On incorporating diffusion and viscosity concepts into compartmental models for analysing faecal marker excretion patterns in ruminants

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Deterministic mathematical equations are derived to describe the pattern of marker excretion in the faeces of ruminants under steady-state conditions when diffusion and viscosity concepts are introduced into a simple two-compartment scheme of the gastrointestinal tract. The basic scheme comprises a pure-mixing pool obeying first-order kinetics and a second compartment exhibiting streamline flow. Introduction of a velocity gradient, longitudinal diffusion or both into the second compartment, even with various simplifying assumptions, yields analytically insoluble equations. The impact of these mechanisms is to be investigated numerically rather than analytically in future work.

Diffusion: Viscosity: Compartmental models: Faecal marker excretion patterns: Ruminants

1. INTRODUCTION

Faecal marker concentration curves, that is plots of concentration (mg marker per g faecal dry matter (DM)) *v.* time (h), are constructed from grab or bulked faecal samples taken at different times folowing single-dose infusion of an indigestible, non-absorbable marker such as ruthenium phenanthroline or ytterbium acetate. The marker is normally administered directly into the rumen or orally, and is assumed to behave ideally (for basic terminology, see France & Siddons, 1986). Compartmental analysis is then used to interpret the concentration data and a number of models of the ruminant gastrointestinal (GI) tract have been proposed for this very purpose, each of which is a sequential, irreversible, multicompartment scheme (for reviews, see France *et al.* 1985, 1988). This methodological approach permits estimation of biological measures such as rate of passage and retention time in the rumen, transit time in the GI tract, and rate of faecal production.

The models proposed generally assume first-order kinetics and discrete time lags. Implicit in these assumptions is that the GI tract can be adequately represented by a series of pure-
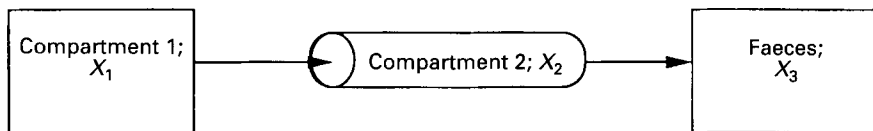
Fig. 1. Two compartment representation of the ruminant gastrointestinal tract. Boxes, pure-mixing pools; cylinder, streamline flow. The faeces are external to the tract.

mixing pools and simple plug flows. However, such a representation is an over-simplification mechanistically since diffusion and viscosity are aspects of flow that are always present, although they may not be important in particular situations. The fact that a number of different models have been used to analyse digesta flow and none found satisfactory in all situations suggests their limitations. The principal objective of the present paper, therefore, is to introduce diffusion and viscosity into this formalism so that their possible roles can be examined, and the limitations of the simple models based on first-order kinetics and discrete time lags may be better understood. Such considerations are now timely, given recent advances in statistical software which facilitate fitting transcendental functions such as the error function and Bessel functions to experimental data.

The basis of the analysis presented herein is a two-compartment scheme for the tract comprising one pure-mixing pool followed by a steamline flow representing events in the proximal and distal tract respectively. In the analysis, attention is focused on the second compartment of the scheme. The solutions obtained by varying the physical flow properties of this compartment are also attempted by treating passage as a stochastic process and convoluting the residence time distributions for the two sections of the tract.

2. COMPARTMENTAL SCHEME

The scheme, shown in Fig. 1, represents perhaps the simplest mechanistically-correct compartmentalization of the ruminant GI tract. The assumption of a mixing pool would seem reasonable for the proximal tract which includes the rumen, as does the assumption of streamline flow for the distal tract which includes the intestines. For this scheme no attempt is made to nominate the sections of the tract more specifically. The third compartment, which is faecal, is taken to be a mixing pool external to the tract. The variables X_1, X_2 and X_3 represent the amounts (mg) of a non-absorbable, indigestible, digesta-flow marker in the compartments concerned at time t (h) post dosing. When $t = 0$ all the marker is in the first compartment and when $t = \infty$ it is all in the faeces, as the marker is administered as a single dose Δ (mg) into the rumen. A fractional rate-constant k (/h) is ascribed to passage out of the first compartment, as efflux from a pure-mixing pool obeys linear kinetics. Diffusion and viscosity concepts are introduced into this scheme by considering four cases for describing flow through the second compartment, namely streamline flow with (section 3) no diffusion and no velocity gradient, (section 4) diffusion but no velocity gradient, (section 5) a velocity gradient but no diffusion, (section 6) diffusion and a velocity gradient.

3. STREAMLINE FLOW WITH NO DIFFUSION AND NO VELOCITY GRADIENT

In this case movement along the distal tract is a simple plug flow. Although aspects of the case have been analysed by Ellis *et al.* (1979), Krysl *et al.* (1985) and others, the complete

analysis is given here *ab initio* in order to establish a consistent and workable formalism for subsequent consideration of diffusion and viscosity.

Let the lag T (h) represent the constant transit time between compartment 1 and faeces. Application of the principle of mass conservation gives the following linear differential equations describing the dynamic behaviour of the system:

$$dX_1/dt = -kX_1, \tag{3.1}$$

$$dX_3/dt = 0, 0 \leqslant t < T, \tag{3.2a}$$

$$= kX_1(t-T), t \geqslant T. \tag{3.2b}$$

Solving equations 3.1 and 3.2 analytically subject to the initial conditions $X_1 = \Delta$ and $X_3 = 0$ gives:

$$X_1 = \Delta e^{-kt}, \tag{3.3}$$

$$X_3 = 0, 0 \leqslant t < T, \tag{3.4a}$$

$$= \Delta(1 - e^{-k(t-T)}), t \geqslant T. \tag{3.4b}$$

On substituting for X_1 in equation 3.2b by using equation 3.3, the rate of appearance of marker in the faeces (mg marker/h) becomes:

$$dX_3/dt = 0, 0 \leqslant t < T, \tag{3.5a}$$

$$= k\Delta e^{-k(t-T)}, t \geqslant T. \tag{3.5b}$$

Assume the rate of production of faeces, F (g DM/h), is constant and let the instantaneous faecal marker concentration be C_3 (mg marker/g DM faeces). Dividing equations 3.5a and 3.5b by F then yields:

$$C_3 = (dX_3/dt)/F = 0, 0 \leqslant t < T, \tag{3.6a}$$

$$= kAe^{-k(t-T)}, t \geqslant T, \tag{3.6b}$$

where A is the area under the marker concentration curve and is equal to Δ/F.

An expression for marker appearance in the faeces can also be obtained by considering transit through the GI tract as a two-stage stochastic process. Let the residence time (h) in the first compartment (proximal tract) be exponentially distributed (mean = SD = k^{-1}) with probability density function:

$$f_1(t) = ke^{-kt}, t \geqslant 0, \tag{3.7}$$

and residence in the second compartment (distal tract) follow a Dirac delta distribution (mean = T) with probability density function:

$$f_2(t) = \delta(t-T), t \geqslant 0. \tag{3.8}$$

Here, δ is the Dirac delta function:

$$\delta(t-T) = \infty, t = T, \tag{3.9a}$$

$$= 0, \text{otherwise.} \tag{3.9b}$$

This generalized function represents a spike of infinite height and infinitesimal width at $t = T$, so that the area under the spike is unity, i.e.

$$\int_0^\infty \delta(t-T)\,dt = 1, \tag{3.10}$$

δ operates so as to select the value of a function $g(t)$ at the point $t = T$:

$$\int_0^\infty g(t)\,\delta(t-T)\,dt = g(T). \tag{3.11}$$

Knowing f_1 and f_2, the probability density function of transit time through the GI tract can be obtained by convoluting f_1 and f_2:

$$f(t) = \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau. \tag{3.12}$$

Using equations 3.7 and 3.8 to substitute for f_1 and f_2 respectively in equation 3.12 gives:

$$f(t) = k \int_0^t e^{-k\tau} \delta(t-\tau-T)\, d\tau. \tag{3.13}$$

Applying equation 3.11 yields:

$$f(t) = ke^{-k(t-T)}, \tag{3.14}$$

provided $t \geqslant T$. The cumulative distribution function for transit time through the whole tract is, therefore:

$$\Phi(t) = \int_T^t f(t)\, dt = 1 - e^{-k(t-T)}. \tag{3.15}$$

Thus, the probability of a particle of marker being excreted in the faeces within t h of dosing is $1 - e^{-k(t-T)}$. Note that, writing $X_3/\Delta \equiv \Phi(t)$, equation 3.15 is identical to equation 3.4b, and these two formulations of the problem are mathematically equivalent. The mean transit time through the tract is given by:

$$\int_T^\infty tf(t)\, dt = 1/k + T. \tag{3.16}$$

The probability density function $f(t)$ can be obtained experimentally by scaling the faecal marker concentration curve C_3 by A, the total area under the curve:

$$f(t) = C_3/A. \tag{3.17}$$

This equation is identical to equation 3.6b.

The parameters k, A and T of equation 3.6b are estimated by fitting a negative exponential equation to the faecal marker concentration data using non-linear least-squares. These may then be used to derive a number of useful biological measures. Mean retention time (MRT) in pool 1 (which might be interpreted as the rumen) is given by k^{-1} (h); transit time through the GI tract by $k^{-1} + T$ (h); total marker clearance time by $3k^{-1} + T$ (h); and the rate of faecal production by the dose Δ over A. The clearance time calculation assumes the range of transit times following an exponential distribution (SD = mean) is within 3 SD (i.e. probability (transit time $\leqslant$ 3 SD) = 97·5 %).

4. STREAMLINE FLOW WITH LONGITUDINAL DIFFUSION BUT NO VELOCITY GRADIENT

We next give an approximate treatment which, amongst other things, partially ignores back-diffusion, assumes that the distal tract is long and that its contents are homogeneous, and also that diffusion is slow compared with convective flow. There is evidence in the literature that liquid and particulate markers do not behave very differently post-ruminally and therefore that the digesta behaves as a relatively homogeneous mixture (Grovum & Williams, 1973; Faichney, 1975).

Consider one-dimensional diffusion, with constant diffusion coefficient D (cm²/h), in an isotropic, homogeneous medium. The continuity equation is given by Fick's second law:

$$\partial C/\partial t = D\partial^2 C/\partial x^2, \tag{4.1}$$

where C (mg/cm^3) is the concentration of the diffusing substance, x (cm) the space co-ordinate and t (h) is time.

The diffusion coefficient D of a sphere of radius a is given by $D = (\mathscr{R}T/N)/(6\pi\eta a)$, where $\mathscr{R}$ is the gas constant, T is the absolute temperature, N is Avogrado's number, and η is the dynamic viscosity (Mahler & Cordes, 1966). The viscosity of water at 37° is 0·0006928 kg/m per s (Diem & Lentner, 1970). Transforming to present units, therefore, D (water, 37°) $= 1·180 \times 10^{-9}/a$ cm^2/h, where a is expressed in cm. A particle of $a = 0·1$ cm has $D = 1·180 \times 10^{-8}$ cm^2/h, and this value can be easily scaled for larger or smaller particles. For comparison we note that some measured values (Diem & Lentner, 1970) are D (albumin, water, 20°) $= 2·124 \times 10^{-3}$ cm^2/h and D (sucrose, water, 20°) $= 0·01872$ cm^2/h. In time $t = 10$ h the distance travelled by diffusion by sucrose is $Dt^{1/2}$, giving 0·43 cm. This is small, though not insignificant, compared with typical distances travelled by convective flow of a few metres per hour. Time taken to traverse the intestines of sheep (roughly 50 m) is 9–26 h (Grovum & Williams, 1973). Using the above 'spherical particle' formula, this gives a (albumin) $= 5·6 \times 10^{-7}$ cm and a (sucrose) $= 6·3 \times 10^{-8}$ cm, which are reasonable values.

A solution to equation 4.1, which can be verified by differentiation, is

$$C(x,t) = K\exp[-x^2/(4Dt)]/t^{1/2}, \tag{4.2}$$

where K is a constant. Let the medium be bounded by an infinite cylinder ($-\infty < x < \infty$) of radius R (cm) and consider an instantaneous plane source of strength ψ (mg) at $x = 0$ and $t = 0$, then:

$$\psi = \int_{-\infty}^{\infty} C(x,t)\,\pi R^2 dx. \tag{4.3}$$

Using equation 4.2 to substitute for C in equation 4.3 and rearranging gives an expression for the arbitrary constant:

$$K = \psi t^{1/2}/[\pi R^2 \int_{-\infty}^{\infty} \exp[-x^2/(4Dt)]\,dx], \tag{4.4a}$$

$$= \psi/[2\pi R^2 D^{1/2} \int_{-\infty}^{\infty} \exp(-z^2)\,dz], \tag{4.4b}$$

$$= \psi/[2\pi^{3/2} R^2 D^{1/2} \operatorname{erf}(\infty)], \tag{4.4c}$$

$$= \psi/(2\pi^{3/2} R^2 D^{1/2}), \tag{4.4d}$$

where erf (x) is the error function and is equal to $2\int_0^x \exp(-z^2)\,dz/\pi^{1/2}$.

Substituting for K in equation 4.2 gives

$$C(x,t) = \psi \exp[-x^2/(4Dt)]/(2\pi^{3/2} R^2 D^{1/2} t^{1/2}). \tag{4.5}$$

For a continuous plane source of strength $\phi(t)$ (mg/h) at $x = 0$, equation 4.5 becomes:

$$C(x,t) = \int_0^t \phi(\tau)\exp\{-x^2/[4D(t-\tau)]\}/[2\pi^{3/2} R^2 D^{1/2}(t-\tau)^{1/2}]d\tau. \tag{4.6}$$

If the diffusion takes place within a uniform stream of velocity U (cm/h) flowing in the x-direction then equation 4.6 is modified to:

$$C(x,t) = \int_0^t \phi(\tau)\exp\{-[x-U(t-\tau)]^2/[4D(t-\tau)]\}/[2\pi^{3/2} R^2 D^{1/2}(t-\tau)^{1/2}]\,d\tau. \tag{4.7}$$

To incorporate this analysis within the model of the GI tract (Fig. 1), the distal tract is assumed to be an open-ended impermeable hollow tube of length L (cm) and radius R (cm).

Stream flow is in the positive x-direction. The transit time for material (i.e. whole digesta) flowing through the tract (in absence of any diffusion) is T (h), i.e. $U = L/T$. Marker (the diffusing substance) enters from compartment 1 at the extremity $x = 0$ and leaves for compartment 3 at $x = L$. Its rate of entry is assumed to be (convective transfer only)

$$\phi(t) = kX_1, t \geqslant 0, \tag{4.8a}$$

$$= k\Delta e^{-kt} \tag{4.8b}$$

(see equations 3.1 and 3.3). We assume that equation 4.7 can be applied in compartment 2. Substituting for ϕ gives marker concentration in compartment 2 as

$$C(x,t) = \int_0^t k\Delta e^{-k\tau} \exp\{-[x - L(t-\tau)/T]^2/[4D(t-\tau)]\}/$$

$$[2\pi^{3/2}R^2 D^{1/2}(t-\tau)^{1/2}]\,d\tau, 0 \leqslant x \leqslant L, \quad (4.9)$$

and $C(L,t)$ gives the time course of marker leaving the compartment. The instantaneous concentration of marker appearing in faeces, C_3 (mg/g DM faeces), is, therefore:

$$C_3 = C(L,t)/\rho, \tag{4.10a}$$

$$= \int_0^t k\Delta e^{-k\tau} \exp\{-[L - L(t-\tau)/T]^2/[4D(t-\tau)]\}/[2\pi^{3/2}R^2 D^{1/2}(t-\tau)^{1/2}]\,d\tau/\rho,$$

$$(4.10b)$$

where ρ (g DM/cm^3) is faecal density. The rate of production of faeces can be calculated by dividing the pulse dose by the area under the faecal marker concentration curve:

$$F = \Delta/\int_0^\infty C_3\,dt. \tag{4.11}$$

Equation 4.10 b can also be derived by considering transit through the GI tract as a two-stage stochastic process. Let residence time (h) in the first compartment (proximal tract) be exponentially distributed (parameter k) with probability density function

$$f_1(t) = ke^{-kt}, t \geqslant 0, \tag{4.12}$$

and residence time in the second compartment (distal tract) follow a distribution whose probability density function is given by

$$f_2(t) = a\exp[-b(t-c)^2/t]/t^{1/2}, t \geqslant 0, \tag{4.13}$$

where a, b and c (all > 0) are parameters. Equation 4.13, as far as we are aware, represents no known statistical distribution though a cursory inspection suggests the lognormal, gamma or chi-squared might provide suitable approximations. The probability density function for time through the whole tract is obtained by convoluting f_1 and f_2:

$$f(t) = \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau, \tag{4.14a}$$

$$= \int_0^t kae^{-k\tau}\exp[-b(t-\tau-c)^2/(t-\tau)]/(t-\tau)^{1/2}\,d\tau, \tag{4.14b}$$

$f(t)$ can be obtained experimentally by scaling the faecal marker concentration curve C_3 by A, the total area under the curve. Therefore,

$$C_3 = Af(t), \tag{4.15a}$$

$$= A\int_0^t kae^{-k\tau}\exp[-b(t-\tau-c)^2/(t-\tau)]/(t-\tau)^{1/2}\,d\tau. \tag{4.15b}$$

Equations 4.15*b* and 4.10*b* are equivalent.

The diffusion model developed in this section suffers the limitation that equation 4.10*b* cannot be integrated analytically. Consequently, the biological measures of rate of passage and retention time in the first compartment, transit time through the whole tract and rate of faecal production cannot be calculated following a non-linear regression analysis of faecal marker concentration data, unlike in the previous section.

5. STREAMLINE FLOW WITH A VELOCITY GRADIENT BUT NO DIFFUSION

In order to introduce a velocity gradient into the formalism (Fig. 1), we treat whole digesta moving along compartment 2 (the distal tract) as an homogeneous fluid moving slowly, so that there is no turbulence, through an open-ended impermeable hollow tube of length L (cm) and radius R (cm). The Reynolds number Re is defined by $Re = v\rho d/\eta$, where v (m/s) is speed, ρ (kg/m^3) is density, d (m) is the dimension associated with the system, and η (kg/m per s) is the viscosity. The Reynolds number represents impinging fluid momentum ρv^2 per unit area and time: the frictional viscous force $(\eta v/d)$ per unit area which balances the fluid momentum. The value of Re determines the nature of the flow pattern. If Re is less than about 2100, the flow is streamline; above this value there are eddies and vortices; above about 3000 there is steady turbulent flow (Fishenden & Saunders, 1950). To estimate Re for the distal tract, we take v 0·001 m/s ($\equiv$ 3·6 m/h), ρ 1000 kg/m^3, d 0·01 m, and η 0·001 kg/m per s, giving Re 10. Therefore, streamline flow is predicted, although this ignores the effects of peristalsis and cilia in promoting mixing and turbulence.

Laminar flow is assumed in this analysis. We are unaware of direct evidence to support such an assumption though such evidence is well documented in relation to blood flow (e.g. Nubar, 1971). Digesta in contact with the wall of the compartment clings to it and is at rest. The thin, cylindrical layer of digesta adjacent to this stationary layer moves very slowly, and successive thin layers move at increasing velocities. Hence, digesta at the centre has the maximum velocity U. The velocity u (cm/h) of an elemental cylindrical layer of digesta of radius r and thickness $2\pi rdr$ is given by (Newman & Searle, 1948):

$$u = U(1 - r^2/R^2). \tag{5.1}$$

Thus, digesta from this elemental layer leaves compartment 2 through a ring of area $2\pi rdr$ at a rate u. Therefore, the total rate at which digesta is voided into compartment 3, F (g DM faeces/h), is:

$$F = \rho \int_0^R 2\pi rudr, \tag{5.2}$$

where ρ (g DM/cm^3) is faecal density. Using equation 5.1 to substitute for u in equation 5.2 and integrating yields:

$$F = \rho \pi R^2 U/2. \tag{5.3}$$

Marker enters compartment 2 from compartment 1 and travels a distance L before being voided into compartment 3 along with digesta. The rate of marker entry is $k\Delta e^{-kt}$, $t \geq 0$ (see equations 3.1 and 3.3). If V_1 (cm^3) is the volume of compartment 1 and as faecal production is constant (equation 5.3), then:

$$k = \pi R^2 U/(2V_1). \tag{5.4}$$

The time taken, T (h), for a particle of marker to traverse the compartment (of length L) depends on the velocity of the stream (elemental cylindrical layer) in which the particle travels, i.e.

$$T = L/u. \tag{5.5}$$

Using equation 5.1 to substitute for u in equation 5.5 yields:

$$T = L/[U(1-r^2/R^2)].\qquad(5.6)$$

We note that $T_{\min} = L/U$ (when $r = 0$) gives the fastest transit time through compartment 2.

The concentration of marker in the digesta leaving compartment 2 through the elemental ring $2\pi r dr$ at time t ($\geqslant T$) equals the concentration of marker entering the compartment through the ring at time $t-T$ (i.e. $\Delta e^{-k(t-T)}/V_1$). The flux of marker leaving compartment 2 at time t ($\geqslant T_{\min}$) is obtained by integrating over the elemental rings. Dividing this flux by the rate of faeces production F gives the instantaneous concentration of marker in the faeces:

$$C_3 = 0, 0 \leqslant t < L/U,\qquad(5.7a)$$

$$C_3 = \int_0^{R^*(t)} (\Delta e^{-k(t-T)}/V_1)\, 2\pi r u\, dr/F, t \geqslant L/U,\qquad(5.7b)$$

where $R^*(t)$ denotes the upper limit on r at time t and is given by $R^* = R\sqrt{[1-L/(Ut)]}$. Therefore (with equations 5.1 and 5.6):

$$C_3 = 2\pi U\Delta e^{-kt} \int_0^{R^*(t)} r(1-r^2/R^2)\exp\{kL/[U(1-r^2/R^2)]\}\,dr/(FV_1), t \geqslant L/U.\qquad(5.8)$$

Unfortunately, the integral contained in equation 5.8 is non-analytical.

Equation 5.8 apparently cannot be derived by considering transit through the GI tract as a two-stage stochastic process. As for the diffusion model (section 4: pp. 372–375), biological measures such as mean retention time in the first compartment cannot be calculated for the viscosity model following non-linear regression analysis of faecal marker concentration data. The rate of faecal production can, however, be determined numerically by dividing the pulse dose by the area under the faecal marker concentration curve (equation 4.11).

6. STREAMLINE FLOW WITH LONGITUDINAL DIFFUSION AND A VELOCITY GRADIENT

Here we combine the considerations of longitudinal diffusion and viscosity of the previous two sections without further biological justification. Consider one-dimensional diffusion in the x-direction with constant diffusion coefficient D as in section 4 (pp. 372–375) and, as in section 4, our treatment is approximate. The solution to the continuity equation (equation 4.1) is given by equation 4.2. Again, let the medium be bounded by an infinite cylinder of radius R, with an instantaneous plane source of strength ψ (mg) at $x = 0$ and $t = 0$. Now consider the diffusion along a cylindrical layer of the medium of radius r and thickness $2\pi r dr$. The effective strength of the source at $x = 0$ for this elemental layer is $2rR^{-2}\psi dr$. The constant K in the concentration equation for the layer (equation 4.2) becomes:

$$K = 2r\psi dr/(2\pi^{3/2}R^4 D^{1/2}),\qquad(6.1)$$

(cf. equation 4.4d). Substituting for K in the concentration equation (equation 4.2) gives:

$$C(x,t) = 2r\psi \exp[-x^2/(4Dt)]\,dr/(2\pi^{3/2}R^4 D^{1/2} t^{1/2}).\qquad(6.2)$$

For a continuous plane source of strength $\phi(t)$ (mg/h) at $x = 0$, equation 6.2 describing concentration in the layer becomes:

$$C(x,t) = \int_0^t 2r\phi(\tau)\exp\{-x^2/[4D(t-\tau)]\}\,dr/[2\pi^{3/2}R^4 D^{1/2}(t-\tau)^{1/2}]\,d\tau.\qquad(6.3)$$

If the medium contained by this elemental layer flows, in the x-direction, with uniform velocity u, then equation 6.3 is modified to:

$$C(x,t) = \int_0^t 2r\phi(\tau) \exp\{-[x-u(t-\tau)]^2/[4D(t-\tau)]\}\, dr/[2\pi^{3/2}\, R^4\, D^{1/2}(t-\tau)^{1/2}]\, d\tau. \quad (6.4)$$

If the flow of the medium is laminar then u is given by equation 5.1. Substituting for u in equation 6.4 using 5.1 and summing over all elemental layers gives the concentration equation for the whole medium:

$$C(x,t) = \int_0^R \int_0^t 2r\phi(\tau) \exp\{-[x-U(1-r^2/R^2)(t-\tau)]^2/[4D(t-\tau)]\}/$$
$$[2\pi^{3/2}\, R^4\, D^{1/2}(t-\tau)^{1/2}]\, d\tau\, dr. \quad (6.5)$$

This analysis can be incorporated into the model of the GI tract (Fig. 1) using the arguments presented earlier in section 4 (p. 374; i.e. equations 4.8–4.10). The instantaneous concentration of marker appearing in faeces, C_3, now becomes:

$$C_3 = C(L,t)/\rho, \quad (6.6a)$$
$$= \int_0^R \int_0^t 2rk\Delta e^{-k\tau} \exp\{-[L-U(1-r^2/R^2)(t-\tau)]^2/[4D(t-\tau)]\}/$$
$$[2\pi^{3/2}R^4\, D^{1/2}(t-\tau)^{1/2}]\, d\tau\, dr/\rho \quad (6.6b)$$

where ρ is faecal density, k is the fractional rate-constant for passage out of the first compartment, Δ is the single dose of marker applied to the first compartment at time zero, and L is the length of the distal tract. Unfortunately, the double integral in equation 6.6b is non-analytical.

Equation 6.6b seemingly cannot be derived by considering transit through the GI tract as a two-stage stochastic process. As for the separate diffusion (section 4, pp. 372–375) and velocity-gradient (section 5, pp. 375–376) models, biological measures such as mean retention time in the first compartment cannot be calculated for this combined model following non-linear regression analysis of faecal marker concentration data, though the rate of faecal production can be determined numerically by dividing pulse dose by area under the marker concentration curve.

DISCUSSION

Compartmental analysis is frequently used to interpret faecal marker excretion patterns in ruminants, and a number of models of the GI tract have been employed for the purpose. The models utilized are usually deterministic ones based on first-order kinetics and discrete time lags, each yielding a simple system of linear differential equations which can be solved analytically to provide a functional relationship describing the time-course of marker appearance in faeces. Estimates of model parameters are derived by fitting the functional relationship, usually using non-linear least squares, to the experimentally-determined faecal marker excretion pattern. The parameter estimates so obtained permit determination of biological measures such as rate of passage and retention time in the rumen, transit time in the GI tract, and the rate of faecal production.

Implicit in the assumptions underlying these models is that the ruminant GI tract can be adequately represented by a series of pure-mixing pools and simple plug flows. This is undoubtedly a simplification and in the present paper more advanced aspects of flow are incorporated into the formalism for the first time. This is done by introducing diffusion and viscosity concepts into the second compartment of a two-compartment scheme comprising a pure-mixing pool followed by a streamline flow representing events in the proximal

and distal lumen respectively. The scheme chosen represents possibly the simplest mechanistically-correct compartmentalization of the ruminant GI tract.

Diffusion concepts are introduced into the formalism by superimposing simple one-dimensional diffusion on convective flow in the second compartment. The mathematical treatment employed is approximate for reasons of pragmatism, assuming the distal tract is long and diffusion is slow compared with convective flow, and partially ignoring back-diffusion. Unfortunately, even this degree of simplification results in an analytically insoluble model, yielding an expression for the time-course of marker appearance in faeces which involves the integral of a combination of two exponentials and an algebraic function (equation 4.10b). Consequently, biological measures (e.g. rate of passage and retention time in the first compartment, transit time through the whole tract, rate of faecal production) cannot be calculated following a non-linear regression analysis of faecal marker concentration data, unlike with the simpler deterministic models based on first-order kinetics and discrete time lags that are currently used in practice.

Viscosity concepts are introduced by applying a velocity gradient to flow in the second compartment. Laminar flow is assumed. In the absence of diffusion, this leads once more to an analytically insoluble model resulting in an expression for the time-course of marker appearance in faeces that incorporates the integral of a product of an exponential and an algebraic function (equation 5.8). Superimposing diffusion on the velocity gradient also leads to analytically intractable mathematics (equation 6.6b). Thus, it is our intention to investigate the impact of these mechanisms numerically, rather than analytically, in further work.

An alternative to the deterministic approach to digesta-flow problems is one based on stochastic assumptions. In the present paper, we illustrate the treatment of passage as a stochastic process by convoluting the distributions of residence time in the proximal and distal tract. Ellis *et al.* (1979) used this approach, but France *et al.* (1985) showed that the results of Ellis *et al.* (1979) could be obtained more simply using a mechanistic deterministic model. Thus, it is quite possible that more advanced aspects of flow such as diffusion and viscosity, treated deterministically, account for some of the digesta-flow phenomena reported, without recourse to stochastic assumptions.

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