics subject classification*: primary 17B01; secondary 20C20.

1. Introduction

1.1. Elimination and Lie powers Lazard elimination is a powerful tool in studying free Lie algebras. Let $L = L(Y)$ be the free Lie algebra on a set Y over a commutative ring K. The Elimination Theorem (see [2, Chapter 2, Section 2.9, Proposition 10]) reads as follows (here, and throughout this paper, we use the left normed convention for Lie brackets).

THEOREM A (Lazard Elimination). *Suppose that $Y = Y_1 \cup Y_2$ is the disjoint union of its subsets Y_1 and Y_2. Then $L = L(Y)$ is the direct sum of its free subalgebra $L(Y_1)$ and the ideal $I(Y_2)$ that is generated by Y_2. Moreover, $I(Y_2)$ is itself a free Lie algebra with free generating set*

$$Y_2 \wr Y_1 = \{[z, y_1, y_2, \dots, y_k];\ z \in Y_2,\ y_1, y_2, \dots, y_k \in Y_1,\ k \geq 0\}.$$

https://doi.org/10.1017/S1446788700002901 Published online by Cambridge University Press

Thus the Elimination Theorem yields a direct decomposition (over K)

$$L = L(Y_1) \oplus L(Y_2 \wr Y_1)$$

which will be referred to as *elimination of the subalgebra* $L(Y_1)$. In the special case where $Y_1 = \{x\}$ is a singleton, this direct decomposition turns into

$$L = \langle x \rangle \oplus L(Y|x), \tag{1.1}$$

where

$$Y|x = (Y \setminus \{x\}) \wr \{x\} = \{[y, x^k];\ y \in Y \setminus \{x\},\ k \geq 0\},$$

$\langle x \rangle$ denotes the K-span of x in $L(Y)$ and

$$[y, x^k] = [y, \underbrace{x, x, \dots, x}_{k}].$$

The direct decomposition (1.1) will be referred to as *elimination of the free generator* x. The proof of the Elimination Theorem (see [2]) is not difficult, and does not require much more than the most elementary facts about free Lie algebras and derivations. However, it has immediate applications of remarkable depths. For example, the well-known Hall basis and other bases of the free Lie algebra can, in particular in the finite rank case, be easily derived using Lazard elimination (see [2, Chapter 2] and Section 3 below), and deriving a free generating set for the derived subalgebra is a trivial exercise on repeated use of elimination (see Section 3). Recently, in [5], Lazard elimination was one of the essential tools in studying modular Lie representations of groups of prime order. Let $L = L(Y)$ as before, and let $L_n = L_n(Y)$ ($n \geq 1$) denote the degree n homogeneous component of L. Furthermore, let G be a group and suppose that G acts by linear automorphisms on $L_1 = \langle Y \rangle$. Then L_1 is a KG-module, V say. The G-action on V extends uniquely to the whole of L, with G acting by graded algebra automorphisms. In particular, the L_n become KG-modules, and these are termed the Lie powers of V. It is common to write $L = L(V)$ and $L_n = L_n(V)$ in this setting, and we shall say that L is freely generated by V. The paper [5] deals with the situation where K is a field of positive characteristic p, G is the group of order p, and V is an arbitrary finite dimensional KG-module. A theory is developed which provides information about the overall module structure of $L(V)$ and gives a recursive method for finding the multiplicities of the indecomposable KG-modules in the Lie powers $L_n(V)$. The significance of these results lies, apart from the intrinsic interest in a natural and easy to state, but at the same time notoriously difficult problem, in the fact that they are the key to understanding the module structure of the Lie powers of the natural module for $GL(2, p)$ (see [14] for partial results for $p = 2$ and $p = 3$; an account of the general case is in preparation [7]). The recent interest in modular and integral Lie powers was initiated by L. G. Kovács, who started this line of research by putting a problem about Lie powers of the regular module for the cyclic group of order 2 into the Kourovka Notebook ([18, Problem 11.47]). The problem was tackled in [3] and [17], and subsequently solved in [9]. This was then the starting point of a period of intensive research on modular and (to a lesser extent) integral Lie powers over the past five years or so, which is still ongoing and resulted up to now in papers [3–17]. From the very beginning Laci Kovács has been at the forefront of these investigations, as an eminent author and tireless promoter of the subject.

1.2. Restricted elimination In this paper we exhibit a variation of Lazard elimination for free restricted Lie algebras, and apply it to two problems about modular Lie powers. We hope to convince the reader that 'restricted elimination' is a useful tool in studying modular Lie representations, and that many more applications will emerge in the future. From now on K is a field of positive characteristic p, and $R = R(Y)$ is the free restricted Lie algebra on Y over K. Throughout this paper we will identify the free Lie algebra $L(Y)$ with the Lie subalgebra that is generated by Y in R. We also use the notation $R(V)$ (similar to the unrestricted case), and write R_n, $R_n(Y)$ and $R_n(V)$ for the restricted Lie powers. An immediate consequence of Theorem A is that under the above assumptions on Y there are direct decompositions

$$(1.2) \qquad R = R(Y_1) \oplus R(Y_2 \wr Y_1), \quad R = \langle x^{p^\alpha}; \quad \alpha \geq 0 \rangle \oplus R(Y \mid x)$$

of the free restricted Lie algebra R. The variation that is specific to free restricted Lie algebras is as follows.

THEOREM B. *Let $x \in Y$, and let J be the ideal of R that is generated by x^p and $Y \setminus \{x\}$. Then R is the direct sum of its subspace $\langle x \rangle$ and the ideal J. Moreover, J is itself a free restricted Lie algebra with free generating set*

$$Y \mid_r x = \{x^p, \; [z, x^\alpha]; \; z \in Y \setminus \{x\}, \; 0 \leq \alpha \leq p-1\}.$$

Thus Theorem B yields a direct decomposition (over K)

$$R = \langle x \rangle \oplus R(Y|_r x)$$

which will be referred to as *restricted elimination of the free generator* x. Theorem B is, in fact, well known. In [1], for example, it is a key ingredient of the proof of Witt's celebrated theorem about the freeness of subalgebras of free restricted Lie algebras. For a proof of Theorem B we refer to the proof of Theorem 2.7.4 in [1]. Note that the second decomposition in (1.2) can, in fact, be obtained by using restricted elimination repeatedly, that is by eliminating the free generators $x, x^p, x^{p^2}, \ldots$ successively in the obvious way. The second decomposition in (1.2) will be referred to as *full elimination of the free generator* x.

1.3. The first application Our first application of restricted elimination refers to free Lie algebras over a field K of characteristic 2. Let G be the cyclic group of order 2 with generator g and let V be a finite dimensional free KG-module. Then the Lie powers $L_n(V)$ and the restricted Lie powers $R_n(V)$ are (like any finite dimensional KG-module) direct sums of isomorphic copies of the regular module KG and the trivial module K. The multiplicities of the indecomposables KG and K in $L_n(V)$ have been determined in [9], and a similar result for $R_n(V)$ has been obtained in [12]. Moreover, the main result of [12] was an explicit construction of G-invariant homogeneous bases for $L(V)$ and $R(V)$, in other words, an explicit decomposition of the Lie and restricted Lie powers of V into the direct sum of indecomposables. Our first application is an alternative construction of such bases using restricted elimination (Theorem 1 in Section 2). It turns out that using this type of elimination simplifies the construction considerably, and the technical difficulties that had to be overcome in [12] disappear altogether. In addition, our new bases have the advantage that they consist of monomials and hence they are not only homogeneous, but multihomogeneous (meaning that each basis element has a well defined multidegree with respect to the original free generating set of $R(V)$). We mention that in the much harder case of free Lie algebras over $\mathbb{Z}$ similar bases (however non-monomial ones) have recently been constructed in [6]. In that case there is of course no hope for simplification via restricted elimination as this tool is not available over $\mathbb{Z}$.

1.4. The second application Our second application is more involved, and inevitably harder to describe. It gives a new, and in our view more natural and considerably simpler way of proving the main technical result that is at the very heart of [5]. Moreover, we prove this result in a slightly more general setting: While [5] dealt with Lie powers of indecomposable modules for the group of order p, we extend the result to indecomposables for the holomorph of this group, a Frobenius group of order $p(p-1)$. This has the additional advantage that the result in question comes directly in a form required for applications in [7]. In order to be more precise, we need to introduce some notation. Let K be a field of characteristic p, and let G be the group

$$G = \langle g, h \mid g^p = h^{p-1} = 1,\ h^{-1}gh = g^l\rangle,$$

where l is a positive integer such that the image of l under the unique homomorphism $\mathbb{Z} \to K$ generates the multiplicative group of the prime subfield $GF(p)$ of K. We will not distinguish between l and its image in K. We let P denote the p-Sylow subgroup of G. Thus $P = \langle g\rangle$, the cyclic subgroup of order p that is generated by g. It is well-known (see, for example, [4] for more detail and references) that there are precisely $p(p-1)$ indecomposable KG-modules, which will be labelled here by $J_{i,r}$ with $i = 0, 1, \ldots, p-2$ and $r = 1, 2, \ldots, p$. In this notation r is the dimension of $J_{i,r}$ and h acts on the top composition factor $J_{i,r}/\operatorname{rad}(J_{i,r})$ as multiplication by the scalar l^i. Each $J_{i,r}$ has a basis $Y^{(i,r)} = \{y_1^{(i,r)}, y_2^{(i,r)}, \dots, y_r^{(i,r)}\}$ (called the standard basis here) such that

$$y_j^{(i,r)} g = y_j^{(i,r)} + y_{j+1}^{(i,r)} \quad \text{for } j = 1, \dots, r-1,$$
$$y_r^{(i,r)} g = y_r^{(i,r)} \quad \text{and} \quad y_1^{(i,r)} h = l^i y_1^{(i,r)}.$$

Note that $y_1^{(i,r)}$ is an eigenvector for h with eigenvalue l^i. If i and r are understood, we will write the standard basis as $Y = \{y_1, \dots, y_r\}$. The result in question refers to a Lie subalgebra of the free restricted Lie algebra $R = R(J_{i,r}) = R(Y)$. Throughout we will assume that $r \geq 2$ (as the case $r = 1$ is of no interest). We use the term *left normed commutator in* Y for a Lie product of the form $[z_1, z_2, \dots, z_n]$ with $z_1, \dots, z_n \in Y$ (to distinguish such Lie products from more complex commutators such as, for example, $[[z_1, z_2, z_3], [z_4, z_5]]$). The above left normed commutator is a left normed *basic* commutator if $z_1 > z_2 \leq \cdots \leq z_n$ with respect to some given order of Y. In this paper we will use the order $y_1 > y_2 > \cdots > y_r$. Let $\hat{L}_p(J_{i,r})$ denote the subspace of the p-th Lie power $L_p(J_{i,r})$ that is generated by all Hall basic commutators of degree p in Y except $[y_1, y_2^{\alpha}, y_1^{p-1-\alpha}]$ with $1 \leq \alpha \leq p-1$. Furthermore, let $x_i = y_1 g^{i-1}$ with $1 \leq i \leq p$. Finally, put

$$\hat{L}(J_{i,r}) = L_2(J_{i,r}) + L_3(J_{i,r}) + \cdots + L_{p-1}(J_{i,r}) + \hat{L}_p(J_{i,r}) + L_{p+1}(J_{i,r}) + \cdots$$

and

$$L_S = \hat{L}(J_{i,r}) + \langle x_1^p, \dots, x_p^p \rangle.$$

It is easily seen that $\hat{L}(J_{i,r})$ and L_S are G-invariant Lie subalgebras of $R(J_{i,r})$. In [5], L_S is called the *shifted subalgebra* of $L(J_{i,r})$. By $M = M(J_{i,r})$ we denote the free metabelian Lie algebra on $J_{i,r}$, that is the quotient of $L = L(J_{i,r})$ by its second derived subalgebra: $M = L/L''$. We write $M_n = M_n(J_{i,r})$ for its homogeneous components, which are termed the free metabelian Lie powers of $J_{i,r}$. The central technical result of [5, Theorem 4.4], that makes the whole theory work, reads (suitably reformulated) as follows.

THEOREM 2. *Let $r \geq 2$. For each $n \geq 2$, there exists a KG-submodule U_n of $R_n(J_{i,r})$ such that*

(i) *L_S is freely generated by $U = U_2 \oplus U_3 \oplus \cdots$,*
(ii) *for $n \neq p$, U_n is a direct summand of $L_n(J_{i,r})$,*
(iii) *U_p has the form $\langle x_1^p, \dots, x_p^p \rangle \oplus V_p$, where V_p is a direct summand of $\hat{L}_p(J_{i,r})$,*
(iv) *for $n < p$, $U_n \cong M_n(J_{i,r})$,*
(v) *for $n \geq p$, U_n is a projective KG-module, and, moreover,*
(vi) *$L_S = \hat{L}(J_{i,r}) \oplus \langle x_1^p, \dots, x_p^p \rangle$.*

Theorem 2 differs from Theorem 4.4 of [5] in that we have incorporated into the statement all relevant results from the earlier Theorem 4.1 of [5], which is, in fact, an inseparable part of Theorem 4.4. We mention that Theorem 4.1 gives also a formula for the dimension of the modules U_n. This information was used in the original proof in [5], but since it is not required for the applications of the Theorem in [5], nor is it used in our proof, we have omitted it from the statement. Our proof of Theorem 2 provides, in fact, additional information about the projective modules U_n for $n \geq p$. With a view to applications in [7], that information will be recorded in a Corollary. The original Theorem 4.4 of [5] can be recovered from Theorem 2 by replacing the group G by its subgroup P and the modules $J_{i,r}$ by the restrictions $J_r = J_{i,r} \downarrow_P$. We wish to emphasize, however, that the main achievement of the present paper is not the generalization of Theorem 4.4 from P-modules to G-modules, but the new method of the proof. The fact that the theorem about the shifted Lie algebra can be extended from P to G was known to the authors of [5] before work on the present paper had started.

1.5. Organization of the paper The paper is organized as follows. Theorem 1, our first application, is proved in Section 2, and the remaining three sections are devoted to our new proof of Theorem 2. In Section 3 we use restricted and ordinary elimination to derive various decompositions of free restricted Lie algebras, and in particular we exhibit a particular free Lie subalgebra $L(D)$ of $R(J_{i,r})$. A slight modification of this algebra, which is carried out at the beginning of Section 5, will give us the shifted algebra together with a convenient free generating set D^* for it. In the rest of Section 5, D^* will be converted into another free generating set F whose span is G-invariant in $L(D^*)$: This span will be the module U in Theorem 2. In Section 4 we assemble some auxiliary results for use in Section 5. We assume that the reader is familiar with basic material about free Lie algebras and free restricted Lie algebras as given in the first two Chapters of [1]. In particular, the fact that if A is a basis of the free Lie algebra $L(Y)$, then $\{a^{p^\alpha};\ a \in A,\ \alpha \geq 0\}$ is a basis of $R(Y)$ (see [1, 2.7.1]) will be frequently used without reference.

2. Invariant bases for Lie powers in characteristic 2

In this section K is a field of characteristic 2, G is the group of order 2 with generator g, and V is a free KG-module. Hence V has a basis of the form $\{u, ug;\ u \in T\}$ where T is a free generating set of V as a KG-module. Our aim is to derive G-invariant bases for $R(V)$ and $L(V)$ in the case where V has finite dimension. Let $u \in T$ and write $T' = T \setminus \{u\}$. Elimination of the free subalgebra generated by u and ug from $R(V)$ (see (1.2)) gives a direct decomposition

$$(2.1) \qquad R(V) = R(u, ug) \oplus R(T_1 \cup T_1 g),$$

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where

$$T_1 = \{[w, v_1, \dots, v_k];\ w \in T',\ v_1, \dots, v_k \in \{u, ug\},\ k \geq 0\}.$$

It is clear that G acts freely on $T_1 \cup T_1 g$ and that T_1 is a transversal of the G-orbits. Now consider $R(u, ug)$. Restricted elimination of u and ug (in that order) gives

$$\begin{aligned} R(u, ug) &= \langle u \rangle \oplus R(u^2, ug, [ug, u]) \\ &= \langle u, ug \rangle \oplus R(u^2, u^2 g, [ug, u], [u^2, ug], [ug, u, ug]). \end{aligned}$$

Since $[ug, u, ug] = [u^2 g, u]$, this can be rewritten as

$$R(u, ug) = \langle u, ug \rangle \oplus R([ug, u], u^2, u^2 g, [u^2, ug], [u^2 g, u]).$$

Here $[ug, u]$ is fixed by g while u^2, $u^2 g$ and $[u^2, ug]$, $[u^2 g, u]$ generate regular KG-modules. Now full elimination of $[ug, u]$ gives

$$R(u, ug) = \langle u, ug, [ug, u]^{2^\alpha};\ \alpha \geq 0 \rangle \oplus R(T_2 \cup T_2 g), \tag{2.2}$$

where

$$T_2 = \{[w, [ug, u]^k];\ w \in \{u^2, [u^2, ug]\},\ k \geq 0\}.$$

Clearly, G acts freely on $T_2 \cup T_2 g$ and T_2 is a transversal of the G-orbits. Substituting (2.2) into (2.1) gives

$$R(V) = \langle u, ug, [ug, u]^{2^\alpha};\ \alpha \geq 0 \rangle \oplus R(T_1 \cup T_1 g) \oplus R(T_2 \cup T_2 g).$$

We call that an *elimination step* for u. Let $\Gamma_1 = T_1 \cup T_2$, $u_2 \in \Gamma_1$, and perform another elimination step for the free restricted Lie algebra $R(T_i \cup T_i g)$ that contains u_2. This results in a direct decomposition

$$E_2: \qquad R(V) = \bigoplus_{i=1}^{2} \langle u_i, u_i g, [u_i g, u_i]^{2^\alpha};\ \alpha \geq 0 \rangle \oplus \bigoplus_{i=1}^{3} R(T_i^{(2)} \cup T_i^{(2)} g),$$

where we have put $u = u_1$ for convenience. We can then pick an element from $\Gamma_2 = \bigcup_{i=1}^{3} T_i$, and perform another elimination step, and so on. After n such steps we arrive at a direct decomposition

$$E_n: \qquad R(V) = \bigoplus_{i=1}^{n} \langle u_i, u_i g, [u_i g, u_i]^{2^\alpha};\ \alpha \geq 0 \rangle \oplus \bigoplus_{i=1}^{n+1} R(T_i^{(n)} \cup T_i^{(n)} g),$$

and then the $(n+1)$-st elimination step can be performed for some element $u_{n+1} \in \Gamma_n = \bigcup_{i=1}^{n+1} T_i^{(n)}$.

Now suppose that V has finite dimension, in other words, T is a finite set. We call a sequence $E_1, E_2, E_3, \dots$ of elimination steps *suitable*, if for all $n \geq 1$ the element $u_n \in \Gamma_n$ is of smallest possible degree.

THEOREM 1. *Let $E_1, E_2, E_3, \ldots$ be a suitable sequence of elimination steps. Then*

$$\Omega = \bigcup_{i=1}^{\infty} \{u_i, u_i g, [u_i g, u_i]^{2^\alpha};\ \alpha \geq 0\}$$

is a G-invariant monomial basis of $R(V)$, and the subset Ω' consisting of all elements of Ω that are not written as a proper power is a G-invariant monomial basis of $L(V)$.

PROOF. It is clear from the very nature of the elimination process that the elements of Ω are linearly independent, that they are monomials in the free generators u, ug ($u \in T$), and that Ω is G-invariant. It remains to show that Ω spans $R(V)$. To see this, we argue as in [12]. Observe that every set Γ_n consists precisely of $\Gamma_{n-1} \setminus \{u_n\}$ plus elements of degree strictly larger than $\deg u_n$. Moreover, every Γ_n contains only finitely many elements of any given degree. It follows easily that if $\delta(n)$ denotes the smallest positive integer such that Γ_n contains an element of degree $\delta(n)$, then $\delta(n) \to \infty$ as $n \to \infty$. Consequently, for each m there exists an n such that the direct sum of free restricted Lie algebras on the right hand side of E_n consists entirely of elements of degree $> m$. This in turn implies that the homogeneous component $R_m(V)$ is contained in the span of the eliminated elements. Hence $R(V) \subseteq \langle\Omega\rangle$, as required. To show that Ω' is a basis of $L(V)$ observe first that Ω' consists of Lie elements, and hence $\langle\Omega'\rangle \subseteq L(V)$, and that $\Omega = \{w^{p^\alpha};\ w \in \Omega',\ \alpha \geq 0\}$. If Ω' is not a basis of $L(V)$, it can be extended to a basis Ω'' of $L(V)$. But then the set $\{w^{p^\alpha};\ w \in \Omega'',\ \alpha \geq 0\}$ would be a basis of $R(V)$, which is impossible as it contains the basis Ω is a proper subset. This completes the proof of Theorem 1. □

In the smallest non-trivial instance where V is a regular KG-module with basis $\{x, y\}$ and G-action given by $xg = y$, $yg = x$, the basis elements in Ω' up to degree 6 are as follows.

$$x, y;\ [y, x];\ [x^2, y], [y^2, x];\ [x^2, [y, x]], [y^2, [y, x]], [y^2, x^2];$$
$$[x^2, y, x^2], [y^2, x, y^2], [x^2, y, y^2], [y^2, x, x^2], [[x^2, y], [y, x]], [[y^2, x], [y, x]];$$
$$[x^4, y^2], [y^4, x^2], [x^2, [y, x], [y, x]], [y^2, [y, x], [y, x]], [x^2, [y, x], x^2],$$
$$[y^2, [y, x], y^2], [x^2, [y, x], y^2], [y^2, [y, x], x^2], [[y^2, x], [x^2, y]].$$

In conclusion of this section we mention that with Theorem 1 at our disposal it is very easy to produce G-invariant bases for free Lie algebras and restricted Lie algebras on arbitrary finite dimensional KG-modules. The relevant construction is outlined in the proof of Corollary 2 in [6].

3. Decompositions by elimination

In this and the remaining sections of this paper we will use the notation introduced in Section 1.4. Unless stated otherwise, we assume throughout that i and r are fixed, and write simply Y instead of $Y^{(i,r)}$ for the standard basis of $J_{i,r}$. However, in the present section we do not make use of the G-action on $\langle Y\rangle$, and the restriction $r \leq p$ is irrelevant here. Recall that $Y = \{y_1, y_2, \dots, y_r\}$ is ordered by $y_1 > y_2 > \cdots > y_r$, and consider $L = L(Y)$. Consecutive elimination of $y_r, y_{r-1}, \dots, y_1$ (in that order) yields a direct decomposition

$$L(Y) = \langle y_1, y_2, \dots, y_r\rangle \oplus L(B),$$

where

$$\begin{aligned} B &= Y \mid y_r \mid \cdots \mid y_1 \\ &= \{[z_1, z_2, \dots, z_k];\ z_1, \dots, z_k \in Y,\ z_1 > z_2 \leq \cdots \leq z_k,\ k \geq 2\} \end{aligned}$$

is the set of all left normed basic commutators of degree at least 2 in Y. It is clear that $L(B) = L'$, the derived algebra of L. Thus, as mentioned in the Introduction, an easy repeated application of Lazard elimination allows us to obtain a free generating set of the derived algebra L'.

REMARK. It is not hard to see that if we continue the elimination process indefinitely in such a way that at each stage a free generator of minimal degree is eliminated (similar to what we did in the proof of Theorem 1), then the set of eliminated elements emerging in the limit is a complete set of Hall basic commutators. Moreover, it follows automatically that this set is indeed a basis of L. This is by far the easiest way of obtaining a basis for a free Lie algebra of finite rank that is known to this author, and it is another convincing demonstration of what a powerful tool Lazard elimination is.

Since B is a free generating set for L', it follows immediately that the set $\{w + L'';\ w \in Y \cup B\}$ is a basis of $M = M(Y)$. This well-known fact will be used below without reference. When working in M, we will allow ourselves the liberty to refer to B and other subsets of L' as subsets of M, meaning, of course, the corresponding sets of cosets.

Now consider the free restricted Lie algebra $R = R(Y)$. Consecutive restricted elimination of $y_r, y_{r-1}, \dots, y_1$, (in that order) yields a direct decomposition

$$R(Y) = \langle y_1, y_2, \dots, y_r\rangle \oplus R(C), \tag{3.1}$$

where C consists of the left normed commutators

$$[z_1, z_2^{\alpha_2}, \dots, z_k^{\alpha_k}] \tag{3.2}$$

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with $z_1, \dots, z_k \in Y$, $z_1 > z_2 < \dots < z_k$, $k \geq 2$ and $1 \leq \alpha_i < p$ $(i = 2, \dots, k)$, and the elements

$$[z_1^p, z_2^{\alpha_2}, \dots, z_k^{\alpha_k}] \tag{3.3}$$

with $z_1, \dots, z_k \in Y$, $z_1 < z_2 < \dots < z_k$, $k \geq 1$ and $1 \leq \alpha_i < p$ $(i = 2, \dots, k)$. Note that the elements (3.2) are left normed basic commutators in Y. Note also that the elements (3.3) include the p-th powers $y_1^p, y_2^p, \dots, y_r^p$, and that the elements (3.2) and (3.3) together include (up to sign) all left normed basic commutators of degrees $2, 3, \dots, p, p+1$. In fact, all these basic commutators except $[y_i, y_j^p]$ with $1 \leq i < j \leq r$ are of the form (3.2), and for the exceptional ones we have $[y_i, y_j^p] = -[y_j^p, y_i]$. Let $L(C)$ denote the Lie subalgebra generated by C in R. It is clear that $L(C)$ is freely generated by C, and it follows immediately from (3.1) that there is a direct decomposition

$$L(C) = \langle y_1^p, \dots, y_r^p \rangle \oplus L'.$$

Full elimination of $y_r^p, y_{r-1}^p, \dots, y_2^p$ (in that order) from $R(C)$, gives a direct decomposition

$$R(Y) = \langle y_1, y_2^{p^\alpha}, y_3^{p^\alpha}, \dots, y_r^{p^\alpha}; \alpha \geq 0 \rangle \oplus R(D), \tag{3.4}$$

where

$$D = \{[w, y_r^{\beta_r p}, \dots, y_3^{\beta_3 p}, y_2^{\beta_2 p}];\ w \in C \setminus \{y_2^p, y_3^p, \dots, y_r^p\},\ \beta_2, \dots, \beta_r \geq 0\}.$$

Now consider $L(D)$, the free Lie subalgebra of R that is generated by D. It is clear that this Lie algebra is freely generated by D and it follows immediately from (3.4) that

$$L(D) = \langle y_1^p \rangle \oplus L'.$$

We will return to the free Lie algebra $L(D)$ in Section 5, but first we turn to some auxiliary results required for the proof.

4. Auxiliary results

4.1. Changing free generating sets Let $L(X)$ be the free Lie algebra on a set X and assume that X is the disjoint union

$$X = X_1 \cup X_2 \cup X_3 \cup \cdots$$

of its finite subsets $X_1, X_2, \dots$. Let $L(< m)$ denote the subalgebra of $L(X)$ that is generated by $X_1, X_2, \dots, X_{m-1}$. For each $n \geq 1$, let $\varphi_n \in GL(\langle X_n \rangle)$ be a linear

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automorphism of the space $\langle X_n \rangle$, and let $\varphi : X \to L(X)$ be a map that is, for all $x \in X_n$, $n = 1, 2, \ldots$ of the form

$$\varphi(x) = \varphi_n(x) + w_x, \tag{4.1}$$

where $w_x \in L(< n)$. We need the following

LEMMA 1. *If $\varphi : X \to L(X)$ is a map of the form* (4.1), *then $\varphi(X)$ is a free generating set of $L(X)$.*

A proof of this simple fact can be found in [6, Section 2.3]. We record an easy consequence of this lemma for reference purposes. Suppose that X is a graded set in which the elements of X_m have degree m. This turns $L(X)$ into a graded Lie algebra with homogeneous components $\Lambda_n = \Lambda_n(X)$, $(n \geq 1)$ where Λ_n is the span of all monomials of degree n in X. Here we use the Greek letter Λ deliberately to distinguish these homogeneous components from the ordinary homogeneous components L_n which are formed assuming that all free generators in X have degree 1. Note that $X_n \subseteq \Lambda_n$ for all n, and that $\Lambda_n \cap L(< n)$ is a vector space complement of $\langle X_n \rangle$ in Λ_n. Now let H be a group and suppose that H acts on $L(X)$ in such a way that each Λ_n is a KH-submodule of $L(X)$. Then the subalgebras $L(< n)$ are also KH-submodules of $L(X)$. Let π_n denote the natural surjection $\Lambda_n \twoheadrightarrow \Lambda_n/(\Lambda_n \cap L(< n))$.

COROLLARY. *Assume that for all $n \geq 1$ the natural surjections π_n split as KH-module homomorphisms. Let ν_n be a spitting map for π_n and put $F_n = \nu_n \pi_n(X_n)$. Then $F = F_1 \cup F_2 \cup F_3 \cup \cdots$ is a free generating set for $L(X)$, and for each n there is an isomorphism*

$$\langle F_n \rangle \cong \Lambda_n/(\Lambda_n \cap L(< n))$$

of KH-modules. In particular, $\langle F_n \rangle$ is a KH-submodule for all n.

PROOF. The map $X \to L(X)$ defined by $x \mapsto \nu_n \pi_n(x)$ for $x \in X_n$ $(n \geq 1)$ is of the form (4.1), and Lemma 1 applies. Hence F is a free generating set of $L(X)$. The asserted isomorphism is obvious. □

4.2. The restricted Lie power $R_p(Y)$ From now on we will work with the notation introduced in Section 1.4. Let $Y = \{y_1, y_2, \ldots, y_r\}$ be the standard basis of $J_{i,r}$ and $x_1, \ldots, x_p$ as in Section 1.4. Then, by an easy calculation, we get for $i = 1, \ldots, p$

$$x_i = y_1 + \binom{i-1}{1} y_2 + \binom{i-1}{2} y_3 + \cdots + \binom{i-1}{r} y_r,$$

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where the binomial coefficients are evaluated in K. Applying the p-power operation of $R(Y)$ gives for $i = 2, \dots, p$,

$$x_i^p = y_1^p + \binom{i-1}{1} y_2^p + \binom{i-1}{2} y_3^p + \cdots + \binom{i-1}{r} y_r^p + l_i, \tag{4.2}$$

where $l_i \in L_p(Y)$. The following result can be found in [5, Lemma 3.2 and Corollary 3.3].

LEMMA 2. (i) *The elements* $x_1^p, x_2^p, \dots x_p^p \in R_p(V)$ *are linearly independent modulo* $\hat{L}_p(Y)$. (ii) *The elements* $l_2, l_3, \dots, l_p$ *from* (4.2) *are linearly independent modulo* $\hat{L}_p(Y)$*, and form a basis of* $L_p(Y)$ *modulo* $\hat{L}_p(Y)$.

4.3. Modules We continue to work with the notation from Section 1.4, but now we write $L(J_{i,r}), M(J_{i,r}), \dots$ to emphasize that the focus here is on the G-action on these objects. The first of three results contained in this subsection is proved in [5, Section 2].

LEMMA 3. *For* $n = 2, 3, \dots, p+1$, *the natural surjection*

$$L_n(J_{i,r}) \to M_n(J_{i,r})$$

splits as a homomorphism of KG*-modules.*

The second result is implicitly contained in [5], but for the convenience of the reader we spell out how to derive it in the compact form we use here. Let E denote the set of all left normed basic commutators of the form

$$[z_1, z_2^{\alpha_2}, \dots, z_k^{\alpha_k}] \tag{4.3}$$

with $z_1, \dots, z_k \in Y$, $z_1 > z_2 < \cdots < z_k$, $k \geq 2$, $\alpha_i \geq 1$ $(i = 2, \dots, k)$, satisfying the additional condition that $\alpha_k < p$ whenever $z_k = y_1$. Let $\hat{E}$ denote the subset of all elements (4.3) in E such that $z_2 \in \{y_3, \dots, y_r\}$, and let $\tilde{E}$ denote the subset of all elements (4.3) in E such that $z_2 = y_r$. It will be convenient to introduce one more piece on notation at this point. If A is a set of homogeneous elements in $R(J_{i,r})$, we write A_n for the subset of elements of degree n in A.

LEMMA 4. (i) *For all* $n \geq p+1$, *and also for* $n = p$ *and* $r \geq 3$, *the subspace* $\langle \tilde{E}_n \rangle \subseteq M_n(J_{i,r})$ *is a projective* KG*-submodule of* $M_n(J_{i,r})$. (ii) *For all* $n \geq p$, *the subspace* $\langle \hat{E}_n \rangle \subseteq M_n(J_{i,r})$ *is a projective* KG*-submodule of* $M_n(J_{i,r})$. (iii) *For all* $n \geq p+1$, *the subspace* $\langle E_n \rangle \subseteq M_n(J_{i,r})$ *is a projective* KG*-submodule of* $M_n(J_{i,r})$.

PROOF. It is easily seen that $\langle E_n\rangle$, $\langle\tilde{E}_n\rangle$ and $\langle\hat{E}_n\rangle$ are submodules, and so it remains to establish projectivity under the relevant conditions. It is shown in [5, (3.20), Section 2], that the restriction $\langle\tilde{E}_n\rangle\downarrow_P$ is a free KP-submodule of $M_n(J_{i,r})$ for $n\geq p+2$, but the proof works equally well for $n=p+1$, and $n=p$ when $r\geq 3$. Hence $\langle\tilde{E}_n\rangle$ is a projective KG-module for the required range of n and r. This proves (i). For the proof of (ii) and (iii), let

$$\delta_n^{(i,r)}: M_n(J_{i,r})\to M_n(J_{i,r-1})$$

denote the natural surjection given by $y_j^{(i,r)}\mapsto y_j^{(i,r-1)}$ for $j=1,\ldots,r-1$ and $y_r^{(i,r)}\mapsto 0$. This is a KG-homomorphism, and it is easily seen that the left normed basic commutators in B (see Section 3) with $z_2=y_r^{(i,r)}$ form a basis of $\ker\delta_n^{(i,r)}$. The restriction of $\delta_n^{(i,r)}$ to $\langle E_n\rangle$ yields a short exact sequence

$$\langle\tilde{E}_n^{(i,r)}\rangle\rightarrowtail\langle E_n^{(i,r)}\rangle\twoheadrightarrow\langle E_n^{(i,r-1)}\rangle \tag{4.4}$$

of KG-modules and a chain of surjective homomorphisms

$$\langle E_n^{(i,r)}\rangle\twoheadrightarrow\langle E_n^{(i,r-1)}\rangle\twoheadrightarrow\cdots\twoheadrightarrow\langle E_n^{(i,2)}\rangle. \tag{4.5}$$

The kernel of the composite $\langle E_n^{(i,r)}\rangle\twoheadrightarrow\langle E_n^{(i,2)}\rangle$ is $\langle\hat{E}_n\rangle$. Since $\langle\tilde{E}_n^{(i,r)}\rangle$ is projective (and hence injective) by (i), (4.4) and (4.5) yield direct decompositions

$$\langle E_n^{(i,r)}\rangle\cong\bigoplus_{j=2}^{r}\langle\tilde{E}_n^{(i,j)}\rangle,\qquad\langle\hat{E}_n^{(i,r)}\rangle\cong\bigoplus_{j=3}^{r}\langle\tilde{E}_n^{(i,j)}\rangle. \tag{4.6}$$

Again, by (i), the $\langle\tilde{E}_n^{(i,j)}\rangle$ in these direct decompositions are projective for the relevant range of n and j. This proves (ii) and (iii). □

The third and final result in this subsection is the following

LEMMA 5. *The subspace* $\langle x_1^p, x_2^p,\ldots,x_p^p\rangle$ *of* $R(J_{i,r})$ *is a projective* KG*-module isomorphic to* $J_{i,p}$.

PROOF. The elements $x_1^p, x_2^p,\ldots,x_p^p$ are linearly independent by Lemma 2 (i). Hence they span a regular KP-module which is generated by x_1^p. As y_1 is an eigenvector for h with eigenvalue l^i, its p-th power $x_1^p=y_1^p$ is also an eigenvector for h with eigenvalue $l^{pi}=l^p$. The result follows now from [4, Lemma 1]. □

5. Proof of Theorem 2

5.1. The proof Consider the free Lie algebra $L(D)$ from Section 3. We follow the strategy outlined in Section 1.5. The first step is a minor intermediate modification of the free generating set D.

STEP 1. *Let D' denote the set obtained from the set D by replacing the elements $[y_1, y_2^{\alpha}, y_1^{p-1-\alpha}] \in D_p$ with $1 \leq \alpha \leq p-1$ by the elements $l_2, l_3, \dots l_p$ from* (4.2). *Then D' is also a free generating set for $L(D)$.*

This follows by a straightforward application of Lemma 1 with $X_i = D_i$ using Lemma 2 (ii).

STEP 2. *Let D^* denote the set obtained from D' by replacing the elements $l_2, l_3, \dots, l_p$ by $x_2^p, x_3^p, \dots, x_p^p$. Then D^* is a free generating set for the (free) Lie algebra it generates.*

Suppose that D^* is not a free generating set. Then there exists a non-trivial linear combination

$$\sum_{i=1}^{m} \alpha_i b_i(D^*) = 0, \tag{5.1}$$

where $\alpha_i \in K$, and the $b_i(D^*)$ are Hall basic commutators in D^*. Recall that $L(D) \subset L(C)$ (see Section 3), and observe that, by construction, the free generators in D are Lie monomials in the free generators of C. Hence every Lie monomial in $L(D)$ has a well defined multidegree with respect to the free generators in C (which have degree 1 in this context). Here we make use of the partial degree in the free generators $y_1^p, y_2^p, \dots, y_r^p \in C$. For $w \in L(C)$ we write

$$\operatorname{Deg} w = \deg_{y_1^p} w + \cdots + \deg_{y_r^p} w$$

for the sum of the partial degrees in those free generators. Note that $\operatorname{Deg} l_i = 0$ since $l_i \in L_p$. Now return to (5.1). We will use the degree Deg to obtain a contradiction. We may assume that the $b_i(D^*)$ involve (some of) the elements $x_2^p, x_3^p, \dots, x_p^p$ since the other elements of D^* are contained in the free generating set D'. Now we replace all entries of $x_2^p, x_3^p, \dots, x_p^p$ in (5.1) by the right hand sides of (4.2), and expand. This results in a linear combination of basic commutators in D' and commutators obtained from those basis commutators by replacing at least one of the entries of $l_2, l_3, \dots, l_p$ by one of $y_1^p, y_2^p, \dots y_r^p$. Since the latter have larger degree Deg than the former, this gives rise to a non-trivial linear combination of basic commutators in D' that is zero. But this is impossible since D' is a free generating set.

STEP 3. (i) $L(D^*) = L_S$. (ii) $L_S = \hat{L}(J_{i,r}) \oplus \langle x_1^p, \ldots, x_p^p \rangle$.

By construction, $L(D^*)$ contains all Hall basic commutators in Y of degrees $2, 3, \ldots, p-1$, all Hall basis commutators in Y of degree p except $[y_1, y_2^\alpha, y_1^{p-1-\alpha}]$ with $1 \leq \alpha \leq p-1$, and also the elements $x_1^p = y_1^p, x_2^p, \ldots x_p^p$. But it also contains all homogeneous components $L_n(J_{i,r})$ with $n > p$ since $|D_n^*| = |D_n|$ for all n, and this clearly implies

$$\dim(L(D^*) \cap R_n(J_{i,r})) = \dim(L(D) \cap R_n(J_{i,r})),$$

but $L(D) \cap R_n(J_{i,r}) = L_n(J_{i,r})$ for $n > p$ (see Section 3). This proves (i). Point (ii) is obvious since the elements $x_1^p, \ldots, x_p^p$ belong to the free generating set D^*.

Our next step consists of a further modification of the free generating set D^*. Consider the elements

$$[y_1, z_2^{\alpha_2}, \ldots, z_{k-1}^{\alpha_{k-1}}, y_1^{p-1}, y_s^{\beta_s p}, y_{s-1}^{\beta_{s-1} p}, \ldots, y_2^{\beta_2 p}] \in D^* \tag{5.2}$$

such that $\beta_s \neq 0$ and $y_s < z_2$, and replace them with

$$\begin{aligned} &[y_1, z_2^{\alpha_2}, \ldots, z_{k-1}^{\alpha_{k-1}}, y_1^{p-1}, y_s^{\beta_s p}, y_{s-1}^{\beta_{s-1} p}, \ldots, y_2^{\beta_2 p}] \\ &\quad - [y_s^p, z_2^{\alpha_2}, \ldots, z_{k-1}^{\alpha_{k-1}}, y_s^{(\beta_s - 1)p}, y_{s-1}^{\beta_{s-1} p}, \ldots, y_2^{\beta_2 p}, y_1^p]. \end{aligned} \tag{5.3}$$

STEP 4. *The set D'' obtained from D^* by replacing the elements* (5.2) *with* (5.3) *is also a free generating set for $L(D^*)$.*

Since the second term in (5.3) is a product of two free generators of $L(D^*)$ of smaller degree (the last factor of the added term is $y_1^p \in D^*$), this follows by a straightforward application of Lemma 1. The reason for this modification will become clear in the next step. Note that all the replaced elements (5.2) are of degree $\geq p+2$, and hence $D_n^* = D_n''$ for all $n \leq p+1$.

Our next goal is to identify the span of the image of the free generating set D^* in the free metabelian Lie algebra $M = M(Y)$. We will use the well-known fact that in M any Lie product $w = [u_1, u_2, u_3, \ldots, u_k]$ with $u_1, \ldots, u_k \in R$ is symmetric with respect to the entries $u_3, \ldots, u_k$. This allows us to permute those entries arbitrarily without changing the value of w. Recall the definition of the set E from Section 4.

STEP 5. *In $M(Y)$, $\langle D_n'' \rangle = \langle E_n \rangle$ for all $n \geq 2$ except $n = p$, where $\langle D_p'' \setminus \{x_1^p, \ldots, x_p^p\} \rangle = \langle \hat{E}_p \rangle$.*

There is nothing to prove for $n \leq p+1$ as the relevant sets coincide for $n \leq p$ and coincide (up to sign) for $n = p+1$. Now let $n \geq p+2$. We show that $\langle E_n \rangle \subseteq \langle D_n'' \rangle$. This is by an easy but tedious case-by-case consideration as follows. Let w be a left normed commutator of the form (4.3), and write $\alpha_i = \beta_i p + \rho_i$ with $0 \le \rho_i < p$, $(i = 2, 3, \dots, k)$. Then

$$w = [z_1, z_2^{\beta_2 p + \rho_2}, \dots, z_k^{\beta_k p + \rho_k}].$$

We will see that w is either itself in $\pm D''$ or a linear combination of two elements from this set. If $\rho_2 \neq 0$, we have

$$w = [z_1, z_2^{\rho_2}, z_3^{\rho_3}, \dots, z_k^{\rho_k}, z_2^{\beta_2 p}, \dots, z_k^{\beta_k p}] \in \pm D''.$$

If $\rho_2 = \dots = \rho_k = 0$, we have

$$w = [z_1, z_2^{\beta_2 p}, \dots, z_k^{\beta_k p}] = -[z_2^p, z_1, z_2^{(\beta_2 - 1)p}, \dots, z_k^{\beta_k p}] \in \pm D''$$

Now suppose that $\rho_2 = \dots = \rho_{s-1} = 0$, but $\rho_s \neq 0$ for some $s \le k$. Then we have

$$w = [z_1, z_2^p, z_s^{\rho_s}, \dots, z_k^{\rho_k}, z_2^{(\beta_2 - 1)p}, \dots, z_k^{\beta_k p}]. \tag{5.4}$$

If $z_1 \le z_s$ in (5.4), we have

$$w = -[z_2^p, z_1, z_s^{\rho_s}, \dots, z_k^{\rho_k}, \dots].$$

This is in $\pm D''$ except when $z_1 = z_s$ and $\rho_s = p - 1$. In the latter case we have

$$w = -[z_2^p, z_1^p, z_{s+1}^{\rho_{s+1}}, \dots, z_k^{\rho_k}, \dots]$$

If $\rho_{s+1} = \dots = \rho_k = 0$, this is in $\pm D''$. If $\rho_{s+1} = \dots = \rho_{t-1} = 0$ but $\rho_t \neq 0$ for some $t \le k$, then we have

$$w = -[z_2^p, z_1^p, z_t, \dots] = [z_1^p, z_t, z_2^p, \dots] - [z_2^p, z_t, z_1^p, \dots].$$

If $z_1 \neq y_1$ both terms in the last line are in $\pm D''$ (as we can move z_2^p in the first term and z_1^p in the second term to the right), and if $z_1 = y_1$, then

$$w = [z_1, z_t, z_1^{p-1}, z_2^p, \dots] - [z_2^p, z_t, z_1^p, \dots]$$

which is of the form (5.3) and hence in $\pm D''$.

Finally, suppose that $z_1 > z_s$ in (5.4). Then

$$w = -[z_2^p, z_s, z_1, \dots] + [z_1, z_s, z_2^p, \dots].$$

If $z_1 = z_k = y_1$ and $\rho_k = p - 1$, this is (up to sign) of the form (5.3), and hence in $\pm D''$, and if this is not the case, then both terms on the right hand side of the last equation are in $\pm D''$. This proves that $\langle E \rangle \subseteq \langle D'' \rangle$. The verification of the inverse inclusion is similar.

STEP 6. *The free Lie algebra* $L_S = L(D'')$ *has a homogenous free generating set* $F = F_2 \cup F_3 \cup \dots$ *such that for all* n, $U_n = \langle F_n \rangle$ *is a* KG*-module. Moreover,* $U_n \cong M_n(J_{i,r})$ *for* $n = 2, \dots, p-1$ *and* U_n *is a projective* KG*-module for* $n \geq p$.

We will apply the corollary to Lemma 1 with $X_i = D''_i$ for $i \geq 2$ and $X_1 = \varnothing$. Then $\Lambda_n = L_n(J_{i,r})$ for all $n \geq 2$ except $n = p$ where $\Lambda_p = \hat{L}_p(J_{i,r}) \oplus \langle x_1^p, \dots, x_p^p \rangle$. In order to apply the corollary we need to verify that the natural surjections $\Lambda_n \twoheadrightarrow \Lambda_n/(\Lambda_n \cap L(< n))$ split. Clearly, $\Lambda_n \cap L(< n) = L_n(J_{i,r}) \cap L''$ for $2 \leq n \leq p+1$. Now, when $2 \leq n \leq p-1$, the elements of D''_n, that is all left normed basic commutators of degree n in Y, generate modulo $L(< n)$ the metabelian Lie power $M_n(J_{i,r})$. The corresponding natural surjection splits by Lemma 3. If $n = p$ we have $D''_p = \hat{E}_p \cup \{x_1^p, \dots, x_p^p\}$. The elements of $\hat{E}_p$ generate modulo $L(< n)$ a projective KG-module by Lemma 4(ii) and the elements $x_1^p, \dots, x_p^p$ generate a projective KG-module by Lemma 5. Thus D''_p generates modulo $L(< p)$ a projective KG-module, and hence the corresponding natural surjection splits. It is plain that we may take it that the splitting map is the identity map on $\langle x_1^p, \dots, x_p^p \rangle$, and that it maps $\hat{E}_p + L(< n)/L(< n)$ into $\hat{L}_p(J_{i,r})$. Finally, if $n > p$, it was established in Step 3 that the image of D''_n under the canonical surjection $L(Y) \to M(Y)$ coincides with the span of E_n in $M_n(Y)$. By Lemma 4(iii), this is a projective KG-module. But then D''_n also generates a projective KG-module modulo $L(< n)$. Hence the natural surjections split for all $n > p$, and consequently the corollary applies to $L(D'')$, thus ensuring the existence of the required sets F_n.

STEP 7. *The* KG*-modules* $U_n = \langle F_n \rangle$ *satisfy all conditions in the statement of Theorem* 2.

All that remains is stocktaking: (i) has been established in Step 6, (ii) and (iii) are obvious consequences of our construction (with $V_p = \langle F_p \setminus \{x_1^p, \dots, x_p^p\} \rangle$), (iv) and (v) are proved in Step 6, and (vi) has been shown in Step 3. This completes the proof of Theorem 2.

5.2. A corollary As mentioned in the introduction, our proof of Theorem 2 yields extra information that will be useful for applications in [7]. For $n \geq 2$ and $r \geq 2$, let $\tilde{M}_n(J_{i,r})$ denote the submodule spanned by the elements of $\tilde{E}_n^{(i,r)}$ (see Section 4.4) in the metabelian Lie power $M_n(J_{i,r})$.

COROLLARY. *For all* $n \geq p+1$, *there are* KG*-isomorphisms*

$$U_n \cong \bigoplus_{j=2}^{r} \tilde{M}_n(J_{i,j}),$$

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and, moreover, in degree p we have

$$U_p = V_p \oplus \langle x_1^p, x_2^p, \dots, x_p^p \rangle \cong \bigoplus_{j=3}^{r} \tilde{M}_p(J_{i,j}) \oplus J_{i,p}.$$

PROOF. The argument in Step 6 of the above proof yields, in fact, isomorphisms

$$U_n \cong \langle E_n \rangle \qquad \text{for } n \geq p+1,$$
$$U_p \cong \langle \hat{E}_p \rangle \oplus \langle x_1^p, x_2^p, \dots, x_p^p \rangle,$$

where the spans of E_n and $\hat{E}_p$ are understood as taken in $M(J_{i,r})$. The result now follows immediately from (4.6) and Lemma 5. □

Acknowledgement

This work was carried out while the author was visiting the Australian National University, Canberra. The author is grateful to the ANU and his host L. G. Kovács for their great hospitality. Financial support from the EPSRC (Overseas Travel Grant GR/M93130) is also gratefully acknowledged.

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https://doi.org/10.1017/S1446788700002901 Published online by Cambridge University Press