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Overconvergent Families of Siegel–Hilbert Modular Forms

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Abstract. We construct one-parameter families of overconvergent Siegel–Hilbert modular forms. This result has applications to the construction of Galois representations for automorphic forms of non-cohomological weights.

1 Introduction

The study of p-adic families of automorphic forms has been carried out in many works. In the case of elliptic modular forms, the overconvergent modular eigenforms of *finite slope* (*i.e.*, with non-zero Hecke eigenvalue at p) are interpolated to be points on a rigid analytic curve, which is known as the Coleman–Mazur eigencurve [CM]. Before this seminal work, the family of ordinary eigenforms was obtained by Hida [Hi86].

Among all the approaches to the construction of eigenvarieties for more general algebraic groups, the work of Kisin–Lai [KL] on overconvergent Hilbert modular forms is most closely related to ours. Their method is a generalization of that of Coleman-Mazur. In both cases, the key point for interpolating modular forms is the *complete continuity* (*cf.* [Co, p. 425] for definition) of the Atkin–Lehner operator on certain spaces of overconvergent forms. In the case of elliptic modular forms, Coleman–Mazur interpolate modular forms by twisting by p-adic analytic families of Eisenstein series. However, in the more general (Siegel–)Hilbert modular case such a theory of Eisenstein series is not yet available. Instead, we lift (a certain power of) the Hasse invariant in characteristic p to be a global section of certain automorphic line bundle over the integral model of the Shimura variety.

We would like to mention certain differences between our method and that of [KL], which are mainly caused by the generality of the Siegel–Hilbert moduli space.

In the Hilbert modular case of [KL], they glue the toroidal compactfication of the Rapoport model [Ra] with the Deligne–Pappas model [DP], because the Rapoport model may not be proper at the places which are ramified in the totally real field. Fortunately, Rapoport's toroidal compactification can be used because the Lie algebra condition, which causes non-properness at finite distance, is automatic in the boundary. In the Siegel–Hilbert case, we have to do more to take care of the rami-

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fied places. There exists the canonical integral model of Pappas–Rapoport [PR] in the Siegel–Hilbert modular case, which has a moduli interpretation. Its toroidal compactifications are, however, not completely understood. Fortunately, the (partial) toroidal compactifications and minimal compactification of the ordinary locus is constructed in [La12] successfully, which will be enough for our use.

Furthermore, we follow the idea of Hida [Hi02] to form the formal Igusa tower over the formal completion of the (compactified) moduli space with level structure away from p, instead of using the "unramified $\Gamma_{00}(Np^n)$ cusps" in [KL]. This seems more convenient in the general Siegel–Hilbert case.

Write $G = \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\,\mathrm{GSp}_{2g}$. The moduli space above is actually for its subgroup $G' = G \times_{\mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_m} \mathbf{G}_m$. Finally, with these strategies and results, we construct one-dimensional families of eigenforms on G', for any totally real field F and $g \geq 1$. More precisely, we obtain, for each classical weight κ, a reduced rigid analytic curve $\mathcal{E}_\kappa$, whose points are in one-to-one correspondence with systems of Hecke eigenvalues of overconvergent automorphic forms on G', whose weights "differ" from that of κ by parallel weights. One of the key properties of the rigid curve $\mathcal{E}_\kappa$ is that the canonical map to the weight space given by weights of modular forms is, locally in the domain, finite flat. We refer the reader to Theorem 4.13 for more details.

Essentially due to the (local) finite flatness of the weight map on $\mathcal{E}_\kappa$ (and the argument in Section 4.4), we have the following theorem.

Theorem 1.1 (See Theorem 4.16) *Let f be a (classical) Siegel–Hilbert modular eigenform on $G = \mathrm{Res}_{\mathcal{O}_F/\mathbf{Z}}\,\mathrm{GSp}_{2g}$ of weight κ with some tame level and level p^n at p. For any positive integers t with large enough p-adic valuation, there exist Siegel–Hilbert modular eigenforms f_t of the same level and of varying weights, whose Hecke eigenvalues converge p-adically to that of f when t goes to zero p-adically.*

This theorem is sufficient for some applications. For example, Theorem 1.1 is one of the main ingredients for attaching Galois representations to automorphic forms π on GL_2 over arbitrary CM fields, as seen in [Mo]. More precisely, in order to construct such a 2-dimensional representation, we first lift π to an automorphic form Π (of non-cohomological type!) on $\mathrm{GSp}_4(\mathbf{A}_F)$. Then the Galois representation ρ_Π associated to Π is obtained by interpolating Galois representations associated to forms on $\mathrm{GSp}_4(\mathbf{A}_F)$ of cohomological type, with the family of cohomological forms supplied by Theorem 1.1. As is mentioned in [Mo], the use of p-adic analytic family of automorphic forms, compared to the use of congruence relations between them, has the advantage that this (less elementary) method allows us to prove local-global compatibility.

We would like to mention that the eigenvariety for Siegel cuspidal eigenforms (not necessarily with parallel weights), *i.e.*, for the group $\mathrm{GSp}_{2g/\mathbf{Q}}$, was recently developed in [AIP]. More recently, the eigenvariety for the group $\mathrm{GL}_{2/F}$ was developed in [AIP2].

The paper is organized as follows.

In Section 2 we recall the results on integral models of PEL Shimura varieties and their compactifications. In the next section, we use the idea of Hida to form the formal Igusa tower. In the last section, we form the spaces of overconvergent

Siegel–Hilbert modular eigenforms and then prove that the $U_{(p)}$-operator is completely continuous on the spaces. Finally by the machinery in [CM], we construct the rigid curves interpolating these overconvergent forms, and then show Theorem 1.1.

Notation

- F is a totally real field of degree d over $\mathbf{Q}$, and $\mathcal{O} = \mathcal{O}_F$ is the ring of integers. We denote by $\mathbf{A} = \mathbf{A}_F$ the ring of adeles of F, and by $\mathbf{A}_f$ the ring of finite adeles of F.
- For a maximal torus $T = T_G$ of a reductive group G over $\mathbf{Z}$, $\mathrm{Nm}\colon \mathrm{Res}^{\mathcal{O}}_{\mathbf{Z}} T \to T$ is the norm map, *i.e.*, for any ring R, $\mathrm{Nm}(R)\colon T(\mathcal{O} \otimes_{\mathbf{Z}} R) \to T(R)$ is given by the norm $N_{F/\mathbf{Q}}$ on F.
- $p \geq 2$ is a fixed rational prime.
- If $K/\mathbf{Q}_p$ is a finite extension, K_0 is the maximal unramified extension of $\mathbf{Q}_p$ in K, and $[K : K_0] = e$. $\bar{\mathbf{Q}}_p$ is a fixed algebraic closure of K, and $\mathbf{C}_p$ is the completion of $\bar{\mathbf{Q}}_p$ for the p-adic topology.
- Let $H \subset G(\mathbf{A}_f)$ denote an open compact subgroup which is of the form $H = H_p H^p$, where $H_p \subset G(\mathbf{Q}_p)$, $H^p \subset G(\mathbf{A}_f^p)$, for $\mathbf{A}_f^p$ the ring of finite adeles over F with trivial p-component.

2 Siegel–Hilbert Moduli Spaces

2.1 PEL Datum

2.1.1 The General Integral PEL Data

Recall the (integral) PEL datum $(\mathcal{O}_B, *, L, \psi, h)$, whose rational part $(B, *, L_{\mathbf{Q}}, \psi_{\mathbf{Q}}, h)$ can give rise to a Shimura datum by 4.1 [Ko].

- B is a finite dimensional semisimple $\mathbf{Q}$-algebra whose center is denoted by F, and is equipped with a positive involution $*$:
$$(ab)^* = b^* a^*, b^{**} = b, \quad \forall a, b \in B,$$
$$\mathrm{Tr}_{B/\mathbf{Q}}(bb^*) > 0, \quad \forall b \neq 0.$$
$\mathcal{O}_B$ is an order of B stabilized by the involution above.
- (L, ψ) is a symplectic $(\mathcal{O}_B, *)$-module over $\mathbf{Z}$, *i.e.*, L is a finite free $\mathbf{Z}$-module carrying an alternating form $\psi\colon L \times L \to \mathbf{Z}$, such that
$$\psi(bx, y) = \psi(x, b^* y), \quad \forall x, y \in L, b \in \mathcal{O}_B.$$
Let G be the group over $\mathbf{Z}$ so that for any $\mathbf{Z}$-algebra R,
$$G(R) = \{g \in \mathrm{GL}_{(\mathcal{O}_B)_R}(L_R) \mid \psi(gx, gy) = \nu(g)\psi(x, y), \nu(g) \in R\}.$$
- Let
$$\tilde{h}\colon \mathbf{C} \to \mathrm{End}_{(\mathcal{O}_B)_{\mathbf{R}}}(L_{\mathbf{R}})$$
be an $\mathbf{R}$-algebra homomorphism that gives a Hodge structure of type $(1,0), (0,1)$ on $L_{\mathbf{R}}$, such that $\psi\big(x, \tilde{h}(\sqrt{-1})y\big)$ is a symmetric positive definite bilinear form on $L_{\mathbf{R}}$. The restriction $\tilde{h}|_{\mathbf{C}^\times}$ can be viewed as a morphism of $\mathbf{R}$-algebraic groups
$$h\colon \mathrm{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_{m,\mathbf{C}} \longrightarrow G_{\mathbf{R}}.$$

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The action of h gives a decomposition

$$L_{\mathbf{C}} = V_{0,\mathbf{C}} \oplus V_{1,\mathbf{C}}, \tag{2.1.1}$$

where h acts on the first factor by the character $z \mapsto \bar{z}$ and on the second one by $z \mapsto z$. The Shimura field is then by definition

$$E = F[\mathrm{Tr}_{\mathbf{C}}(b|V_{0,\mathbf{C}}), b \in B].$$

The decomposition (2.1.1) is then defined over the subfield E of $\mathbf{C}$.

2.1.2 PEL Data for Symplectic Groups

Let $B = F$ be a totally real field of degree d. Let $\mathcal{O}_B = \mathcal{O}$ and $* = \mathrm{Id}$ be the trivial involution. Let L be a finite free $\mathbf{Z}$-module of rank $2dg$ equipped with an $\mathcal{O}$-module structure, together with the standard symplectic form

$$\varphi\colon L \times L \to \mathcal{O}$$

given by the antisymmetric matrix $J = \begin{pmatrix} & -I_{dg} \\ I_{dg} & \end{pmatrix}$. Set

$$\psi = \mathrm{Tr}_{\mathcal{O}/\mathbf{Z}} \circ \varphi.$$

The $\mathbf{C}$-algebra homomorphism $\tilde{h}$ is

$$a + bi \mapsto \begin{pmatrix} aI_{dg} & -bI_{dg} \\ bI_{dg} & aI_{dg} \end{pmatrix}.$$

We have the PEL datum $(\mathcal{O}, \mathrm{Id}, L, \psi, h = \tilde{h}|_{\mathbf{C}^\times})$. In this case

$$G = \mathrm{Res}_{\mathcal{O}/\mathbf{Z}} \mathrm{GSp}_{2g},$$

where GSp_{2g} is the split reductive group of symplectic similitudes respecting the matrix J.

The Shimura field in this case is $E = \mathbf{Q}$.

2.2 The Siegel–Hilbert Moduli Space over the Shimura Field

Keep the Shimura datum $(\mathcal{O}, \mathrm{Id}, L, \psi, h)$ as above. Let $H \subset G(\hat{\mathbf{Z}})$ be an open compact subgroup. We recall the moduli problem from [Ko, Section 5] and [La08, 1.4.1.4].

Let $\mathcal{M}_H$ be the functor that assigns to a $\mathbf{Q}$-scheme S the isomorphism classes of the tuples $(A, i, \lambda, \alpha_H)$ of one of the following kinds:

- A is an abelian scheme over S of relative dimension dg, equipped with an $\mathcal{O}$-action called the *real multiplication*: $i\colon \mathcal{O} \hookrightarrow \mathrm{End}_S(A)$.
- The requirement of Kottwitz determinant condition

  $$\det_{\mathcal{O}_S}(b|\,\mathrm{Lie}\,A) = \det_E(b|V_0), \quad \forall b \in F$$

  as *polynomial functions*, for which both sides of the equality are considered as morphisms of S-schemes (*cf.* [Ko, Section 5] for details).
- $\lambda\colon A \to A^\vee$ is a polarization. Recall that a symmetric homomorphism $A \to A^\vee$ is a *polarization* if (locally for the étale topology) it comes from a line bundle over A that is ample over S (*cf.* [GIT, 6.2]).

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- α_H is an H-level structure of type $(L_{\hat{\mathbf{Z}}}, \psi)$ analogous to that defined in Section 2.4.1 (*cf.* [La08, 1.3.7.6] for more details).

The functor $\mathcal{M}_H$ is represented by a separated smooth algebraic stack of finite type over $E = \mathbf{Q}$, by Artin's theory and Grothendieck's theory of Hilbert schemes. We denote the moduli stack by $\mathcal{M}_H$ again. If H is neat, then $\mathcal{M}_H$ is a smooth quasi-projective scheme over $\mathbf{Q}$, by [GIT] (and the theory of Hilbert schemes). As a special case of the construction of $\mathcal{M}_H$, we have the functor $\mathcal{M}_{H^p}$ with the level structure α_H being the prime to p level structure H^p.

We denote the universal abelian scheme over $\mathcal{M}_H$ by $\mathcal{A}$, and denote by ω the pull-back along the unit section of the relative differentials $\Omega^1_{\mathcal{A}/\mathcal{M}_H}$.

Remark 2.1 Let X denote the $G(\mathbf{R})$-conjugacy classes of $\tilde{h}$. The complex manifold $G(\mathbf{Q})\backslash X \times G(\mathbf{A}^\infty)/H$ descends to a quasi-projective scheme Sh_H over $\mathbf{Q}$, which is commonly called the Shimura variety. We have a canonical open and closed immersion

$$Sh_H \hookrightarrow [\mathcal{M}_H]$$

of the Shimura variety into the coarse moduli space of the algebraic stack $\mathcal{M}_H$. The moduli $[\mathcal{M}_H]$ is in fact the Shimura variety for the group $G' = G \times_{\mathrm{Res}_{F/\mathbf{Q}} \mathbf{G}_m} \mathbf{G}_m$, the subgroup of G whose determinants lie in $\mathbf{G}_m$.

2.3 Integral Models and Compactifications

In [La12], Lan constructs a normal and flat algebraic stack $\vec{\mathcal{M}}_H$ over $\mathbf{Z}_{(p)}$ that comes with a canonical isomorphism

$$\vec{\mathcal{M}}_H \times_{\mathrm{Spec}\, \mathbf{Z}_{(p)}} \mathrm{Spec}\, \mathbf{Q} \simeq \mathcal{M}_H.$$

We recall the construction briefly.

We first find an auxiliary Shimura datum that can provide the canonical integral model and toroidal compactification. In fact, we can embed the $\mathbf{Z}$-module L into another finite free $\mathbf{Z}$-module L_{aux}, which comes with an alternating pairing ψ_{aux} whose restriction to L is ψ. The $\mathbf{R}$-algebra homomorphism $\tilde{h}$ then induces another $\mathbf{R}$-algebra homomorphism $\tilde{h}_{\mathrm{aux}}$, whose restriction to $\mathbf{C}^\times$ is denoted by h_{aux}. Moreover, we have a subring $\mathcal{O}_{\mathrm{aux}} \subset \mathcal{O}$ for which the embedding $L \hookrightarrow L_{\mathrm{aux}}$ is $\mathcal{O}_{\mathrm{aux}}$-linear. The point is that, for the auxiliary Shimura datum $(\mathcal{O}_{\mathrm{aux}}, \mathrm{Id}, L_{\mathrm{aux}}, \psi_{\mathrm{aux}}, h_{\mathrm{aux}})$, the prime p is a *good* prime to which the main results of [La08] apply.

Now we have an induced homomorphism of algebraic groups over $\mathbf{Z}$,

$$t\colon G \longrightarrow G_{\mathrm{aux}},$$

where the second group is defined by the auxiliary Shimura datum in the same way as before. The auxiliary Shimura datum provides a moduli stack $\mathcal{M}_{G_{\mathrm{aux}}(\hat{\mathbf{Z}}^p)}$, which is separated smooth and of finite type over $\mathbf{Z}_{(p)}$. By the fact that p is a good prime for $\mathcal{M}_{G_{\mathrm{aux}}(\hat{\mathbf{Z}}^p)}$, we can show that there is a canonical isomorphism

$$\mathcal{M}_{G_{\mathrm{aux}}(\hat{\mathbf{Z}}^p) \times G_{\mathrm{aux}}(\mathbf{Z}_p)} \simeq \mathcal{M}_{G_{\mathrm{aux}}(\hat{\mathbf{Z}}^p)} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Q}.$$

More generally, for any open compact subgroup $H_{\mathrm{aux}} = H^p_{\mathrm{aux}} G_{\mathrm{aux}}(\mathbf{Z}_p) \subset G_{\mathrm{aux}}(\hat{\mathbf{Z}})$ such that H^p is mapped to H^p_{aux} under the morphism $t\colon G(\mathbf{Z}^p) \to G_{\mathrm{aux}}(\mathbf{Z}^p)$, we have similarly a moduli stack $\mathcal{M}_{H^p_{\mathrm{aux}}}$ for which p is a good prime and a morphism

$$\mathcal{M}_H \to \mathcal{M}_{H^p_{\mathrm{aux}}} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Q}, \tag{2.3.1}$$

compatible with the map between the two PEL data, which is finite on the coarse moduli spaces.

Proposition 2.2 ([La12, Proposition 2.2.1.1]) *The normalization $\vec{\mathcal{M}}_H$ of $\mathcal{M}_{H^p_{\mathrm{aux}}}$ in $\mathcal{M}_H$ is a normal flat algebraic stack over $\mathbf{Z}_{(p)}$ whose generic fibre is canonically isomorphic to $\mathcal{M}_H$. The normalization of $[\mathcal{M}_{H^p_{\mathrm{aux}}}]$ in $[\mathcal{M}_H]$ under the map of coarse moduli spaces induced by (2.3.1) is canonically isomorphic to $[\vec{\mathcal{M}}_H]$, which is a quasi-projective normal flat scheme over $\mathbf{Z}_{(p)}$. Hence $\vec{\mathcal{M}}_H \simeq [\vec{\mathcal{M}}_H]$ is a scheme if H is neat.*

From now on, we always assume H is neat.

Let $\mathcal{M}^{\mathrm{tor}}_H$ be the toroidal compactification of $\mathcal{M}_H$ for a fixed admissible smooth rational polyhedral cone decomposition datum Σ for $\mathcal{M}_H$.

Proposition 2.3 ([La12, Propositions 2.2.1.2, 2.2.2.1, and 2.2.2.3])

(i) *There is an admissible smooth rational polyhedral cone decomposition datum Σ_{aux} for $\mathcal{M}_{H^p_{\mathrm{aux}}}$ (hence the toroidal compactification $\mathcal{M}^{\mathrm{tor}}_{H^p_{\mathrm{aux}}}$ of $\mathcal{M}_{H^p_{\mathrm{aux}}}$), which is compatible with Σ in a natural way, and induces a canonical morphism*

$$\mathcal{M}^{\mathrm{tor}}_H \longrightarrow \mathcal{M}^{\mathrm{tor}}_{H^p_{\mathrm{aux}}} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Q}, \tag{2.3.2}$$

which is compatible with the stratifications on both sides (in particular, extending (2.3.1)) and the pullback of universal objects.

(ii) *Let $\mathcal{M}^{\min}_H$ and $\mathcal{M}^{\min}_{H^p_{\mathrm{aux}}}$ be the corresponding minimal compactifications. Then the morphism (2.3.2) induces a natural morphism*

$$\mathcal{M}^{\min}_H \longrightarrow \mathcal{M}^{\min}_{H^p_{\mathrm{aux}}} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Q},$$

which is compatible with the stratifications on both sides. The normalization $\vec{\mathcal{M}}^{\min}_H$ of $\mathcal{M}^{\min}_{H^p_{\mathrm{aux}}}$ in $\mathcal{M}^{\min}_H$ is a projective normal flat scheme over $\mathbf{Z}_{(p)}$ whose generic fibre is canonically isomorphic to $\mathcal{M}^{\min}_H$. It contains $\vec{\mathcal{M}}_H$ as an open dense subscheme.

(iii) *In the case that Σ is projective, there is an integral model $\vec{\mathcal{M}}^{\mathrm{tor}}_H$ for the toroidal compactication $\mathcal{M}^{\mathrm{tor}}_H$, which is by construction the normalization of the blow-up of certain coherent ideal sheaf on $\vec{\mathcal{M}}^{\min}_H$. It is a projective normal flat scheme over $\mathbf{Z}_{(p)}$, such that $\vec{\mathcal{M}}^{\mathrm{tor}}_H \otimes_{\mathbf{Z}_{(p)}} \mathbf{Q} \simeq \mathcal{M}^{\mathrm{tor}}_H$ in a canonical way. If $H' \subset H$ is an open compact subgroup, then there is a canonical map $\vec{\mathcal{M}}^{\mathrm{tor}}_{H'} \to \vec{\mathcal{M}}^{\mathrm{tor}}_H$, compatible with the canonical map $\mathcal{M}_{H'} \to \mathcal{M}_H$.*

For the integral model $\vec{\mathcal{M}}_{H^p}$ with prime to p level, we have the following stronger result.

Theorem 2.4 ([PR]) *The canonical map $\vec{\mathcal{M}}_{H^p} \to \mathcal{M}_{H^p_{\mathrm{aux}}}$ is a closed embedding.*

In particular, we have a moduli interpretation for $\vec{\mathcal{M}}_{H^p}$, with PEL data as part of the moduli problem.

Proof By Theorem 12.2 of [PR], the flat scheme-theoretic image in $\vec{\mathcal{M}}_{H^p_{\mathrm{aux}}}$ of the generic fibre $\mathcal{M}_{H^p}$ is normal, hence is canonically isomorphic to $\vec{\mathcal{M}}_{H^p}$. In other words, the integral model $\vec{\mathcal{M}}_{H^p}$ defined above coincides with the canonical integral model of Pappas–Rapoport [PR].

For the last claim, the reader is referred to [PR, Section 15] for more details. ■

2.4 Ordinary Loci and Partial Compactifications

2.4.1 Level Structures Prime to p

We recall certain results from [La12, Chapter 3]. Let S be a scheme over $\mathbf{Z}_{(p)}$. Let A be an abelian scheme over S, equipped with polarization λ and $\mathcal{O}$-endomorphism i as before. Let $H^p \subset G(\hat{\mathbf{Z}}^p)$ be an open compact. Let $N \geq 4$ be a natural number prime to p such that $H^p \supset U(N)$, the principal mod N congruence subgroup. A principal level N structure of (A, λ, i) of type $(L_{\hat{\mathbf{Z}}^p}, \psi)$ is the pair (α_N, ν_N) defined as follows:

- $\alpha_N \colon L/NL \xrightarrow{\sim} A[N]$ is an $\mathcal{O}$-linear isomorphism of group schemes over S, such that
  - (i) the symplectic pairing $L/NL \times L/NL \longrightarrow \mathbf{Z}/N\mathbf{Z}$ and the λ-Weil pairing $A[N] \times A[N] \longrightarrow \mu_N$ induced by the polarization λ are compatible for a chosen isomorphism of group schemes $\nu_N \colon \mathbf{Z}/N\mathbf{Z} \xrightarrow{\sim} \mu_N$ with respect to a fixed primitive N-th root of unity ζ_N.
  - (ii) α_H is symplectic liftable: there is a tower of finite étale surjections
$$(S_M \twoheadrightarrow S_N = S)_{N|M, p\nmid M}$$
and $\mathcal{O}$-linear isomorphisms $\alpha_M \colon L/ML \xrightarrow{\sim} A[M]$ with respect to an isomorphism $\nu_M \colon \mathbf{Z}/M\mathbf{Z} \xrightarrow{\sim} \mu_M$ such that for any valid indices $M'|M''$,
$$(\alpha_{M'}, \nu_{M'}) = (\alpha_{M''}, \nu_{M''}) \bmod M'.$$

  (This condition is required so that α_N lifts, at any geometric point s of S, to an $\mathcal{O}$-linear symplectic isomorphism between $L_{\hat{\mathbf{Z}}^p}$ and the Tate module of A_s.)

Consider all natural numbers N such that $p \nmid N$ and $H^p \supset U(N)$. A level H^p structure of (A, λ, i) of type $(L_{\hat{\mathbf{Z}}^p}, \psi)$ is a collection of $H^p/U(N)$-orbits of principal level N structures (α_N, ν_N) for all N as above.

2.4.2 Ordinary Level Structures at p

Let
$$0 = D^1 \subset D^0 \subset D^{-1} = L_{\mathbf{Z}_p}$$
be a filtration of $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Z}_p$-modules, such that $\mathrm{Gr}_D^{-1} := D^{-1}/D^0$ is torsion-free as a $\mathbf{Z}_p$-module, and under the pairing ψ D^0 is totally isotropic and is its own annihilator.

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Such a filtration determines a filtration

$$0 = D^{\vee,1} \subset D^{\vee,0} \subset D^{\vee,-1} = L^{\vee}_{\mathbf{Z}_p}$$

on the dual lattice $L^{\vee}_{\mathbf{Z}_p}$. We have the natural map

$$\varphi^0_D \colon D^0 \to D^{\vee,0}$$

whose reduction mod p^n is denoted by φ^0_{D,p^n}.

Let $P_D \subset G_{\mathbf{Z}_p}$ be the stabilizer of D. Let M_D be the group over $\mathbf{Z}_p$, whose R-points, for any $\mathbf{Z}_p$-algebra R, are $(g,c) \in \mathrm{GL}_{\mathcal{O}\otimes_{\mathbf{Z}} R}(\mathrm{Gr}_D \otimes_{\mathbf{Z}_p} R) \times \mathbf{G}_m(R)$ such that $\psi(gx, gy) = c\psi(x,y)$. We denote by U_D the kernel of the natural morphism from P_D to M_D. Now for any integer $n \in \mathbf{Z}_{\geq 0}$, we set

$$U_{p,0}(p^n) = \big(G(\mathbf{Z}_p) \to G(\mathbf{Z}/p^n\mathbf{Z})\big)^{-1} P_D(\mathbf{Z}/p^n\mathbf{Z}),$$
$$U^{\mathrm{bal}}_{p,1}(p^n) = \big(G(\mathbf{Z}_p) \to G(\mathbf{Z}/p^n\mathbf{Z})\big)^{-1} U_D(\mathbf{Z}/p^n\mathbf{Z}).$$

Let S be a scheme over $\mathbf{Z}$. Let A be an abelian scheme together with a polarization λ and an $\mathcal{O}$-endomorphism i. An ordinary principal level p^n structure of (A, λ, i) of type $(L_{\mathbf{Z}_p}, \psi, D)$ is the following data:

- An $\mathcal{O}$-linear homomorphism $\alpha^0_{p^n} \colon (D^0/p^nD^0)^{\mathrm{mult}} \to A[p^n]$ of group schemes over S.
- An $\mathcal{O}$-linear homomorphism $\alpha^{\vee,0}_{p^n} \colon (D^{\vee,0}/p^nD^{\vee,0})^{\mathrm{mult}} \to A^{\vee}[p^n]$ of group schemes over S.
- A section ν_{p^n} of $(\mathbf{Z}/p^n\mathbf{Z})^{\times} \simeq \mathrm{Isom}_S(\mu_{p^n}, \mu_{p^n})$ so that the homomorphism of multiplicative group schemes

  $$\nu_{p^n} \circ (\varphi^0_{D,p^n})^{\mathrm{mult}} \colon (D^0/p^nD^0)^{\mathrm{mult}} \to (D^{\vee,0}/p^nD^{\vee,0})^{\mathrm{mult}}$$

  is compatible with λ under $\alpha^0_{p^n}$ and $\alpha^{\vee,0}_{p^n}$, and such that the scheme theoretic images $\mathrm{Im}(\alpha^0_{p^n})$ and $\mathrm{Im}(\alpha^{\vee,0}_{p^n})$ nullify each other under the λ-Weil pairing on $A[p^n] \times A^{\vee}[p^n]$.
- The requirement that α_{p^n} is symplectic liftable: there is a tower of quasi-finite étale surjections

  $$(S_{p^{n'}} \twoheadrightarrow S_{p^n} = S)_{n' \geq n}$$

  and triples $(\alpha^0_{p^{n'}}, \alpha^{\vee,0}_{p^{n'}}, \nu_{p^{n'}})$ as above such that for any $n'' \geq n'$,

  $$(\alpha^0_{p^{n''}}, \alpha^{\vee,0}_{p^{n''}}, \nu_{p^{n''}}) \bmod p^{n'} = (\alpha^0_{p^{n'}}, \alpha^{\vee,0}_{p^{n'}}, \nu_{p^{n'}}).$$

Let $H_p \subset G(\mathbf{Z}_p)$ be an open compact subgroup such that $U^{\mathrm{bal}}_{p,1}(p^n) \subset H_p \subset U_{p,0}(p^n)$ for some integer $n \geq 0$. An ordinary level H_p structure of (A, λ, i) of type $(L_{\mathbf{Z}_p}, \psi, D)$ is an $H_p/U^{\mathrm{bal}}_{p,1}(p^n)$-orbit of ordinary principal level p^n structure of (A, λ, i) of type $(L_{\mathbf{Z}_p}, \psi, D)$.

2.4.3 Integral Models with Ordinary Level Structures

Let $H = H^pH_p \subset G(\hat{\mathbf{Z}})$ be an open compact subgroup such that $U^{\mathrm{bal}}_{p,1}(p^n) \subset H_p \subset U_{p,0}(p^n)$ for some integer $n \geq 0$. Let $\mathcal{M}^{\mathrm{ord,naive}}_H$ be the functor that assigns to a $\mathbf{Z}_{(p)}$-scheme S the isomorphism classes of the tuples $(A, i, \lambda, \alpha_{H^p}, \alpha_{H_p})$ as follows:

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- A is an abelian scheme over S of relative dimension dg, equipped with an $\mathcal{O}$-endomorphism $i\colon \mathcal{O} \hookrightarrow \mathrm{End}_S(A)$.
- $\lambda\colon A \to A^\vee$ is a polarization.
- α_{H^p} is a level H^p structure of type $(L_{\hat{\mathbf{Z}}^p}, \psi)$.
- α_{H_p} is a level H_p structure of type $(L_{\mathbf{Z}_p}, \psi, D)$.

The functor $\mathcal{M}_H^{\mathrm{ord,naive}}$ is represented by a scheme of finite type over $\mathbf{Z}_{(p)}$.

Let r_H be the fixed nonnegative integer as in [La12] that is determined by the PEL data and the filtration D of $L_{\mathbf{Z}_p}$. We can check that, over any $\mathbf{Q}[\zeta_{p^{r_H}}]$-scheme S, there is a natural assignment from the level H structures of $(A, i, \lambda)_S$ to the pairs of H^p-level structure of type $(L_{\hat{\mathbf{Z}}^p}, \psi)$ and H_p-level structure of type $(L_{\mathbf{Z}_p}, \psi)$, which is in fact injective. As a consequence, we have an open and closed immersion

$$\mathcal{M}_H \otimes_{\mathbf{Q}} \mathbf{Q}[\zeta_{p^{r_H}}] \longrightarrow \mathcal{M}_H^{\mathrm{ord,naive}} \otimes_{\mathbf{Z}} \mathbf{Q}[\zeta_{p^{r_H}}]$$

whose image, which is an open and closed subscheme of $\mathcal{M}_H^{\mathrm{ord,naive}} \otimes_{\mathbf{Z}} \mathbf{Q}[\zeta_{p^{r_H}}]$, is denoted by $\mathcal{M}_H^{\mathrm{ord}}$.

Proposition 2.5 ([La12, Theorem 3.4.2.5]) *The normalization $\vec{\mathcal{M}}_H^{\mathrm{ord}}$ of $\mathcal{M}_H^{\mathrm{ord,naive}}$ in $\mathcal{M}_H^{\mathrm{ord}}$ under the natural morphism $\mathcal{M}_H^{\mathrm{ord}} \to \mathcal{M}_H^{\mathrm{ord,naive}}$ is a scheme smooth quasi-projective separated of finite type over $\mathbf{Z}_{(p)}[\zeta_{p^{r_H}}]$, which is an open subscheme of $\vec{\mathcal{M}}_H \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_{(p)}[\zeta_{p^{r_H}}]$.*

2.4.4 Partial Compactifications of Ordinary Loci

Keep the data as before.

Theorem 2.6 (Theorem 5.2.1.1, [La12]) *There is a scheme $\vec{\mathcal{M}}_H^{\mathrm{ord,tor}}$, quasi-projective smooth separated of finite type over $\mathbf{Z}_{(p)}[\zeta_{p^{r_H}}]$, containing $\vec{\mathcal{M}}_H^{\mathrm{ord}}$ as an open dense subscheme. The universal tuple $(\mathcal{A}, i, \lambda, \alpha_{H^p}, \alpha_{H_p})$ on $\vec{\mathcal{M}}_H^{\mathrm{ord}}$ extends to $\vec{\mathcal{M}}_H^{\mathrm{ord,tor}}$. The boundary $\vec{\mathcal{M}}_H^{\mathrm{ord,tor}} \backslash \vec{\mathcal{M}}_H^{\mathrm{ord}}$ is a relative Cartier divisor with normal crossing.*

We have the Hodge line bundle $(\det\omega)_{\vec{\mathcal{M}}_H^{\mathrm{ord}}}$ over $\vec{\mathcal{M}}_H^{\mathrm{ord}}$, and $(\det\omega)_{\vec{\mathcal{M}}_H^{\mathrm{ord,tor}}}$, its extension to $\vec{\mathcal{M}}_H^{\mathrm{ord,tor}}$. Form

$$\vec{\mathcal{M}}_H^{\mathrm{ord,min}} = \mathrm{Proj}_{\oplus_{k\geq 0}} \Gamma\big(\vec{\mathcal{M}}_H^{\mathrm{ord,tor}}, (\det\omega)^k_{\vec{\mathcal{M}}_H^{\mathrm{ord,tor}}}\big).$$

This is in general not projective, as the partial toroidal compactification $\vec{\mathcal{M}}_H^{\mathrm{ord,tor}}$ is not proper.

Theorem 2.7 ([La12, Theorem 6.2.1.1]) *There exists a canonical proper morphism*

$$\vec{\mathcal{M}}_H^{\mathrm{ord,tor}} \longrightarrow \vec{\mathcal{M}}_H^{\mathrm{ord,min}}.$$

The scheme $\vec{\mathcal{M}}_H^{\mathrm{ord,min}}$ is quasi-projective normal and flat over $\mathbf{Z}_{(p)}[\zeta_{p^{r_H}}]$, which contains $\vec{\mathcal{M}}_H^{\mathrm{ord}}$ as an open dense subscheme.

Remark 2.8 For the moduli space $\vec{\mathcal{M}}_{H^p}$ with prime to p level, the integral model $\vec{\mathcal{M}}^{\mathrm{ord}}_{H^p}$ is simply the ordinary locus of $\vec{\mathcal{M}}_{H^p} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_{(p)}[\zeta_{p^{r_H}}]$. It then comes with a moduli interpretation, by Theorem 2.4.

2.5 Hecke Correspondences

2.5.1 The Double-coset Hecke Algebra

Let q be a prime number and $v \mid q$ a place in F. For the completion F_v of F at the place v, we denote by $\mathcal{O}_v$ the integer ring and fix a uniformizer ϖ_v. We define the spherical Hecke algebra $\mathcal{H}^{\mathrm{sph}}_v$ for $\mathrm{GSp}_{2g}(F_v)$ with coefficients in $\mathbf{Z}$ to be the algebra of $\mathbf{Z}$-valued functions on $\mathrm{GSp}_{2g}(F_v)$ that are bi-invariant under $\mathrm{GSp}_{2g}(\mathcal{O}_v)$. It is generated by the characteristic functions on the following double cosets:

$$T_{v,1} = \mathrm{GSp}_{2g}(\mathcal{O}_v) \begin{pmatrix} I_g & \\ & \varpi_v I_g \end{pmatrix} \mathrm{GSp}_{2g}(\mathcal{O}_v),$$

$$T_{v,i} = \mathrm{GSp}_{2g}(\mathcal{O}_v) \begin{pmatrix} I_{g-i+1} & & & \\ & \varpi_v I_{i-1} & & \\ & & \varpi_v^2 I_{g-i+1} & \\ & & & \varpi_v I_{i-1} \end{pmatrix} \mathrm{GSp}_{2g}(\mathcal{O}_v), \quad 2 \le i \le g,$$

$$S_v = \varpi_v \, \mathrm{GSp}_{2g}(\mathcal{O}_v).$$

2.5.2 Weights and Automorphic Sheaves

Through the end of this section, let $\mathcal{M}$ denote $\vec{\mathcal{M}}_{H^p}$, $\vec{\mathcal{M}}^{\mathrm{ord}}_{H^p}$, or $\mathcal{M}_H$.

We again denote the universal abelian scheme over $\mathcal{M}$ by $\mathcal{A}$, and denote by ω the pullback of $\Omega^1_{\mathcal{A}/\mathcal{M}}$ along the unit section. We remark that ω is locally free over $\mathcal{O}_{\mathcal{M}}$, but is not locally free as an $\mathcal{O}_{\mathcal{M}} \otimes_{\mathbf{Z}} \mathcal{O}_F$-module if p is ramified in F, for the integral models.

We only give the construction of automorphic sheaves of $\mathcal{M}_H$, and those of $\mathcal{M} = \vec{\mathcal{M}}_{H^p}$ when p is a *good* prime for the moduli, so that ω is locally free over $\mathcal{O}_{\mathcal{M}} \otimes_{\mathbf{Z}} \mathcal{O}_F$. The latter is enough for the auxiliary moduli. The automorphic sheaves over $\vec{\mathcal{M}}_{H^p}$ and $\vec{\mathcal{M}}^{\mathrm{ord}}_{H^p}$ in the general case are then defined by restriction via the closed immersion in Theorem 2.4 and the inclusion $\vec{\mathcal{M}}^{\mathrm{ord}}_{H^p} \subset \vec{\mathcal{M}}_{H^p} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_{(p)}[\zeta_{p^{r_H}}]$. We refer the reader to [La12, Chapter 8] for more details, including the cases with level structure at p.

Let $T_{g/\mathcal{O}}$ be the standard diagonal maximal torus of $\mathrm{GSp}_{2g/\mathcal{O}}$. Putting $G = \mathrm{Res}^{\mathcal{O}}_{\mathbf{Z}} \mathrm{GSp}_{2g/\mathcal{O}}$ and $T = \mathrm{Res}^{\mathcal{O}}_{\mathbf{Z}} T_{g/\mathcal{O}}$, take the standard Borel B of G with unipotent radical U and identify $T = B/U$. Let M be the Levi of the standard Siegel parabolic of G. Then $M \supset T$.

Consider a character

$$\kappa \colon T \longrightarrow \mathbf{G}_m.$$

We may regard κ as a character of $B \cap M$ that is trivial on $U \cap M$. The character κ is called *dominant* with respect to B, if the induced representation $\mathrm{Ind}^M_{B\cap M} \kappa^{-1}$ inside

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the rational functions of the scheme $(M/U \cap M)$ is non-zero. The Bruhat–Tits decomposition shows that the subspace $(\mathrm{Ind}_{B\cap M}^{M}\,\kappa^{-1})^{U\cap M}$ is one-dimensional, and T acts on a generator by $-w_0\kappa$, where w_0 is the longest element in the Weyl group (with respect to T). The M-translation of the generator generates a sub-representation

$$\rho_\kappa^* \subset \mathrm{Ind}_{B\cap M}^{M}\,\kappa^{-1},$$

where an element m in the standard Levi M acts as $m \cdot f(x) = f(m^{-1}x)$. The R-dual ρ_κ of ρ_κ^* is called the *rational representation of highest weight* κ, which has the universal property that for any M-module X,

$$\mathrm{Hom}_M(\rho_\kappa, X) \simeq \mathrm{Hom}_M(X^*, \rho_\kappa^*) \simeq \mathrm{Hom}_B(X^*, -\kappa) \simeq \mathrm{Hom}_B(\kappa, X).$$

We define the *automorphic sheaf* of weight κ on $\mathcal{M}$ to be the contraction product

$$\omega^\kappa = \mathrm{Isom}_{\mathcal{M}}(\mathcal{O}_{\mathcal{M}}^{dg}, \omega) \times^M \rho_\kappa,$$

that is, the quotient of the product $\mathrm{Isom}_{\mathcal{M}}(\mathcal{O}_{\mathcal{M}}^{dg}, \omega) \times \rho_\kappa$ by the equivalence relation $(\varphi \circ m, w) \sim (\varphi, m \cdot w)$, for $\varphi \in \mathrm{Isom}_{\mathcal{M}}(\mathcal{O}_{\mathcal{M}}^{dg}, \omega)$, $m \in M$ and $w \in \rho_\kappa$.

The construction above then provides the automorphic sheaves on $\mathcal{M}_H$ and the ones on the auxiliary moduli in the integral cases, and then those on the integral models $\mathcal{M} = \vec{\mathcal{M}}_{H^p}, \vec{\mathcal{M}}_{H^p}^{\mathrm{ord}}$ without the assumption that p is a good prime for the moduli problems. We always denote the automorphic sheaves over $\mathcal{M}$ by the same symbol ω^κ.

By the results of [La12, Chapter 8], the automorphic sheaf ω^κ extends from the moduli schemes to the total (resp. partial) compactifications in a canonical way, which is compatible with the restrictions to the ordinary loci of the total objects.

2.5.3 Geometric Correspondences

As in the previous section, we may assume p is a good prime for the moduli $\mathcal{M}$.

Let $\mathfrak{a}$ be an ideal of $\mathcal{O}$. Let $\mathcal{M}^{\mathfrak{a}}$ be the moduli stack of isogenies between objects in $\mathcal{M}$, that is, the algebraic stack representing the functor $\mathcal{M}^{\mathfrak{a}}$ that assigns to any base scheme S over $\mathbf{Q}$ (resp. $\mathbf{Z}_{(p)}$, resp. $\mathbf{Z}_{(p)}[\zeta_{p^{r_H}}]$) the category in groupoids in which an object is an isogeny

$$f\colon A \to B$$

between two polarized abelian schemes with endomorphisms and level structures (A, i_A, λ_A) and (B, i_B, λ_B), whose kernel is (étale locally) $\mathcal{O}$-linearly isomorphic to $(\mathcal{O}/\mathfrak{a}\mathcal{O})^g$ and intersects with (the image of) the level structure only along the unit section, is compatible with the $\mathcal{O}$-endomorphisms, and respects the polarizations on both sides.

Here we obtain the representability of the functor $\mathcal{M}^{\mathfrak{a}}$ by the use of the fact that $\mathcal{M}$ is representable and by the theory of Hilbert schemes (*cf.* [FC, p. 251]). In particular, since H is assumed to be neat, the functor $\mathcal{M}^{\mathfrak{a}}$ is represented by a quasi-projective scheme over $\mathbf{Q}$ (resp. $\mathbf{Z}_{(p)}$, resp. $\mathbf{Z}_{(p)}[\zeta_{p^{r_H}}]$), which is denoted by the same symbol, as usual. The universal isogeny over $\mathcal{M}^{\mathfrak{a}}$ is denoted by $\mathcal{I}^{\mathfrak{a}}$. Assigning such an isogeny to its source (resp. target), we have two natural projections

$$\mathcal{M} \xleftarrow{\pi_{1,\mathfrak{a}}} \mathcal{M}^{\mathfrak{a}} \xrightarrow{\pi_{2,\mathfrak{a}}} \mathcal{M},$$

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whose restrictions to any connected component Z of $\mathcal{M}^{\mathfrak{a}}$ are proper, by the valuative criterion.

In the case that p is invertible in the base scheme S, the two projections

$$\pi_{i,(p)}\colon \mathcal{M}^{(p)} \to \mathcal{M}, \quad i = 1, 2$$

are finite étale. In this case, for $v \mid q$ a prime ideal in $\mathcal{O}$, we have the bijection between the connected components of $\mathcal{M}^{v}$ and the double cosets γ_v in the spherical Hecke algebra $\mathcal{H}_v^{\mathrm{sph}}$. Denote the corresponding connected component of $\mathcal{M}^{v}$ by $\mathcal{M}^{\gamma_v}$, over which the universal isogeny is said to be of type γ_v. We have the two projections

$$\pi_{i,\gamma_v}\colon \mathcal{M}^{\gamma_v} \to \mathcal{M}, \quad i = 1, 2,$$

of type γ_v.

For $\mathcal{M} = \vec{\mathcal{M}}^{\mathrm{ord}}_{H^p}, \vec{\mathcal{M}}_{H^p}$ over a scheme S in characteristic p, and $v|p$ a prime ideal in $\mathcal{O}$, we again have the connected component of $\mathcal{M}^{\gamma_v}$ and the two projections

$$\pi_{i,\gamma_v}\colon \mathcal{M}^{\gamma_v} \to \mathcal{M}, \quad i = 1, 2,$$

of type γ_v. (We refer the reader to [FC, Chapter VII] for details on the facts above.)

In the two cases above, consider the commutative diagram

$$\begin{array}{ccc} \mathcal{J}^{\mathfrak{a}} & \longrightarrow & \mathcal{A} \\ {\scriptstyle f_{\mathfrak{a}}}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ Z = \mathcal{M}^{\mathfrak{a}} & \xrightarrow{\pi_{i,\mathfrak{a}}} & \mathcal{M}. \end{array}$$

Over the base S, we have a natural map of $\mathcal{O} \otimes_{\mathbf{Z}} \mathcal{O}_Z^{\times}$-torsors

$$\pi_{2,\mathfrak{a}}^{*}\omega = \pi_{2,\mathfrak{a}}^{*}(f_{*}\Omega^1_{\mathcal{A}/\mathcal{M}}) \to f_{\mathfrak{a}*}\Omega^1_{\mathcal{J}^{\mathfrak{a}}/Z} \xrightarrow{\sim} f_{\mathfrak{a}*}(\pi_{1,\mathfrak{a}}^{*}\Omega^1_{\mathcal{A}/\mathcal{M}}) \xrightarrow{\sim} \pi_{1,\mathfrak{a}}^{*}\omega,$$

hence the induced map

$$\theta\colon \pi_{2,\mathfrak{a}}^{*}\omega^{\kappa} \longrightarrow \pi_{1,\mathfrak{a}}^{*}\omega^{\kappa}.$$

Applying $\pi_{1,\mathfrak{a}*}$ and composing with the trace map

$$\mathrm{Tr}\colon \pi_{1,\mathfrak{a}*}\pi_{1,\mathfrak{a}}^{*}\omega^{\kappa} \to \omega^{\kappa},$$

we obtain the map

$$\pi_{1,\mathfrak{a}*}\pi_{2,\mathfrak{a}}^{*}\omega^{\kappa} \xrightarrow{\pi_{1,\mathfrak{a}*}\theta} \pi_{1,\mathfrak{a}*}\pi_{1,\mathfrak{a}}^{*}\omega^{\kappa} \xrightarrow{\mathrm{Tr}} \omega^{\kappa}.$$

Taking global sections and composing with the natural map

$$H^0(\mathcal{M}_R, \omega^{\kappa}) \to H^0(\mathcal{M}_R, \pi_{1,\mathfrak{a}*}\pi_{2,\mathfrak{a}}^{*}\omega^{\kappa}),$$

we get the desired endomorphism

$$T_{\mathfrak{a}}\colon H^0(\mathcal{M}_R, \omega^{\kappa}) \to H^0(\mathcal{M}_R, \omega^{\kappa}),$$

which will be denoted by $U_{(p)}$ in the case $\mathfrak{a} = (p)$. We remark that the Hecke operator $U_{(p)}$ corresponds to the product of the double cosets $T_{v,1}$, $v|p$.

We have the same construction for $Z = \mathcal{M}^{\gamma_v}$, the connected component of $\mathcal{M}^{v}$ of type γ_v. In these cases, the Hecke operators corresponding to the double cosets $T_{v,i}$ (resp. S_v) will be denoted by $T_{v,i}$ (resp. S_v) again.

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2.6 Hasse Invariants and Liftings

2.6.1 Hasse Invariants on Abelian Schemes in Characteristic p

Let A be an abelian scheme over S, a scheme in characteristic p. We have $A^{(p)}$, the pullback of $A \to S$ via the absolute Frobenius Frob_S on S, and $V_{A/S}\colon A^{(p)} \to A$, the Verschiebung isogeny. The latter then induces the map

$$\mathcal{C}_{A/S}\colon \Omega^1_{A/S} \to \Omega^1_{A^{(p)}/S} \simeq \mathrm{Frob}_S^* \Omega^1_{A/S},$$

whose highest exterior power gives

$$\mathcal{C}_{A/S}\colon \det \omega_{A/S} \to (\det \omega_{A/S})^{\otimes p},$$

hence a section $h \in H^0\big(A, (\det \omega_{A/S})^{\otimes(p-1)}\big)$. Applying this to the universal abelian scheme on the special fibre of $\vec{\mathcal{M}}_{H^p}$, we then have a global section

$$h \in H^0\big((\vec{\mathcal{M}}_{H^p})_{\mathbf{F}_p}, (\det \omega)^{\otimes p-1}\big),$$

which is known as the Hasse invariant of the moduli space. We have its extensions to $\vec{\mathcal{M}}^{\mathrm{tor}}_{H^p}$ and $\vec{\mathcal{M}}^{\mathrm{min}}_{H^p}$, and denote them by h again. The reader is referred to [La12, Section 6.3] for more details.

Remark 2.9 For A/S an abelian scheme of dimension n, the Hasse invariant $h(A)$ is non-vanishing if and only if A is ordinary, which means that $A[p]$ has p^n elements at every geometric point of S.

Lemma 2.10 *Recall the notation from Section 2.5.3. The natural map of sheaves on* $(\vec{\mathcal{M}}^{\mathfrak{a}}_{H^p})_{\mathbf{F}_p}$ *(resp.* $(\vec{\mathcal{M}}^{\gamma_\nu}_{H^p})_{\mathbf{F}_p}$*) with* p *coprime to* $\mathfrak{a}$ *(resp.* ν*)*

$$\theta\colon \pi_2^*(\det \omega)^{\otimes p-1} \to \pi_1^*(\det \omega)^{\otimes p-1}$$

satisfies

$$\theta(\pi_2^* h) = \pi_1^* h.$$

Proof This follows from the functoriality of the Cartier operator.

Let R be an $\mathbf{F}_p$-algebra. Note that

$$\theta(\pi_2^* h)(A \to B, \mathbf{v}, \mathbf{v}') = h(B, \mathbf{v}'), \quad (\pi_1^* h)(A \to B, \mathbf{v}, \mathbf{v}') = h(A, \mathbf{v}),$$

where $\mathbf{v}$ (resp. $\mathbf{v}'$) is a chosen basis of $H^0(A, \Omega^1_{A/R})$ (resp. $H^0(B, \Omega^1_{B/R})$) so that $\pi_2^*(\mathbf{v}') = \mathbf{v}$. Thus we only need to show $h(B, \mathbf{v}') = h(A, \mathbf{v})$.

On the other hand, writing $\pi_2^{(p)}$ as the pullback of π_2 via Frob_R, we have

$$\begin{aligned} h(B, \mathbf{v}') \,\mathrm{Frob}_R^*(\det \mathbf{v}) &= h(B, \mathbf{v}') \,\mathrm{Frob}_R^*\big(\pi_2^*(\det \mathbf{v}')\big) = h(B, \mathbf{v}') \pi_2^{(p)*} \,\mathrm{Frob}_R^*(\det \mathbf{v}') \\ &= \pi_2^{(p)*}\big(h(B, \mathbf{v}') \,\mathrm{Frob}_R^*(\det \mathbf{v}')\big) = \pi_2^{(p)*}\big(\mathcal{C}_{B/R}(\det \mathbf{v}')\big) \\ &= \mathcal{C}_{A/R}\big(\pi_2^*(\det \mathbf{v}')\big), \end{aligned}$$

where the last equality is obtained by the étaleness of the projection π_2. Moreover, we have

$$\mathcal{C}_{A/R}\big(\pi_2^*(\det \mathbf{v}')\big) = \mathcal{C}_{A/R}(\det \mathbf{v}) = h(A, \mathbf{v}) \,\mathrm{Frob}_R^*(\det \mathbf{v}),$$

which, combined with the chain of equalities above, gives

$$h(B, \mathbf{v}') = h(A, \mathbf{v}),$$

as desired. ∎

2.6.2 Lifting Hasse Invariants to Characteristic Zero

Consider the Hasse invariant

$$h \in H^0\big((\vec{\mathcal{M}}^{\min}_{H^p})_{\mathbf{F}_p}, (\det \omega)^{\otimes p-1} \big).$$

Since the line bundle $(\det \omega)^{\otimes p-1}$ on $(\vec{\mathcal{M}}^{\min}_{H^p})_{\mathbf{F}_p}$ is ample, for sufficiently large integer k the line bundle $(\det \omega)^{\otimes k(p-1)}$ is very ample.

Proposition 2.11 *For $k \in \mathbf{Z}_{\geq 0}$ big enough, the section*

$$h^k \in H^0\big((\vec{\mathcal{M}}^{\min}_{H^p})_{\mathbf{F}_p}, (\det \omega)^{\otimes (p-1)k} \big)$$

lifts to a section

$$\widetilde{h^k} \in H^0\big(\vec{\mathcal{M}}^{\min}_{H^p}, (\det \omega)^{\otimes (p-1)k} \big).$$

Proof It is standard to show by Serre vanishing that $H^1\big(\vec{\mathcal{M}}^{\min}_{H^p}, (\det \omega)^{\otimes (p-1)k} \big) = 0$ when k is sufficiently large, which in turn gives the surjectivity of

$$H^0\big(\vec{\mathcal{M}}^{\min}_{H^p}, (\det \omega)^{\otimes (p-1)k} \big) \to H^0\big((\vec{\mathcal{M}}^{\min}_{H^p})_{\mathbf{F}_p}, (\det \omega)^{\otimes (p-1)k} \big). \qquad ∎$$

From now on, we fix such a lift as in Proposition 2.11, $E := \widetilde{h^{k_0}}$, such that $k_0 \gg 0$ and $p \nmid k_0$.

3 Analytification of Siegel–Hilbert moduli schemes

3.1 Preliminary

We recall certain definitions and results from [KL] and [Lü], which will be applied to the rigid analytifications of the Siegel–Hilbert moduli schemes, as well as their formal models and the automorphic sheaves.

3.1.1 Relative Compactness

Let K be a finite extension of $\mathbf{Q}_p$ and $\mathcal{O}_K$ the ring of integers. For $\mathfrak{X}$ a formal scheme over $\operatorname{Spf} \mathcal{O}_K$, we denote by $\mathfrak{X}^{\mathrm{rig}}$ the rigid analytic space associated to it, and by $\mathfrak{X}_0$ its special fibre. For $\mathfrak{U} \subset \mathfrak{X}$ an admissible open, we let $]\mathfrak{U}_0[$ denote the tube of $\mathfrak{U}$, *i.e.*, the pre-image in $\mathfrak{X}^{\mathrm{rig}}$ of the $\mathfrak{U}_0$ under the natural specialization $\mathfrak{X}^{\mathrm{rig}} \to \mathfrak{X}_0$, which is surjective. If $f\colon X \to Y$ is morphism of rigid spaces, we call a morphism of ϖ-adic $\mathcal{O}_K$-flat formal schemes $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ a formal model of f, if f is the rigidification of $\mathfrak{f}$.

Definition 3.1 Let $f\colon X \to V$ be a quasi-compact morphism of rigid analytic spaces, and $U \subset X$ a quasi-compact (relative to X) admissible open. We say U is

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relatively compact in X over V, denoted by

$$U \Subset_V X,$$

if there exists an admissible covering of quasi-compact subsets $\{V_i\}$ of V such that locally (over each V_i) there is a closed V-immersion of X into an n-dimensional unit ball $\mathbf{D}^n_V(1)$ under which U maps into a ball $\mathbf{D}^n_V(\epsilon)$ for some $\epsilon < 1$.

If $V = \operatorname{Sp} K$, we simply write $U \Subset_V X$ as $U \Subset X$.

The notion of relative compactness in Definition 3.1 is independent of the choice of covering $\{V_i\}$, essentially by Raynaud's theorem that the category of quasi-compact rigid spaces is equivalent to that of quasi-compact admissible formal schemes localized by admissible blow-ups.

Lemma 3.2 ([KL, 2.1.8]) *Let $i\colon Y \hookrightarrow Y'$ and $j\colon X \hookrightarrow X'$ be admissible open inclusions, all of which are quasi-compact rigid spaces over the quasi-compact rigid space V, and $f\colon Y' \to X'$ be a proper morphism. Suppose we have the following Cartesian diagram*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} \\ Y' & \xrightarrow{f} & X' , \end{array}$$

and suppose $X \Subset_V X'$. Then $Y \Subset_V Y'$.

3.1.2 Overconvergence

Let $\bar{X}$ be a quasi-compact rigid space over K, with a formal model $\bar{\mathfrak{X}}$. Let $D \subset \bar{\mathfrak{X}}_0$ be a Cartier divisor. Choose a finite covering $\{\mathfrak{U}_i\}_{i=1,\dots,n}$ of $\bar{\mathfrak{X}}_0$ so that for any i the ideal of $D|_{\mathfrak{U}_i}$ is generated by a single section $h_i \in \mathcal{O}_{\bar{\mathfrak{X}}_0}$. Choose for each h_i a lifting $\tilde{h}_i \in \Gamma(\bar{\mathfrak{X}}, \mathcal{O}_{\bar{\mathfrak{X}}})$. For any $r \in (|p|^{1/e}, 1]$, define

$$\bar{X}(r) = \bigcup_{1 \le i \le n} \{x \in \bar{X} \mid |\tilde{h}_i| \ge r\},$$

which is independent of the choice of $\tilde{h}_i$.

Proposition 3.3 ([KL, 2.3.2]) *If $\bar{X}(1) \Subset_{\bar{X}} X'$ for X' a quasi-compact admissible open of $\bar{X}$, then for any r close enough to 1 we have $\bar{X}(r) \subset X'$.*

3.2 Lifting Frobenius

Let K be a finite extension of $\mathbf{Q}_p$. Let X be a scheme that is locally of finite type over $\mathcal{O}_K$, and $\mathfrak{X}$ the formal completion of X along its special fiber. Denote by $\mathfrak{X}^{\mathrm{rig}}$ the rigid fibre of $\mathfrak{X}$ via Raynaud's functor.

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3.2.1 Passing to the Rigid Analytic Setting

We glue $\vec{\mathcal{M}}_{H^p} \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_{(p)}[\zeta_{p^{r_H}}]$ with $\vec{\mathcal{M}}^{\mathrm{ord,tor}}_{H^p}$ (resp. $\vec{\mathcal{M}}^{\mathrm{ord,min}}_{H^p}$) and denote the resulting scheme by $\bar{\mathcal{M}}_{H^p}$ (resp. $\mathcal{M}^*_{H^p}$).

Let $\mathfrak{M}^*_{H^p}$ (resp. $\bar{\mathfrak{M}}_{H^p}$) be the formal completion of $\mathcal{M}^*_{H^p}$ (resp. $\bar{\mathcal{M}}_{H^p}$) along the special fiber over p. The same procedure of completion along the special fibre gives the universal semi-abelian scheme

$$\bar{\mathfrak{A}} \to \bar{\mathfrak{M}}_{H^p}.$$

Combining with rigidification, we get the universal semi-abelian scheme

$$\bar{\mathfrak{A}}^{\mathrm{rig}} \to \bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}.$$

Take $\mathfrak{M}_{H^p}$ to be the open formal subscheme of $\mathfrak{M}^*_{H^p}$ whose points are in $\vec{\mathcal{M}}_{H^p}$, which is hence equipped with the universal object

$$\mathfrak{A} = \bar{\mathfrak{A}}|_{\mathfrak{M}_{H^p}} \to \mathfrak{M}_{H^p}.$$

Set

$$\mathfrak{M}^{\mathrm{rig}}_{H^p} = \bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p} \cap (\mathcal{M}_{H^p})^{\mathrm{rig}}_{\mathbf{Q}_p},$$

with $(\mathcal{M}_{H^p})^{\mathrm{rig}}_{\mathbf{Q}_p}$ the rigid space associated to the $\mathbf{Q}_p$-scheme $(\mathcal{M}_{H^p})_{\mathbf{Q}_p}$, which then comes with the restriction

$$\mathfrak{A}^{\mathrm{rig}} = \bar{\mathfrak{A}}^{\mathrm{rig}}|_{\mathfrak{M}^{\mathrm{rig}}_{H^p}} \to \mathfrak{M}^{\mathrm{rig}}_{H^p}.$$

By adding the superscript ord, we have the analogous construction and notation for the ordinary locus.

3.2.2 Canonical Subgroup and Frobenius

The complement D of the ordinary locus of the special fibre $(\bar{\mathcal{M}}_{H^p})_0$ of $\bar{\mathcal{M}}_{H^p}$ is a Cartier divisor, which is the vanishing locus of the Hasse invariant h, by Theorems 2.4 and 2.6. We apply the construction in Section 3.1.2 to $\bar{X} = \bar{\mathcal{M}}^{\mathrm{rig}}_{H^p}$, and then have for $r \in (|p|^{1/e}, 1]$ the quasi-compact admissible opens $\bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}(r) \subset \bar{\mathcal{M}}^{\mathrm{rig}}_{H^p}$. In particular, we see by definition that $\bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}(1) = \bar{\mathfrak{M}}^{\mathrm{ord,rig}}_{H^p}$.

It is elementary to check by Definition 3.1 and [Lü, Corollary 5.11] that, for $r, s \in (|p|^{1/e}, 1)$ with $s < r$,

$$\bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}(r) \Subset \bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}(s) \tag{3.2.1}$$

and

$$\bar{\mathfrak{M}}^{\mathrm{ord,rig}}_{H^p} \Subset \bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}(r).$$

We have the following result from [Fa] (*cf.* [AIP, 4.1.3] for extension to semi-abelian schemes.)

Theorem 3.4 ([Fa, Theorem 6]) *For each $n \in \mathbf{Z}_{\geq 1}$ and r sufficiently close to* 1 *(depending on n), there is a canonical subgroup of level n $\mathcal{H}_n(r) \subset \bar{\mathfrak{A}}^{\mathrm{rig}}[p^n]$ the p-divisible group $G = \bar{\mathfrak{A}}^{\mathrm{rig}}[p^\infty]$ over $\bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}(r)$, which is locally free of rank p^{dg} over*

$\mathcal{O}_F$, and whose restriction to the ordinary locus $\mathfrak{M}^{\mathrm{ord,rig}}_{H^p}$ is the multiplicative subgroup $\mathfrak{A}^{\mathrm{rig}}[p^n]^{\mathrm{mult}} \subset \mathfrak{A}^{\mathrm{rig}}[p^n]$.

Moreover, the level-one canonical subgroup $\mathcal{H}_1(r)$ is the kernel of the Frobenius on G, and for any $1 \le m \le n$, $\mathcal{H}_n(r)/\mathcal{H}_m(r)$ is the canonical subgroup of $\bar{\mathfrak{A}}^{\mathrm{rig}}[p^n]/\mathcal{H}_m(r)$ of level $n-m$.

The multiplicative subgroup $\mathfrak{A}[p^n]^{\mathrm{mult}} \subset \mathfrak{A}[p^n]$ is a finite flat group scheme of order p^{ndg}. We have, by the proof of [Ka, 1.11.6], that $\mathfrak{A}/\mathfrak{A}[p^n]^{\mathrm{mult}}$ gives an element in $\mathfrak{M}^{\mathrm{ord}}_{H^p}$. We thus have a canonical map

$$\varphi^n \colon \mathfrak{M}^{\mathrm{ord}}_{H^p} \to \mathfrak{M}^{\mathrm{ord}}_{H^p}, \quad \mathfrak{A} \mapsto \mathfrak{A}/\mathfrak{A}[p^n]^{\mathrm{mult}}.$$

Proposition 3.5 *If r is close enough to 1, then the morphism φ induces the following morphism, which is finite flat of degree p^{ndg}:*

$$\overline{\varphi^n}(r) \colon \bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}(r) \to \bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}(r^{p^n}).$$

Proof We have the map $\overline{\varphi^n} \colon \bar{\mathfrak{M}}^{\mathrm{ord}}_{H^p} \to \bar{\mathfrak{M}}^{\mathrm{ord}}_{H^p}$ defined by $\bar{\mathfrak{A}} \mapsto \bar{\mathfrak{A}}/\bar{\mathfrak{A}}[p^n]^{\mathrm{mult}}$, and then the map $\overline{\varphi^n}^{\mathrm{rig}} \colon \bar{\mathfrak{M}}^{\mathrm{ord,rig}}_{H^p} \to \bar{\mathfrak{M}}^{\mathrm{ord,rig}}_{H^p}$ induced by the first one on the rigid fibre.

The claim then follows from the argument in the proof of [KL, 3.1.7], together with the observation on abelian schemes in [Ka, 1.11.4]. ■

Corollary 3.6 *For r sufficiently close to 1, the sheaf $\bar{\mathfrak{A}}^{\mathrm{rig}}[p^n]/\mathcal{H}_n(r)$ is finite flat.*

Proof This is by Proposition 3.5. ■

3.3 The Formal Igusa Tower

Following the idea of Hida (see, *e.g.*, [Hi02]), we define $\bar{\mathfrak{M}}^{\mathrm{ur,ord}}_{H^p p^n}$ to be the Galois cover of $\bar{\mathfrak{M}}^{\mathrm{ord}}_{H^p}$ trivializing the étale sheaf $\bar{\mathfrak{A}}[p^n]/\bar{\mathfrak{A}}[p^n]^{\mathrm{mult}}$. The pre-image of $\mathfrak{M}^{\mathrm{ord}}_{H^p}$ under the covering map is written as $\mathfrak{M}^{\mathrm{ur,ord}}_{H^p p^n}$. We then have a proper map

$$\bar{\mathfrak{M}}^{\mathrm{ur,ord}}_{H^p p^n} \to \bar{\mathfrak{M}}^{\mathrm{ord}}_{H^p} \to \mathfrak{M}^{*,\mathrm{ord}}_{H^p},$$

for which the Stein factorization is written as $\mathfrak{M}^{*,\mathrm{ur,ord}}_{H^p p^n}$.

Similarly as before, we have the universal semi-abelian scheme

$$\bar{\mathfrak{A}}_n \to \bar{\mathfrak{M}}^{\mathrm{ur,ord}}_{H^p p^n}$$

and the universal abelian scheme

$$\mathfrak{A}_n \to \mathfrak{M}^{\mathrm{ur,ord}}_{H^p p^n}$$

by restriction. Consider the associated rigid space $\bar{\mathfrak{M}}^{\mathrm{ur,ord,rig}}_{H^p p^n}$ and the open subspace $\mathfrak{M}^{\mathrm{ur,ord,rig}}_{H^p p^n}$ by taking intersection with $(\mathcal{M}^{\mathrm{ord}}_{H^p p^n})^{\mathrm{rig}}_{\mathbf{Q}_p}$, the rigid space associated to the $\mathbf{Q}_p$-scheme $(\mathcal{M}^{\mathrm{ord}}_{H^p p^n})_{\mathbf{Q}_p}$. We then have the finite étale map

$$\mathrm{Ig}_n \colon \bar{\mathfrak{M}}^{\mathrm{ur,ord,rig}}_{H^p p^n} \to \bar{\mathfrak{M}}^{\mathrm{ord,rig}}_{H^p},$$

which is a Galois cover of $\bar{\mathfrak{M}}^{\mathrm{ord,rig}}_{H^p}$.

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For r close enough to 1, we set

$$\bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(r) \to \bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}(r)$$

to be the Galois cover of $\bar{\mathfrak{M}}^{\mathrm{rig}}_{H^p}(r)$ trivializing the finite flat sheaf $\bar{\mathfrak{A}}^{\mathrm{rig}}[p^n]/\mathcal{H}_n(r)$ (see Corollary 3.6). In particular, the ordinary locus $\mathfrak{M}^{\mathrm{ur,ord,rig}}_{H^p p^n} = \bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(1)$, which justifies the notation. Set

$$\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r) = \bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(r) \cap (\mathcal{M}_{H^p p^n})^{\mathrm{rig}}_{\mathbf{Q}_p}$$

for r close enough to 1, with $(\mathcal{M}_{H^p p^n})^{\mathrm{rig}}_{\mathbf{Q}_p}$ the rigid space associated to the $\mathbf{Q}_p$-scheme $(\mathcal{M}_{H^p p^n})_{\mathbf{Q}_p}$. For $s < r$ with s close enough to 1, we have

$$\bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(r) \Subset \bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(s), \tag{3.3.1}$$

by Lemma 3.2 and (3.2.1).

Again by the proof of [Ka, 1.11.6], we have a well-defined map

$$\varphi_n \colon \mathfrak{M}^{\mathrm{ur,ord}}_{H^p p^n} \to \mathfrak{M}^{\mathrm{ur,ord}}_{H^p p^n}, \quad \mathfrak{A}_n \mapsto \mathfrak{A}_n/\mathfrak{A}_n[p]^{\mathrm{mult}}.$$

sitting in the following commutative diagram

$$\begin{array}{ccc} \mathfrak{M}^{\mathrm{ur,ord}}_{H^p p^n} & \xrightarrow{\varphi_n} & \mathfrak{M}^{\mathrm{ur,ord}}_{H^p p^n} \\ \downarrow \mathrm{Ig}_n & & \downarrow \mathrm{Ig}_n \\ \mathfrak{M}_{H^p} & \xrightarrow{\varphi^1} & \mathfrak{M}_{H^p}. \end{array}$$

Proposition 3.7 *For r close enough to 1, the map φ_n induces the following morphism, which is finite flat of degree p^{dg}:*

$$\bar{\varphi}_n(r) \colon \bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(r) \to \bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(r^p).$$

Proof This follows from Proposition 3.5 and [KL, 2.2.1]. ∎

4 Overconvergence of Siegel–Hilbert Modular Forms

4.1 *P*-adic Banach Spaces

Set $\bar{\mathcal{Z}}^{\mathrm{ord,rig}} \subset \bar{\mathfrak{M}}^{\mathrm{ur,ord,rig}}_{H^p p^n}$ to be the tube of $(\bar{\mathfrak{M}}^{\mathrm{ur,ord}}_{H^p p^n})_0 \backslash (\mathfrak{M}^{\mathrm{ur,ord}}_{H^p p^n})_0$, and set $\mathcal{Z}^{\mathrm{ord,rig}}$ to be its intersection with $\mathfrak{M}^{\mathrm{ur,ord,rig}}_{H^p p^n}$.

Lemma 4.1 (Köcher Principle) *Assume $dg > 1$ (to exclude the modular curve case). For $r < 1$ sufficiently close to 1, we have the natural isomorphisms*

$$H^0\big(\bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(r), \omega^\kappa\big) \xrightarrow{\sim} H^0\big(\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r), \omega^\kappa\big) \xrightarrow{\sim} H^0\big(\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r) \backslash \mathcal{Z}^{\mathrm{ord,rig}}, \omega^\kappa\big).$$

Proof It suffices to show that on formal affine open subsets, the natural map

$$H^0(\bar{\mathfrak{M}}^{\mathrm{ur,ord}}_{H^p p^n}, \omega^\kappa) \to H^0(\mathfrak{M}^{\mathrm{ur,ord}}_{H^p p^n}, \omega^\kappa)$$

is an isomorphism. Furthermore, by the existence of the Galois covering of $\mathfrak{M}^{\mathrm{ord}}_{H^p}$ by $\mathfrak{M}^{\mathrm{ur,ord}}_{H^p p^n}$, we just need to show the isomorphism for $n = 0$. This then reduces to

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the Köcher principle for classical forms without level at p, which can be shown as in [FC, Proposition 1.5, Chapter V]. ∎

Recall we have the connected component $\mathcal{M}^{\gamma_\nu}_{H^p p^n}$ of $\mathcal{M}^{\nu}_{H^p p^n}$ for any prime ideal $\nu \subset \mathcal{O}$ and double coset γ_ν.

Define $\mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r)$ by the Cartesian diagram

$$\begin{array}{ccc} \mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r) & \longrightarrow & (\mathcal{M}^{\gamma_\nu}_{H^p p^n})^{\mathrm{rig}}_{\mathbf{Q}_p} \\ \big\downarrow & & \big\downarrow {\scriptstyle \pi_{1,\gamma_\nu}} \\ \mathfrak{M}^{\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r) & \longrightarrow & (\mathcal{M}_{H^p p^n})^{\mathrm{rig}}_{\mathbf{Q}_p}. \end{array}$$

Note that the left vertical map is finite étale, being the base change of the finite étale map π_{1,γ_ν}. We also denote it by π_{1,γ_ν} and the other projection by π_{2,γ_ν}.

Proposition 4.2 *Let $r < 1$ be sufficiently close to 1. We have that $\mathcal{O}\big(\mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r)\big)$ is a p-adic Banach space with respect to the norm*

$$|f|_r := \sup_{x \in \mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r)} |f(x)|$$

for a function $f \in \mathcal{O}\big(\mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r)\big)$. This is the same as the norm

$$|f|^\circ_r := \sup_{x \in \mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r) \setminus \pi^{-1}_{1,\gamma_\nu}(\mathcal{Z}^{\mathrm{ord},\mathrm{rig}})} |f(x)|.$$

Proof We use the argument in the proof of [KL, 4.1.6].

First note that $|f|^\circ_r$ is a well-defined norm on the p-adic Banach space

$$\mathcal{O}\big(\mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r) \setminus \pi^{-1}_{1,\gamma_\nu}(\mathcal{Z}^{\mathrm{ord},\mathrm{rig}})\big),$$

with the latter space being finite over $\bar{\mathfrak{M}}^{\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r) \setminus \bar{\mathcal{Z}}^{\mathrm{ord},\mathrm{rig}}$ and hence quasi-compact. (If $\mathcal{F}$ is a coherent sheaf on a quasi-compact rigid space X, then $\mathcal{F}(X)$ is a p-adic Banach space.)

Then we only need to show that

$$|f|_r = |f|^\circ_r, \quad \forall f \in \mathcal{O}\big(\mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r)\big), \tag{4.1.1}$$

which will then realize

$$\mathcal{O}\big(\mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r)\big) \hookrightarrow \mathcal{O}\big(\mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{rig}}_{H^p p^n}(r) \setminus \pi^{-1}_{1,\gamma_\nu}(\mathcal{Z}^{\mathrm{ord},\mathrm{rig}})\big)$$

as a closed subspace. For this, recall we have the Köcher principle and the fact that the natural projection

$$\mathfrak{M}^{\gamma_\nu,\mathrm{ur},\mathrm{ord},\mathrm{rig}}_{H^p p^n} \xrightarrow{\pi_{1,\gamma_\nu}} \mathfrak{M}^{\mathrm{ur},\mathrm{ord},\mathrm{rig}}_{H^p p^n} \xrightarrow{\mathrm{Ig}_n} \mathfrak{M}^{\mathrm{rig}}_{H^p}$$

is finite étale. The former is Lemma 4.1, and the latter holds because both π_{1,γ_ν} and Ig_n are finite étale by construction. Then by the argument in the proof of [KL, 4.1.6], we are reduced to show (4.1.1) for $r = 1$, $n = 0$, and may assume f extends to $f \in \mathcal{O}(\bar{\mathfrak{M}}^{\mathrm{ord}}_{H^p})$ (by Lemma 4.1). In this case we may assume that its image $f_0 \in \mathcal{O}(\bar{\mathfrak{M}}^{\mathrm{ord}}_{H^p})_0$ is non-zero. If $|f|^\circ_1 < |f|_1 = 1$, then f_0 is nilpotent on the open subset

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$(\mathfrak{M}_{H^p}^{\mathrm{ord}})_0$ of $(\bar{\mathfrak{M}}_{H^p}^{\mathrm{ord}})_0$. Thus it vanishes on $(\mathfrak{M}_{H^p}^{\mathrm{ord}})_0$ and hence on the whole $(\bar{\mathfrak{M}}_{H^p}^{\mathrm{ord}})_0$ by Theorem 2.6, a contradiction. ■

Corollary 4.3 *Let $\theta\colon \pi_{2,\gamma_v}^*\big((\det\omega)^{k_0(p-1)}\big) \to \pi_{1,\gamma_v}^*\big((\det\omega)^{k_0(p-1)}\big)$ be the canonical map of sheaves on $\mathfrak{M}_{H^p p^n}^{\gamma_v,\mathrm{ur},\mathrm{rig}}$. Then for r sufficiently close to 1,*

$$\theta\big(\pi_{2,\gamma_v}^*(E)\big) = f_{k_0}\pi_{1,\gamma_v}^*(E) \in H^0\Big(\mathfrak{M}_{H^p p^n}^{\gamma_v,\mathrm{ur},\mathrm{rig}}(r), \pi_{1,\gamma_v}^*\big((\det\omega)^{k_0(p-1)}\big)\Big)$$

for some $f_{k_0} \in \mathcal{O}\big(\mathfrak{M}_{H^p p^n}^{\gamma_v,\mathrm{ur},\mathrm{rig}}(r)\big)$. Here we have used the same symbol E to denote the pullback of E from $\mathfrak{M}_{H^p}^{\mathrm{rig}}$ to its Galois cover $\mathfrak{M}_{H^p p^n}^{\mathrm{ur},\mathrm{rig}}$.

Moreover, we have that $f_{k_0} - 1$ is topologically nilpotent. In fact, for any $0 < \epsilon \le 1$, there is an $r \le 1$ such that $|f_{k_0} - 1|_r \le |p|^\epsilon$.

Proof We see that the proof of [KL, 4.1.7] applies. We only give a sketch here. As before, we reduce to the case $n = 0$. The first assertion then follows from the fact that the special fibre $(\mathfrak{M}_{H^p}^{*,\mathrm{ord}})_0$ is normal, by Theorem 2.7. To show the second assertion, first note that the case $r = 1$ follows from Lemma 2.10 and Proposition 4.2. The case for a general r then follows from the $r = 1$ case, as well as Proposition 3.3. ■

Using Corollary 4.3 we now show the following.

Proposition 4.4

(i) *We have the inclusion*

$$\pi_{2,\gamma_v}(\mathfrak{M}_{H^p p^n}^{\gamma_v,\mathrm{ur},\mathrm{ord},\mathrm{rig}}) \subset \mathfrak{M}_{H^p p^n}^{\mathrm{ur},\mathrm{ord},\mathrm{rig}},$$

and the inclusion

$$\pi_{2,\gamma_v}\big(\mathfrak{M}_{H^p p^n}^{\gamma_v,\mathrm{ur},\mathrm{rig}}(r)\big) \subset \mathfrak{M}_{H^p p^n}^{\mathrm{ur},\mathrm{rig}}(r)$$

for r sufficiently close to 1.

In particular, the inclusions above hold for $\pi_{2,\mathfrak{a}}$ on $\mathfrak{M}_{H^p p^n}^{\mathfrak{a},\mathrm{ur},\mathrm{ord},\mathrm{rig}}$, with $\mathfrak{a} \subset \mathcal{O}$ an ideal.

(ii) *For $r \to 1^-$, $\pi_{2,(p)}$ induces a map*

$$\mathfrak{M}_{H^p p^n}^{(p),\mathrm{ur},\mathrm{rig}}(r^p) \to \mathfrak{M}_{H^p p^n}^{\mathrm{ur},\mathrm{rig}}(r).$$

Proof (i) The first assertion follows from the construction of $\mathfrak{M}_{H^p p^n}^{\mathrm{ur},\mathrm{ord},\mathrm{rig}}$.

For the second inclusion, we argue as in the proof of [KL, 4.1.10]. First observe that it is enough to show this for E-valued points for any finite extension $E/\mathbf{Q}_p$. By the construction of $\mathfrak{M}_{H^p p^n}^{\gamma_v,\mathrm{ur},\mathrm{rig}}(r)$ we may assume $n = 0$. Let $f\colon A \to B$ be an element in $\mathfrak{M}_{H^p}^{\gamma_v,\mathrm{ur},\mathrm{rig}}(r)(E)$. We may enlarge E so that A extends to a semi-abelian scheme $\bar{A}$ over $\mathcal{O}_E$. We may and do assume that $\bar{A}$ is an abelian scheme, because otherwise $r = 1$ and we go back to the first assertion. Then we can extend $\mathrm{Ker}(f)$ to a finite flat subgroup scheme of $\bar{A}$. The quotient of $\bar{A}$ by this subgroup scheme is denoted by $\bar{B}$, and the projection $\bar{A} \to \bar{B}$ by pr. Let $\mathbf{v}_{\bar{A}}$ (resp. $\mathbf{v}_{\bar{B}}$) be a basis of $H^0(\bar{A}, \Omega^1_{\bar{A}/\mathcal{O}_E})$ (resp. $H^0(\bar{B}, \Omega^1_{\bar{B}/\mathcal{O}_E})$). Then we must have

$$\mathrm{pr}^*(\det \mathbf{v}_{\bar{B}}) = a \det \mathbf{v}_{\bar{A}}$$

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for some $a \in \mathcal{O}_E$.

Now, by the definition of overconvergence, it suffices to show that

$$|E(A, \det \mathbf{v}_{\bar{A}})| \le |E(B, \det \mathbf{v}_{\bar{B}})|.$$

By Corollary 4.3 we have

$$E(B, \det \mathbf{v}_{\bar{B}}) = f_{k_0} E\big(A, \mathrm{pr}^*(\det \mathbf{v}_{\bar{B}})\big),$$

with $f_{k_0} - 1$ topologically nilpotent. On the other hand, we have the equality

$$E\big(A, \mathrm{pr}^*(\det \mathbf{v}_{\bar{B}})\big) = a^{(1-p)k_0} E(A, \det \mathbf{v}_{\bar{A}}).$$

Now the result follows because $|f_{k_0}(A, \mathbf{v}_{\bar{A}})| = 1$ and $|a^{(1-p)k_0}| \ge 1$.

(ii) This is proved as in [KL, 4.3.3]. By Proposition 3.7, for s close enough to 1 we have the natural map

$$\bar{\varphi}_n(s)\colon \mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(s) \to \mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(s^p)$$

by restricting to the intersection of the map in Proposition 3.7 with $(\mathcal{M}_{H^p p^n})^{\mathrm{rig}}_{\mathbf{Q}_p}$. For $r \to 1^-$, by part (i) we see $\pi_{2,(p)}$ induces a map

$$\mathfrak{M}^{(p),\mathrm{ur,rig}}_{H^p p^n}(r^p) \to \mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(s)$$

for some $r^p \le s$. If $s \ge r$, we are done. Now assume $s < r$. Thus it suffices to show that their composition

$$\mathfrak{M}^{(p),\mathrm{ur,rig}}_{H^p p^n}(r^p) \xrightarrow{\pi_{2,(p)}} \mathfrak{M}^{\mathrm{ur,ord,rig}}_{H^p p^n}(s) \xrightarrow{\bar{\varphi}_n(s)} \mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(s^p)$$

factors through $\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r^p) \subset \mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(s^p)$, for which it is in turn enough to show the further composition with $\mathfrak{M}^{\mathrm{rig}}_{H^p}(s^p)$

$$\mathfrak{M}^{(p),\mathrm{ur,rig}}_{H^p p^n}(r^p) \xrightarrow{\pi_{2,(p)}} \mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(s) \xrightarrow{\bar{\varphi}_n(s)} \mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(s^p) \xrightarrow{\mathrm{Ig}_n} \mathfrak{M}^{\mathrm{rig}}_{H^p}(s^p)$$

factors through $\mathfrak{M}^{\mathrm{rig}}_{H^p}(r^p)$.

Tracing the construction we know that for $r \to 1^-$ the latter composition is induced by the map

$$\mathfrak{M}^{(p),\mathrm{ur,ord}}_{H^p p^n} \xrightarrow{\pi_{1,(p)}} \mathfrak{M}^{\mathrm{ur,ord}}_{H^p p^n} \xrightarrow{\mathrm{Ig}_n} \mathfrak{M}^{\mathrm{ord}}_{H^p} \xrightarrow{[\cdot p]} \mathfrak{M}^{\mathrm{ord}}_{H^p},$$

where the map $[\cdot p]$ is the multiplication by p on the level structure of the universal abelian scheme. To see this, using Proposition 3.3, we only have to check the statement for the case $r = 1$, which is easily seen. Meanwhile, we notice that $[\cdot p]$ induces the identity map on Hasse invariant, because the Hasse invariant is independent of level structures. Hence it maps $\mathfrak{M}^{\mathrm{rig}}_{H^p}(r^p)$ to itself. This concludes the proof. ■

4.2 Hecke Operators on Overconvergent Siegel–Hilbert Modular Forms

Let $L/\mathbf{Q}_p$ be a finite extension and $\mathcal{R}$ an L-affinoid algebra with a fixed sub-multiplicative semi-norm extending the norm on L, and $Y \in \mathcal{R}$ such that

$$|Y| < |p|^{\frac{1}{p-1}-1}.$$

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We use the convention that for a $\mathbf{Q}_p$-analytic space X, $X_{\mathcal{R}} = X \times_{\mathrm{Sp}\,\mathbf{Q}_p} \mathrm{Sp}\,\mathcal{R}$. Define

$$\begin{aligned} M_{H^p p^n,\kappa,r}(L) &= H^0\left(\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r)_L, \omega^\kappa\right), \\ M_{H^p p^n,\kappa+Y,r}(\mathcal{R}) &= H^0\left(\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r)_{\mathcal{R}}, \omega^\kappa\right), \\ M^{\dagger}_{H^p p^n,\kappa}(L) &= \varinjlim_{r\to 1^{-1}} M_{H^p p^n,\kappa,r}(L), \\ M^{\dagger}_{H^p p^n,\kappa+Y}(\mathcal{R}) &= \varinjlim_{r\to 1^{-1}} M_{H^p p^n,\kappa+Y,r}(\mathcal{R}). \end{aligned}$$

Remark 4.5 Let U_F^+ be the group of totally positive units in $\mathcal{O}_F$, and $U_{F,N}$ its subgroup of elements that are congruent to 1 modulo N. As noted in [KL, 1.11.8] and Remark 2.1, the moduli spaces we have constructed are for $G' = G \times_{\mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_m} \mathbf{G}_m$, where the two maps in the product are the determinantal and diagonal ones. This is because multiplying by U_F^+ on the polarizations of tuples in $\mathcal{M}_{H^p p^n}$ is an isomorphism for the subgroup $U_{F,N}^2 \subset U_F^+$. We thus get an action of the finite group $U_F^+/U_{F,N}^2$ on $\mathcal{M}_{H^p p^n}$, which induces a natural action on $H^0(\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r), \omega^\kappa)$. Hence the $U_F^+/U_{F,N}^2$-invariants of the spaces above are the spaces of forms on $G = \mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{GSp}_{2g}$.

By abuse of notation, we call these spaces the spaces of overconvergent Siegel–Hilbert modular forms of level $H^p p^n$ and of weight κ (resp. $\kappa + Y$) with coefficients in L (resp. $\mathcal{R}$).

By Proposition 4.4 (i), we have two projections

$$\pi_{1,\gamma_v}, \pi_{2,\gamma_v} \colon \mathfrak{M}^{\gamma_v,\mathrm{ur,rig}}_{H^p p^n}(r) \longrightarrow \mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r).$$

We then get the (pullback of) canonical map of sheaves on $\mathfrak{M}^{\gamma_v,\mathrm{ur,rig}}_{H^p p^n}(r)_{\mathcal{R}}$

$$\pi^*_{2,\gamma_v}\omega^\kappa \longrightarrow \pi^*_{1,\gamma_v}\omega^\kappa,$$

whose composition with multiplication by $f_{k_0}^{\frac{Y}{k_0(p-1)}}$ gives the map

$$\pi^*_{2,\gamma_v}\omega^\kappa \longrightarrow \pi^*_{1,\gamma_v}\omega^\kappa \xrightarrow{\cdot f_{k_0}^{\frac{Y}{k_0(p-1)}}} \pi^*_{1,\gamma_v}\omega^\kappa.$$

Here

$$f_{k_0}^{\frac{Y}{k_0(p-1)}} := \exp\Big(\frac{Y}{k_0(p-1)} \log f_{k_0}\Big)$$

is a well-defined element in $\mathcal{O}(\mathfrak{M}^{\gamma_v,\mathrm{ur,rig}}_{H^p p^n}(r)_{\mathcal{R}})$ for r such that $|f_{k_0} - 1|_r \leq |p|^\epsilon$, where ϵ is chosen to satisfy $|Y|\,|p|^\epsilon < |p|^{1/(p-1)}$. We remark that the analyticity of $f_{k_0}^{\frac{Y}{k_0(p-1)}}$ follows from the assumption that $|Y| < |p|^{\frac{1}{p-1}-1}$ (and the assumption that $p \nmid k_0$).

Applying $\pi_{1,\gamma_v *}$ to the above map, and pre-composing it with the map from ω^κ and composing it with the trace map, we obtain

$$\omega^\kappa \to \pi_{1,\gamma_v*}\pi^*_{2,\gamma_v}\omega^\kappa \longrightarrow \pi_{1,\gamma_v*}\pi^*_{1,\gamma_v}\omega^\kappa \xrightarrow{\pi_{1,\gamma_v*}\circ\cdot f_{k_0}^{\frac{Y}{k_0(p-1)}}} \longrightarrow \pi_{1,\gamma_v*}\pi^*_{1,\gamma_v}\omega^\kappa \xrightarrow{\mathrm{Tr}} \omega^\kappa.$$

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Then taking global sections, we get an endomorphism T_{γ_v} of $H^0\big(\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r)_{\mathcal{R}}, \omega^\kappa\big)$. The resulting Hecke operator will denoted by $T_{v,i}$ (resp. S_v) if $\gamma_v = T_{v,i}$ (resp. S_v).

Letting $r \to 1^-$, we obtain the Hecke operator on $M^\dagger_{H^p p^n,\kappa+Y}(\mathcal{R})$, which is denoted by the same symbol. The ring of endomorphisms generated by all the Hecke operators (with v and γ_v varying) is denoted by $\mathbf{T}^\dagger_{H^p p^n,\kappa+Y}$. The product of $T_{v,1}$'s for all $v|p$ is denoted by $U_{(p)}$.

The construction made above and Corollary 4.3 thus give rise to the following.

Proposition 4.6 *Let $\psi_t\colon \mathcal{R} \to L'$ be the character to a finite extension L'/L that sends Y to $(p-1)k_0 t$ for some $t \in \mathbf{Z}_{\geq 0}$. We have the following commutative diagram compatible with the actions of Hecke operators T:*

(4.2.1)
$$\begin{array}{ccccc} M^\dagger_{H^p p^n,\kappa+Y}(\mathcal{R}) & \xrightarrow{\mathrm{Id}\otimes\psi_t} & M^\dagger_{H^p p^n,\kappa}(L') & \xrightarrow{\cdot E^t} & M^\dagger_{H^p p^n,\kappa\cdot \mathrm{Nm}^{(p-1)k_0 t}}(L') \\ {\scriptstyle T}\big\downarrow & & & & {\scriptstyle T}\big\downarrow \\ M^\dagger_{H^p p^n,\kappa+Y}(\mathcal{R}) & \xrightarrow{\mathrm{Id}\otimes\psi_t} & M^\dagger_{H^p p^n,\kappa}(L') & \xrightarrow{\cdot E^t} & M^\dagger_{H^p p^n,\kappa\cdot \mathrm{Nm}^{(p-1)k_0 t}}(L') \end{array}$$

Proof Let $T = T_{\gamma_v}$. Then we are supposed to check that for v$f\otimes x \in M^\dagger_{H^p p^n,\kappa+Y}(\mathcal{R})$,

(4.2.2)
$$\psi_t\big(T(f\otimes x)\big) \cup E^t = T\big(f\otimes \psi_t(x) \cup E^t\big).$$

(This suffices for the proof, since the T_{γ_v}'s generate the Hecke algebra.)

Now, unwinding the definition of the Hecke operator T, we see that for $A \to B$ an isogeny in $\mathfrak{M}^{\gamma_v,\mathrm{ur,rig}}_{H^p p^n}(r)$,

$$T(f\otimes x)(A) = f(B) \otimes f_{k_0}^{\frac{Y}{k_0(p-1)}} x.$$

Then, applied to $(A \to B)$, the left hand side of the equality (4.2.2) is equal to

$$f(B) \otimes \psi_t(x) \cup E^t(B),$$

while the right hand side is equal to

$$\big(f(B) \otimes \psi_t(x)\big) \cup f_{k_0}^t E^t(A),$$

since $\psi_t(Y) = k_0(p-1)t$. Now the result follows from Corollary 4.3. ∎

Proposition 4.7 *For $s < r < 1$ with s sufficiently close to 1, the following natural inclusion is completely continuous:*

$$\mathrm{Res}(s,r)\colon H^0\big(\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(s), \omega^\kappa\big) \hookrightarrow H^0\big(\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r), \omega^\kappa\big).$$

Proof It is equivalent to showing this for the natural inclusion

$$H^0\big(\bar{\mathfrak{M}}^{\mathrm{ur,ord,rig}}_{H^p p^n}(s), \omega^\kappa\big) \hookrightarrow H^0\big(\bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(r), \omega^\kappa\big)$$

by the Köcher principle Lemma 4.1.

Recall $\bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(r) \Subset \bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(s)$ from (3.3.1). Now we conclude by [KL, Proposition 2.4.1]. ∎

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Lemma 4.8 *Suppose r is close enough to* 1*. The Hecke operator* $U_{(p)}$ *on the* $\mathcal{R}$*-module* $M_{H^p p^n,\kappa+Y,r}(\mathcal{R})$ *can be constructed as the composition of the natural inclusion* $\operatorname{Res}(r^p, r)$ *and the following map induced by* $\pi_{2,(p)}$*:*

$$H^0\big(\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r), \omega^\kappa\big) \to H^0\big(\mathfrak{M}^{\mathrm{ur,rig}}_{H^p p^n}(r^p), \omega^\kappa\big). \tag{4.2.3}$$

Proof This follows from Proposition 4.4 (ii). ■

Corollary 4.9 *For r close enough to* 1*, the action of* $U_{(p)}$ *on* $M_{H^p p^n,\kappa+Y,r}(\mathcal{R})$ *is completely continuous.*

Proof We know by Proposition 4.7 that the map $\operatorname{Res}(r, r^p)$ is completely continuous. Moreover, the map (4.2.3) is continuous. Since a composition of a continuous map followed by a completely continuous one is again completely continuous, we are done. ■

Remark 4.10 It is the eigenvalues of the Hecke operator $U_{(p)}$ that we will interpolate, since we only construct a one-parameter family of overconvergent Siegel–Hilbert eigenforms.

4.3 Constructing Families of Overconvergent Siegel–Hilbert Modular Forms

4.3.1 The Setup

Recall that T_g is the standard diagonal maximal torus of $\mathrm{GSp}_{2g/\mathbf{Z}}$. Denote by $c\colon T_g \to \mathbf{G}_m$ the character:

$$c\colon \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_g & & & \\ & & & ba_1^{-1} & & \\ & & & & \ddots & \\ & & & & & ba_g^{-1} \end{pmatrix} \mapsto a_1 \cdots a_g b^{-2}.$$

Let $\mathcal{W}$ be the rigid space whose E-valued points are continuous homomorphisms in $\operatorname{Hom}_{\mathrm{cont}}\big(T_g(\mathcal{O}\otimes_{\mathbf{Z}}\mathbf{Z}_p), E^\times\big)$ for any (not necessarily finite) field extension $E/\mathbf{Q}_p$.

In the rest of the paper, we fix a classical weight κ and a finite extension $L/\mathbf{Q}_p$. For our purpose we only need the part of the weight space that "differs" from our fixed weight κ by parallel weights. Thus let $\mathcal{W}_\kappa$ be the admissible subspace of $\mathcal{W}$ whose E-valued points, for $E \subset \mathbf{C}_p$ a closed subfield containing L, are

$$\mathcal{W}_\kappa(E) = \{\chi = \kappa\cdot(\tau\circ c\circ \mathrm{Nm})\colon T_g(\mathcal{O}\otimes_{\mathbf{Z}}\mathbf{Z}_p) \to E^\times\}$$

for some continuous character $\tau\colon \mathbf{Z}_p^\times \to E^\times$ satisfying

$$v_p\big(1-\tau(t)\big) > \frac{1}{p-1}, \quad t\in\mathbf{Z}_p^\times.$$

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We define a rigid analytic function Y on $\mathcal{W}_\kappa$ as follows: if $\chi \in \mathcal{W}_\kappa(E)$ is as above, and is associated to $\tau\colon \mathbf{Z}_p^\times \to E^\times$, then the value of Y at χ is given by

$$Y(\chi) = \frac{\log \tau(t)}{\log t}$$

for $t \in \mathbf{Z}_p^\times$ sufficiently close to the identity.

By the construction above, we have that $|Y| < |p|^{\frac{1}{p-1}-1}$, hence the Banach module $M^\dagger_{H^p p^n,\kappa+Y}(\mathcal{R})$ of overconvergent forms is well-defined, for any $\operatorname{Sp}\mathcal{R} \subset \mathcal{W}_\kappa$. Let $\mathbf{T}^{\dagger,\mathrm{ur}}_{H^p p^n,\kappa+Y}$ be the closure of the ring of endomorphisms on $M^\dagger_{H^p p^n,\kappa+Y}(\mathcal{R})$ generated by the Hecke operators at the places away from the level, under the norm defined in Proposition 4.2.

Proposition 4.11 *The $\mathcal{R}$-algebra $\mathbf{T}^{\dagger,\mathrm{ur}}_{H^p p^n,\kappa+Y}$ is commutative.*

Proof Let $\mathcal{W}^{\mathrm{cl}}_\kappa$ be a Zariski dense set of integral weights in the Y-neighbourhood of κ, which can be achieved by taking the parameters t to be sufficiently large powers of p, by Proposition 4.6. By the analyticity obtained in Proposition 4.6, each element in $\mathbf{T}^{\dagger,\mathrm{ur}}_{H^p p^n,\kappa+Y}$ is determined by the Zariski dense set $\mathcal{W}^{\mathrm{cl}}_\kappa$. We then have the injection

$$\mathbf{T}^{\dagger,\mathrm{ur}}_{H^p p^n,\kappa+Y} \hookrightarrow \prod_{w\in\mathcal{W}^{\mathrm{cl}}_\kappa} \mathbf{T}^{\dagger,\mathrm{ur}}_{H^p p^n,w},$$

where each factor is the specialization. On the other hand, each Hecke ring $\mathbf{T}^{\dagger,\mathrm{ur}}_{H,p^n,w}$ with the fixed integral weight w is commutative, being the completion of a commutative algebra of Hecke correspondences. Thus the product over $\mathcal{W}^{\mathrm{cl}}_\kappa$ is commutative, so is its subring $\mathbf{T}^{\dagger,\mathrm{ur}}_{H^p p^n,\kappa+Y}$. ■

Let

$$Z_\kappa = \operatorname{Sp}\mathbf{T}^{\dagger,\mathrm{ur}}_{H^p p^n,\kappa+Y}$$

be the rigid space over L associated to the $\mathcal{R}$-algebra $\mathbf{T}^{\dagger,\mathrm{ur}}_{H^p p^n,\kappa+Y}$. It comes with the weight map $\underline{w}\colon Z_\kappa \to \operatorname{Sp}\mathcal{R} \subset \mathcal{W}_\kappa$. Define $X_\kappa = Z_\kappa \times \mathbb{G}_m$. Write x_p for the canonical co-ordinate on $\mathbb{G}_m$.

4.3.2 Construction by the Coleman–Mazur Machinery

Now we are ready to define the one-parameter families of overconvergent Siegel–Hilbert modular forms of level $H^p p^n$ as Coleman and Mazur proceed in [CM]. For our purpose, it is enough to construct it over any affinoid quasi-compact subset $\operatorname{Sp}\mathcal{R} \subset \mathcal{W}_\kappa$. We fix such an $\mathcal{R}$ from now on.

Set $\mathcal{H}$ to be the (topological) commutative ring generated by the formal variables $X_{(p)}$, together with $X_{v,i}, Y_v$ (here $i = 1,\dots,g$) for all prime ideals $v \subset \mathcal{O}$ away from the level. Let $\iota\colon \mathcal{H} \to \mathcal{O}(X_\kappa)$ be the map sending

$$X_{v,i} \mapsto t_{v,i}, \quad Y_v \mapsto s_v, \quad X_{(p)} \mapsto x_p.$$

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Here we have denoted by $t_{v,i}$ and s_v the image of the Hecke operators $T_{v,i}$ and S_v in $\mathcal{O}(X_\kappa)$ respectively, regarded as functions on Z_κ. Then $\mathcal{H}$ acts on $M^{\dagger}_{H^p p^n,\kappa+Y}(\mathcal{R})$ with the action factoring through $\iota(\mathcal{H})$.

For r sufficiently close to 1, we know by Corollary 4.9 that $U_{(p)}$ acts completely continuously on $(M_{H^p p^n,\kappa+Y,r})(\mathcal{R})$. This implies that the action of $\iota(\alpha)U_{(p)}$ on $M_{H^p p^n,\kappa+Y,r}(\mathcal{R})$ is completely continuous for any $\alpha \in \mathcal{H}$. Following [CM, Section 4], for each $\alpha \in \mathcal{H}$, we can form the Fredholm series

$$P_\alpha(T) = \det_{\mathcal{R}}\big(1 - \iota(\alpha)U_{(p)}T | M_{H^p p^n,\kappa+Y,r}(\mathcal{R})\big) \in \mathcal{R}[[T]],$$

which is independent of the choice of r (for r sufficiently close to 1) by the following lemma.

Lemma 4.12 *Let $0 < r < r' < 1$ with r sufficiently close to 1. Then the Banach $\mathcal{R}$-module $M_{H^p p^n,\kappa+Y,r}(\mathcal{R})$ admits an orthogonal basis that is also an orthogonal basis for the $\mathcal{R}$-submodule $M_{H^p p^n,\kappa+Y,r'}(\mathcal{R})$.*

Proof Since $M_{H^p p^n,\kappa+Y,r}(\mathcal{R}) = M_{H^p p^n,\kappa+Y,r}(\mathbf{Q}_p)\hat{\otimes}_{\mathbf{Q}_p}\mathcal{R}$, we may assume $\mathcal{R} = \mathbf{Q}_p$ and $Y = 0$. Recall that, as r is sufficiently close to 1, the natural map $\mathfrak{M}^{*,\mathrm{ur,rig}}_{H^p p^n}(r) \to \mathfrak{M}^{*,\mathrm{rig}}_{H^p}(r)$ is finite étale. We can conclude the proof by [KL, 2.4.5], namely in the notation of *loc. cit.* we let $\mathcal{F}$ be the push-forward of ω^κ under the composition (recall the notation from Section 3.3)

$$\bar{\mathfrak{M}}^{\mathrm{ur,rig}}_{H^p p^n}(r) \to \mathfrak{M}^{*,\mathrm{ur,rig}}_{H^p p^n}(r) \to \mathfrak{M}^{*,\mathrm{rig}}_{H^p}(r),$$

which is proper. Moreover, we have the line bundle $\mathcal{L} = (\det\omega)^{p-1}$, which is ample over $\mathcal{M}^*_{H^p}$, and $\mathcal{D} \subset (\mathcal{M}^*_{H^p})_{\mathbf{F}_p}$, the divisor where the Hasse invariant h vanishes. Then [KL, 2.4.5] gives the result we require. ∎

Set $\mathcal{E}_\kappa = \mathcal{E}^{\mathrm{red}}_\kappa \subset X_\kappa$ to be the nilreduction of the Zariski-closed subspace of X_κ cut out by the ideal generated by the functions $P_\alpha\big((x_p\iota(\alpha))^{-1}\big)$ for all the $\alpha \in \mathcal{H}$ such that $\iota(\alpha)$ is a unit. Alternatively, we can define $\mathcal{E}_\kappa$ as follows.

The entire series associated to $\alpha \in \mathcal{H}$, $P_\alpha(T) \in \mathcal{R}[[T]]$, defines a closed subspace

$$\mathcal{Z}_\alpha \subset \operatorname{Sp}\mathcal{R} \times \mathbf{A}^1,$$

where T is regarded as the co-ordinate on $\mathbf{A}^1$. For each $\alpha \in \mathcal{H}$ such that $\iota(\alpha)$ is unit, we can define the map

$$r_\alpha\colon X_\kappa = Z_\kappa \times \mathbf{G}_m \to \operatorname{Sp}\mathcal{R} \times \mathbf{A}^1\colon x = (z,s) \mapsto \left(\underline{w}(z), \frac{s}{\big(\iota(\alpha)\big)(x)}\right),$$

where we have regarded $\iota(\alpha)$ as a function on X_κ. Then we set $\mathcal{E}_\kappa$ to be the nilreduction of

$$\bigcap_{\substack{\alpha\in\mathcal{H},\\ \iota(\alpha)\in\mathcal{O}(X_\kappa)^\times}} r_\alpha^{-1}(\mathcal{Z}_\alpha).$$

The following theorem is obtained from the above construction formally, as in [CM].

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Theorem 4.13

(i) *Let $E \subset \mathbf{C}_p$ be a closed subfield containing L. For an E-valued point $x \in \mathcal{E}_\kappa(E)$, there is a non-zero simultaneous eigenvector $f_x \in M^{\dagger}_{H^p p^n, \kappa+Y(x)}(E)$ for all the Hecke operators in $\mathbf{T}^{\dagger}_{H^p p^n, \kappa+Y}$ such that the Hecke eigenvalues $\lambda_{T_{v,i}}(z)$, $\lambda_{S_v}(x)$, $\lambda_{U_{(p)}}(x)$ for the operators $T_{v,i}$, S_v and $U_{(p)}$ satisfy*

$$\lambda_{T_{v,i}}(x) = t_{v,i}(x), \quad \lambda_{S_v}(x) = s_v(x), \quad \lambda_{U_{(p)}}(x) = x_p(x).$$

For a fixed $Y_0 \in E$ with $v_p(Y_0 - 1) > \frac{1}{p-1}$, the above assignment induces a bijection between the points $\{x \in \mathcal{E}_\kappa(E)\}_{Y(x)=Y_0}$ and systems of $\mathbf{T}^{\dagger}_{H^p p^n, \kappa+Y}$-eigenvalues of an eigenvector $f \in M^{\dagger}_{H^p p^n, \kappa+Y_0}(E)$ of finite slope *at p.*

(ii) *The rigid analytic space $\mathcal{E}_\kappa$ is a curve. The weight map $\underline{w}\colon \mathcal{E}_\kappa \to \mathcal{W}_\kappa$ is, locally in the domain, finite flat. The image of any component of $\mathcal{E}_\kappa$ under this map misses at most finitely many points in $\mathcal{W}_\kappa$.*

The following theorem plays a similar (yet weaker) role as the expected result that classical Siegel–Hilbert eigenforms are Zariski dense in the rigid analytic space $\mathcal{E}_\kappa$.

Theorem 4.14 *Let f be a classical Siegel–Hilbert modular eigenform of weight κ and of level $H^p p^n$. There exists, for any positive integer t with $v_p(t)$ large enough, a Siegel–Hilbert modular eigenform f_t of weight $\kappa \cdot \mathrm{Nm}^{(p-1)k_0 t}$ and of the same level, such that the Hecke eigenvalues on the f_t's converge p-adically to that of f, as $v_p(t) \to +\infty$. Furthermore, if f is cuspidal, then the f_t can also be taken to be cuspidal.*

Proof The proof is completely similar to that of [KL, 4.5.6].

As before, we take the weight space $\mathcal{W}_\kappa$ centered in κ to be $\mathrm{Sp}\,\mathcal{R}$ since the construction is local. By the construction of $\mathcal{E}_\kappa$, when $\iota(\alpha)$ is unit, we have a map

$$r_\alpha\colon \mathcal{E}_\kappa \longrightarrow \mathcal{Z}_\alpha.$$

By the method of [CM, Chapter 7], we see the projection r_α is finite.

Let $x \in \mathcal{E}_\kappa(L)$ be the point corresponding to f. By the arguments of [CM, 6.2.2 and 6.3.2] we may assume

$$r_\alpha^{-1}\big(r_\alpha(x)\big) = \{x\}. \tag{4.3.1}$$

By this property, we only need to find a family of elements in $\mathcal{Z}_\alpha(L)$ converging to $r_\alpha(x) := x_0$.

Let $w \in \mathrm{Sp}\,\mathcal{R}$ denote the weight of x. Let $x_1, \dots, x_r$ be the points in $\mathcal{Z}_\alpha$ that lie over the weight w and correspond to other (finitely many by [CM, 1.3.7]) roots of $P_\alpha(T)_w \in L[T]$, the specialization by w of $P_\alpha(T) \in \mathcal{R}[[T]]$. The $\iota(\alpha)U_{(p)}$-eigenvalue of x_i ($0 \le i \le r$) is denoted by λ_i. By the (local) finite flatness of the weight map $\underline{w}$ (shrinking $\mathrm{Sp}\,\mathcal{R}$ if necessary) we may assume there are disjoint connected components $\{\mathcal{Z}_i\}_{i=0,\dots,r}$ of $\mathcal{Z}_\alpha$, such that for any $0 \le i \le r$,

- x_i is the only point in $\mathcal{Z}_i$ among the points $\{x_0, \dots, x_r\}$.
- T/λ_0 is topologically unipotent on $\mathcal{Z}_0$, and is topologically nilpotent on $\mathcal{Z}_i$ for any $i \ge 1$.
- $\mathcal{Z}_i$ is finite over $\mathrm{Sp}\,\mathcal{R}$.

Thus $\bigcup_{i=1}^{r} \mathcal{Z}_i$ is finite flat over $\mathrm{Sp}\,\mathcal{R}$, hence corresponds to a polynomial $F(T) \in \mathcal{R}[[T]]$ dividing $P_\alpha(T)$. By the construction, we have the following well-defined idempotent operator

$$e = \lim_{n\to\infty} \Big(\frac{\iota(\alpha)U_{(p)}}{\lambda_0}\Big)^{n!} \frac{F(\alpha U_{(p)})}{F(\lambda_0)^{-1}},$$

which is easily checked to be the identity on a point in $\mathcal{Z}_0$ and kill any points in $\mathcal{Z}_i$ for $i \geq 1$.

Consider the integers t such that $Y^{-1}\big(Y(x) + (p-1)k_0 t\big) \in \mathrm{Sp}\,\mathcal{R}$. We form the Siegel–Hilbert modular eigenform of level $H^p p^n$ and weight $\kappa + (p-1)k_0 t$:

$$g_t = e(E^t \cdot f) = e(E^t \cdot g_0).$$

By Proposition 4.6 (applying the first row of diagram (4.2.1) to f) and the continuity of the Hecke action on $M^{\dagger}_{H^p p^n, \kappa+Y}(\mathcal{R})$, we have that

$$g_t \neq 0, \quad \text{if } v_p(t) \gg 0.$$

We can write $E^t \cdot g_0$ as a finite sum of classical eigenforms. If f is cuspidal to begin with, then $E^t \cdot g_0$ can be written as a finite sum of cuspidal eigenforms. Pick one of them so that the associated point $x_t \in \mathcal{E}_\kappa$ has image in $\mathcal{Z}_0$ under the projection r_α. By this construction, the point x is the limit of $r_\alpha(x_t)$ when t goes to 0 p-adically. Now by (4.3.1), we have that x is the limit of x_t. Let f_t be the classical Siegel–Hilbert modular form corresponding to x_t. This finally concludes the proof. ∎

Remark 4.15 Using the result of Bijakowski [Bi] on classicality of overconvergent automorphic forms on PEL Shimura varieties of type (A) and (C) in the unramified case, we should be able to prove the density of classical points in the eigenvariety $\mathcal{E}$ when p is a good prime for the moduli problem.

4.4 Complement

In this final subsection we give a complement to Theorem 4.14 that is needed for the application [Mo].

Thus let v be a prime of $\mathcal{O}$ with $v \nmid p$. Fix a Bernstein component $\mathcal{B}_v$ of $\mathrm{GSp}_{2g}(F_v)$. Recall that $\mathcal{B}_v$ is given by the (equivalence class of) data given by a pair (M, τ), where M is a Levi subgroup of GSp_{2g/F_v}, and τ is a supercuspidal representation of $M(F_v)$, up to twisting by unramified characters of $M(F_v)$. Let E be a number field over which $\mathcal{B}_v$ is defined, and denote by $\mathfrak{z}_v = E[\mathcal{B}_v]$ the affine coordinate ring of $\mathcal{B}_v$, which is known as the Bernstein centre of $\mathcal{B}_v$. We have an idempotent element $e_v \in \mathfrak{z}_v$, such that for any irreducible admissible representation π_v of $\mathrm{GSp}_{2g}(F_v)$, we have π_v belongs to the component $\mathcal{B}_v$ if and only if $e_v \cdot \pi_v \neq 0$.

Now we come back to the context of the previous subsections. Let l be the rational prime below the prime v, and let m be the exact power such that l^m divides N. We assume that n is sufficiently large that the following holds: denoting by $K_v(m)$ the principal congruence subgroup at the prime v of level n, with associated idempotent $e_{K_v(m)}$. Then we assume that n is large enough that $e_{K_v(m)} \cdot e_v = e_v$, with e_v being the idempotent associated to the Bernstein component $\mathcal{B}_v$ as above. We may also assume that the extension $L/\mathbf{Q}_p$ in the last subsection to be large enough to contain the number field E.

We note that in the particular case where $\mathcal{B}_v$ is the Iwahori component associated to the (standard) Iwahori subgroup $I_v \subset \mathrm{GSp}_{2g}(F_v)$, then for any π_v belonging to $\mathcal{B}_v$, we have $U_{(p)}$ acts invertibly on $\pi_v^{I_v}$ (*cf.* [BC, Section 6.4.1]).

Back to the fixed Bernstein component $\mathcal{B}_v$ as above. The Bernstein centre $\mathfrak{z}_v$ acts on the space of overconvergent Siegel–Hilbert modular forms $M^{\dagger}_{H^p p^n, \kappa+Y}(\mathcal{R})$. Indeed by the theory of Bernstein centre it suffices to see that the local Hecke algebra of $\mathrm{GSp}_{2g}(F_v)$ with respect to the congruence subgroup $K_v(m)$ acts on $M^{\dagger}_{H^p p^n, \kappa+Y}(\mathcal{R})$. Since $v \nmid p$, this follows by a similar argument as in Section 4.2.

As in [BC, Chapter 7], we can then form the space of overconvergent Siegel–Hilbert modular forms associated to the idempotent e_v:

$$e_v M^{\dagger}_{H^p p^n, \kappa+Y}(\mathcal{R}).$$

Then the same argument as in the proof of Theorem accum4.14 but with the constructions applied to the space $e_v M^{\dagger}_{H^p p^n, \kappa+Y}(\mathcal{R})$, yields the following.

Theorem 4.16 *Let f be a classical cuspidal Siegel–Hilbert modular eigenform of weight κ and of level $H^p p^n$. Let π be the cuspidal automorphic representation of $\mathrm{GSp}_{2g}(\mathbf{A}_F)$ generated by f, and assume that π_v belongs to $\mathcal{B}_v$. There exists, for any positive integer t with $v_p(t)$ large enough, a Siegel–Hilbert cuspidal eigenform f_t of weight $\kappa \cdot \mathrm{Nm}^{(p-1)k_0 t}$ and of the same level, such that the Hecke eigenvalues on the f_t's converge p-adically to that of f, as $v_p(t) \to +\infty$. Furthermore the f_t can be taken to have the following property: denoting by π_t the cuspidal automorphic representation of $\mathrm{GSp}_{2g}(\mathbf{A}_F)$ generated by f_t. Then $\pi_{t,v}$ belongs to $\mathcal{B}_v$.*

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