Can. J. Math., Vol. XXX, No. 1, 1978, pp. 66–84

# FORMAL POWER SERIES OVER COMMUTATIVE $N$-ALGEBRAS

ERNST AUGUST BEHRENS

**Introduction.** A Banach algebra $P$ over $\mathbf{C}$ with identity element is called an *$N$-algebra* if any closed ideal in $P$ is the intersection of maximal ideals. An example is given by the algebra $\mathscr{C}_{\mathbf{C}}(X)$ of the continuous $\mathbf{C}$-valued functions on a compact Hausdorff space $X$ under the supremum norm; two others are discussed in § 3. The closure $\mathfrak{a} \circ \mathfrak{b} = \Gamma(\mathfrak{a} \bullet \mathfrak{b})$ of the product of closed ideals $\mathfrak{a}$ and $\mathfrak{b}$ is equal to their intersection. This result implies the distributivity of the lattice $V(P)$ of the closed ideals of $P$, where the lattice operations are the intersection $\mathfrak{a} \cap \mathfrak{b}$ and the closure $\mathfrak{a} \dotplus \mathfrak{b}$ of the sum $\mathfrak{a} + \mathfrak{b}$. The arithmetic of the topologically arithmetical ring $P$, i.e. the structure of the lattice ordered semigroup $V(P)$ under $\circ$, is therefore simply the structure of $V(P)$, considered as a lattice.

The formal power series in an indeterminate $\omega$ with coefficients in the $N$-algebra $P$ form an algebra $R = P[[\omega]]$ over the ring $\mathbf{P} = \mathbf{C}[[\omega]]$ of the formal power series in $\omega$ with coefficients in the field $\mathbf{C}$ of complex numbers. $P[[\omega]]$ is complete under the sequence

$$(1) \qquad q_n^* : \sum_{i \geqq 0} \phi_i \omega^i \rightarrow ||\phi_0|| + \ldots + ||\phi_n||$$

of submultiplicative seminorms $q_n$, $n = 0, 1, 2 \ldots$, $|| \cdot ||$ being the norm in $P$; in other words, $P[[\omega]]$ is a locally-$m$-convex and complete algebra.

If $P$ is *commutative* then $R$ is topologically arithmetical, i.e. the lattice $V(R)$ of the closed ideals in $R$ is distributive, as it is shown in § 1 by associating with any ideal in $V(R)$ an ascending chain of ideals in $V(P)$. The arithmetic of $R$ is developed in § 1, Theorem 1.7.: Any $A \in V(R)$ is the intersection of powers $M^{\alpha(M)}$ of the maximal ideals $M$ in $R$; and if this representation of $A$ is normalized, by taking the exponents maximal, then $A \circ B$ corresponds to $\alpha(M) + \beta(M)$, $M$ in the set $\mathbf{M}$ of the maximal ideals in $R$. The mappings $\alpha : M \rightarrow \alpha(M)$ for $M \in \mathbf{M}$ are the $\mathbf{N}_{0,\infty}$-valued upper semicontinuous functions on the set $\mathbf{M}$ in its (Jacobson) hull-kernel-topology. Here $\mathbf{N}_{0,\infty}$ means the chain consisting of the non-negative rational integers and $\infty$.

Theorem 1.8 gives a characterization of the $N$-algebras $P$. It is based on the following fact: Every closed ideal $\mathfrak{a}$ of $P$ is contained in the intersection $K(H\mathfrak{a})$ of its hull $H\mathfrak{a}$ which is the set of those maximal ideals which contain $\mathfrak{a}$. The algebra $P$ contains closed ideals $\mathfrak{a}$ with hull $\mathfrak{A}$, say, which are the only ones which possess $\mathfrak{A}$ as its hull, i.e. $\mathfrak{a}' \in V(P)$ and $H\mathfrak{a}' = \mathfrak{A}$ implies $\mathfrak{a}' = K(\mathfrak{A})$.

Received September 10, 1976. This research was supported by NRC Grant 214-1558.

By Theorem 1.8, these ideals form a sublattice $\mathfrak{H}$ of $V(P)$ and not only a subset of $V(P)$). The set $\mathfrak{H}$ is very important for the Spectral Synthesis (Benedetto [5]).

§2 deals with a characterization, up to "isometric" isomorphism $\sigma$ of the **P**-algebras $R = P[[\omega]]$ over those $N$-algebras $P$ which satisfy $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$ (and not only $\mathfrak{m} \circ \mathfrak{m} = \mathfrak{m}$), a property which is enjoyed by the examples in §3. The characterization is mainly based on the fact, that $R$ is a commutative, complete, locally-$m$-convex **P**-algebra which is topologically arithmetical (or, equivalently satisfies $\dim_{\mathbf{C}} M/M^2 = 1$ for all $M \in \mathbf{M}$), and on the fact that the powers $\omega^n R$ of its radical $\omega R$ are completely distributive elements in $V(R)$ in the sense

$$(3) \qquad \omega^n R \dotplus \bigcap_{i \in I} A_i = \bigcap_{i \in I} (\omega^n R \dotplus A_i) \quad \text{for } A_i \in V(R)$$

where $I$ is an arbitrary index set. Here the adjective "isometric" needs an explanation: If the locally-$m$-convexity of $R$ is given by a sequence $\{q_n; n = 0, 1, 2, \ldots\}$ of seminorms which extend the norm $\|\cdot\|$ of the factor algebra of $R$ over its radical and which take care of the fact that $R$ is an algebra over $\mathbf{P} = \mathbf{C}[[\omega]]$ (and not only over $\mathbf{C}$) then $q_n^*(\sigma a) = q_n a$ is valid for $a \in R$ and $n = 0, 1, 2, \ldots$ (Theorem 2.9).

The work on $N$-algebras goes back to the papers of G. Šilov [10] and of A. B. Willcox [**11**]. The arithmetic of $P[[\omega]]$ in the case $P = \mathscr{C}_{\mathbf{C}}(X)$ was developed in E. A. Behrens [**3**], but the importance of the complete distributivity (3) of the powers of the radical has shown up at first in the purely algebraic semigroup theoretical contribution (Behrens [**4**]).

**1. The arithmetic of** $P[[\omega]]$**.** Let $P$ be a commutative $N$-algebra with identity element $e$, i.e. a Banach algebra with the property that any closed ideal $\mathfrak{a}$ in $P$ is the intersection of maximal ideals $\mathfrak{m}$. Three examples are mentioned in the introduction and will be discussed in §3. The closed ideals in $P$ form a lattice ordered semigroup $V(P)$ under the following operations: the meet of $\mathfrak{a}$ and $\mathfrak{b}$ in $V(P)$ is their intersection $\mathfrak{a} \cap \mathfrak{b}$, the join of $\mathfrak{a}$ and $\mathfrak{b}$ is the closure $\mathfrak{a} \dotplus \mathfrak{b}$ of their sum $\mathfrak{a} + \mathfrak{b}$ and the o-product of $\mathfrak{a}$ and $\mathfrak{b}$ is the closure $\mathfrak{a} \circ \mathfrak{b}$ of their product $\mathfrak{a} \cdot \mathfrak{b}$. The o-product

$$(1) \qquad \mathfrak{a} \circ \mathfrak{b} \quad \text{is equal to} \quad \mathfrak{a} \cap \mathfrak{b}$$

as the following arguments shows: Let $\mathfrak{X}$ be the set of maximal ideals $\mathfrak{m}$ in $P$ and define the hull of $\mathfrak{a} \in V(P)$ by

$$(2) \qquad H(\mathfrak{a}) = \{\mathfrak{m} \in \mathfrak{X}; \mathfrak{a} \subseteq \mathfrak{m}\}.$$

Then

$$(3) \qquad \mathfrak{a} = \bigcap \{\mathfrak{m}; \mathfrak{m} \in H\mathfrak{a}\}$$

and

(4) $H(\mathfrak{a} \circ \mathfrak{b}) = H\mathfrak{a} \cup H\mathfrak{b} = H(\mathfrak{a} \cap \mathfrak{b})$

because the maximal ideals in $P$ are prime.

Equation (1) implies that $P$ is a topologically arithmetical ring, i.e. the lattice $V(P)$ is distributive.

Now, using the concept of a locally-$m$-convex algebra (E. A. Michael [8]) we get

THEOREM 1.1. *Let $P$ be a commutative $N$-algebra with identity element $e$ and $\|\cdot\|$ as its norm. Then the set $R = P[[\omega]]$ of all formal power series*

(5) $f = \sum_{i \geqq 0} \phi_i \omega^i, \quad \phi_i \in P,$

*in the indeterminate $\omega$ is a complete, locally-m-convex algebra under forming the sum coefficientwise and using the Cauchy product and taking the following sequence of seminorms*

(6) $q_n : f \rightarrow \sum_{0 \leqq i \leqq n} \|\phi_i\| \quad$ *for* $n = 0, 1, 2, \ldots.$

*$R$ is a $\mathbf{P}$-algebra over the complete, locally-m-convex algebra $\mathbf{P} = \mathbf{C}[[\omega]]$ of all formal power series in $\omega$, with complex coefficients, and with*

(7) $p_n : \sum_{i \geqq 0} c_i \omega^i \rightarrow |c_0| + \ldots + |c_n|, \quad n = 0, 1, 2, \ldots$

*as its seminorms.*

*Proof.* $P$ is complete with respect to its norm $\|\cdot\|$.

The closed ideals $A$, $B$ in $R$ form a lattice ordered semigroup $V(R)$ under the intersection $A \cap B$, the closure $A \dot{+} B$ of their sum $A + B$ and the closure $A \circ B$ of their product $A \cdot B$.

To any $A \in V(R)$ there corresponds a chain

(8) $\Phi A = \{\mathfrak{a}_i \in V(P); \mathfrak{a}_i \subseteq \mathfrak{a}_{i+1} \text{ and } i = 0, 1, 2, \ldots\}$

where

(9) $\mathfrak{a}_i = \{\phi \in P; \text{there exists an } f = \sum \phi_j \omega^j \in A \text{ with } \phi_i = \phi\}.$

The equations $\Gamma \mathfrak{a}_i = \mathfrak{a}_i$, where $\Gamma \mathfrak{a}_i$ is the closure of $\mathfrak{a}_i$, follow immediately from the definition (6) of the seminorms $q_n$ on $R$ by the norm $\|\cdot\|$ on $P$. Now $f \in A$ implies $\omega f = \omega e \cdot f \in A$; that proves $\mathfrak{a}_i \subseteq \mathfrak{a}_{i+1}$.

Conversely, any chain

(10) $\{\mathfrak{a}_i \in V(P); \mathfrak{a}_i \subseteq \mathfrak{a}_{i+1}, i = 0, 1, 2, \ldots\}$

is mapped by

(11) $\Psi(\{\mathfrak{a}_i, i\}) = \left\{ \sum_{i \geqq 0} \phi_i \omega^i;\ \phi_i \in \mathfrak{a}_i \right\}$

to a closed ideal $A$ in $R$.

By the definition of $\Phi$ it is clear that $\Phi\Psi$ maps any chain (10) to itself. But the proof of the equality

(12) $\Psi\Phi = \mathrm{id}_{V(R)}$

needs a double induction, based on the fact that in $P$ the closure $\Gamma(\phi P)$ of the principal ideal $\phi P$, generated by $\phi$, is equal to its square (observe (1)). Let $A \in V(R)$ and assume that the vector spaces

(13) $\mathfrak{a}_i \omega^i \subseteq A \quad$ for $0 \leqq i \leqq n - 1$.

Let $\phi = \sum_{i \geqq 0} \phi_i \omega^i \in \Psi\Phi A$ and therefore $\phi_n \in \mathfrak{a}_n$. It is to show that $\phi_n \omega^n \in A$. By the definition of $\mathfrak{a}_n$ there exists an element $\sum_{i \geqq 0} \eta_i \omega^i$ in $A$ with $\eta_n = \phi_n$. By virtue of (13) we can assume $\eta_0 = \ldots = \eta_{n-1} = 0$, in other words

(14) $\phi_n \omega^n + \eta\omega^{n+1} + \ldots \in A$.

This formula is the case $m = 1$ in the following induction assumption on $m$.

There exists an $\eta \in P$ such that

$$g = \phi_n \omega^n + 0\omega^{n+1} + \ldots + 0\omega^{n+m-1} + \eta\omega^{n+m} + \ldots \in A.$$

Then $A$, as an ideal, contains both

$$g \cdot (\phi_n \omega^0 + \eta\omega^m) = (\phi_n{}^2 + 2\phi_n \eta\omega^m + \ldots)\omega^n$$

and

$$g \cdot \phi_n \omega^0 = (\phi_n{}^2 + \phi_n \eta\omega^m + \ldots)\omega^n.$$

Therefore $A$ contains also

(15) $\phi_n{}^2 \omega^n + 0\omega^{n+m} + \ldots.$

Because of (1), $\phi_n$ is contained in $\Gamma(\phi_n{}^2 P)$. Therefore, for a given $\epsilon > 0$ there exists an $\rho_\epsilon \in P$ with $||\phi_n - \phi_n{}^2 \rho_\epsilon|| < \epsilon$. The product $h_\epsilon$ of (15) with $\rho_\epsilon \omega^0$ is an element of $A$ and it differs from the element $\phi_n \omega^n$ by less than $\epsilon$ with respect to the seminorm $q_{n+m}$, which annihilates $\omega^{n+m+1}R$. That proves the case $m + 1$ of the induction assumption. In other words to every pair $(\epsilon, \mathfrak{m})$ there exists an element $h_{\epsilon,\mathfrak{m}} \in A$ such that $q_{n+m}(\phi_n \omega^n - h_{\epsilon,\mathfrak{m}}) < \epsilon$ is valid. That proves $\phi_n \omega^n \in \Gamma A = A$. This argument shows also the validity of the case $n = 0$ in the induction on $n$ and finishes the proof of the equality (12).

THEOREM 1.2. *If $R$ is the ring of Theorem* 1.1, *then:*

(i) *The closed ideals $A$ in $R = P[[\omega]]$ correspond one-to-one to the ascending chains*

$$\{\mathfrak{a}_i \in V(P);\ \mathfrak{a}_i \subseteq \mathfrak{a}_{i+1},\ i = 0, 1, 2, \ldots\}$$

*of closed ideals in $P$ under the mapping*

$$(16)\quad \Psi: \{\mathfrak{a}_i; i = 0, 1, \ldots\} \to \left\{\sum_{i\geqq 0} \phi_i\omega^i; \phi_i \in \mathfrak{a}_i\right\}$$

*and its converse $\Phi$.*

(ii) *Define* $\{\mathfrak{a}_i; i\} \dotplus \{\mathfrak{b}_i; i\} = \{\mathfrak{a}_i \dotplus \mathfrak{b}_i; i\}$ *and the intersection componentwise also. Then*

$$\Phi(A \dotplus B) = \Phi A \dotplus \Phi B \quad \textit{and} \quad \Phi(A \cap B) = \Phi A \cap \Phi B.$$

(iii) *Define* $\{\mathfrak{a}_i; i\} \circ \{\mathfrak{b}_i; i\} = \{\mathfrak{c}_i; i\}$, *where*

$$(17)\quad \mathfrak{c}_i = (\mathfrak{a}_0 \cap \mathfrak{b}_i) \dotplus (\mathfrak{a}_1 \cap \mathfrak{b}_{i-1}) \dotplus \ldots \dotplus (\mathfrak{a}_i \cap \mathfrak{b}_0), i = 0, 1, \ldots$$

*Then*

$$(18)\quad \Phi(A \circ B) = \Phi A \circ \Phi B.$$

(iv) $R = P[[\omega]]$ *is topologically arithmetical.*

*Proof.* (i) is proved above. (ii) and (iii) are easy to verify using (i). Compare (iii) with the purely set theoretical lemma before Theorem 2.3 in Behrens [**3**]. (iv) follows from the fact that ideals in $P$ satisfy

$$(19)\quad \mathfrak{a} \cap (\mathfrak{b} \dotplus \mathfrak{c}) = (\mathfrak{a} \cap \mathfrak{b}) \dotplus (\mathfrak{a} \cap \mathfrak{c})$$

and an application of (ii).

COROLLARY 1.3. *The $n$-th $\circ$-power of a maximal ideal $M$ in $R$ is*

$$\sum_{0\leqq i\leqq n-1} \mathfrak{m}^i\omega^i + \sum_{j\geqq n} P\omega^j,$$

*where the maximal ideal $\mathfrak{m}$ in $P$ is the image of $M$ under the natural epimorphism of $R$ onto $R/\omega R \simeq P$.*

COROLLARY 1.4. *The $n$-th $\circ$-power of the Jacobson radical $J$ of $R$ is*

$$(20)\quad J^n = \omega^n R = \sum_{j\geqq n} P\omega^j.$$

COROLLARY 1.5. *For any natural number $n$, the $n$-th $\circ$-power $J^n$ of the radical $J$ of $R$ is a completely distributive element in the lattice $V(R)$, i.e.*

$$(21)\quad J^n \dotplus \bigcap_{\iota\in I} B_\iota = \bigcap_{\iota\in I} (J^n \dotplus B_\iota), \quad I \textit{ an arbitrary set.}$$

*Proof.* By Corollary 1.4, the components $\mathfrak{a}_0, \ldots, \mathfrak{a}_{n-1}$ of $J^n \dotplus B_\iota$ are the same as the corresponding components of $B_\iota$, and the later components $\mathfrak{a}_k$, $k \geqq n$, are all equal to $P$. Analogously the components of index less than $n$ of $J^n \dotplus B_\iota$ are the same as those of $\bigcap B_\iota$ and the others are all equal to $P$.

*Remark.* This property of the radical is important for the characterization of our rings. On the other hand it is shared by the rings with segregated radical (i.e. the additive group of $R$ is the direct sum of a subring of $R$ and $J$) and

https://doi.org/10.4153/CJM-1978-006-1 Published online by Cambridge University Press

artinian factoring $R/J$, even if $V(R)$ is a not necessarily distributive lattice, as it is remarked in Behrens [**4**].

The representation (3) of the closed ideals in $P$ as the intersection of maximal ideals in $P$ induces a representation of the closed ideals in $R$ as the intersection of powers of maximal ideals in $R$: Any $A \in V(R)$ is equal to a sum $\sum_{i\geqq 0} \mathfrak{a}_i\omega^i$ where $\mathfrak{a}_i \subseteq \mathfrak{a}_{i+1}$, $\mathfrak{a}_i \in V(P)$. By (3), $\mathfrak{a}_i \subseteq \mathfrak{m} \in \mathfrak{X}$ if and only if $\mathfrak{m} \in H(\mathfrak{a}_i)$, and, by Corollary 1.3, this is equivalent to $A \subseteq M^{i+1}$. Therefore with $\mathbf{M}$ being the set of the maximal ideals in $R$,

(22) $A = \bigcap \{M^{\alpha(M)}; M \in \mathbf{M}\}$,

where

(23) $\alpha(M) = \begin{cases} 0 & \text{if } A \not\subseteq M \\ \sup \{i; \mathfrak{a}_{i-1} \subseteq \mathfrak{m}\}. \end{cases}$

It is convenient to set

(24) $N_M = \bigcap \{M^i; i \geqq 0\}$,

a closed ideal in $R$. Then

(25) $\alpha(M) = \infty \Leftrightarrow A \subseteq N_M$.

Let $A = \bigcap M^{\alpha(M)}$, $B = \bigcap M^{\beta(M)}$ and $A \circ B = \bigcap M^{\gamma(M)}$. Then, by Theorem 2, (iii),

(26) $$\begin{aligned} \gamma(M) &= \sup\{i; \mathfrak{a}_j \cap \mathfrak{b}_{i-1-j} \subseteq \mathfrak{m}; 0 \leqq j \leqq i-1\} \\ &= \alpha(M) + \beta(M) \end{aligned}$$

because $\mathfrak{a}_j \cap \mathfrak{b}_{i-1-j} \subseteq \mathfrak{m}$ is valid if and only if $\mathfrak{a}_j \subseteq \mathfrak{m}$ or $\mathfrak{b}_{i-1-j} \subseteq \mathfrak{m}$.

These considerations are collected in the following theorem, where $R$ is again the ring of the Theorem 1.1.

THEOREM 1.6. (i) *Any closed ideal $A = \sum \mathfrak{a}_i\omega^i$ in $R = P[[\omega]]$ can be represented by*

(27) $A = \bigcap \{M^{\alpha(M)}; M \in \mathbf{M}\}$ *with* $\alpha(M) = \sup\{i; \mathfrak{a}_{i-1} \subseteq \mathfrak{m}\}$.

(ii) *The exponents $\gamma(M)$ of the product $A \circ B$ are*

(28) $\gamma(M) = \alpha(M) + \beta(M)$.

If $\xi$ is an arbitrary function on $\mathbf{M}$ with values in the set $\mathbf{N}_{0,\infty}$ of all non-negative integers together with $\infty$, then $A = \bigcap \{M^{\xi(M)}; M \in \mathbf{M}\}$ is a closed ideal in $R$ and therefore $A = \bigcap \{M^{\alpha(M)}; M \in \mathbf{M}\}$ also, where the function $\alpha$ is given by (23). This normalization of the representation of $A$ as an intersection of powers of maximal ideals means a condition for the function $\alpha$ which can be formulated in the following way:

Let $\mathfrak{S}$ be an arbitrary subset of the set $\mathfrak{X}$ of the maximal ideals in $P$. Then the closed ideal $\mathfrak{a} = \bigcap \{\mathfrak{m}; \mathfrak{m} \in \mathfrak{S}\}$ has the following normalized representation

$$(29) \quad \mathfrak{a} = \bigcap \{\mathfrak{m}; \mathfrak{a} \subseteq \mathfrak{m} \in \mathfrak{X}\} = \bigcap \mathfrak{m}^{\alpha_0(\mathfrak{m})}.$$

In other words, if we define the kernel $K(\mathfrak{S})$ of a subset $\mathfrak{S}$ of $\mathfrak{X}$ as the intersection of those maximal ideals of $P$ which are members of $\mathfrak{S}$ and recall (2), that the hull $H\mathfrak{a}$ of an ideal $\mathfrak{a}$ in $P$ consists of the set of all maximal ideals which contain $\mathfrak{a}$, then

$$(30) \quad \mathfrak{a} = \bigcap \{\mathfrak{m}; \mathfrak{m} \in HK\mathfrak{S}\} = \bigcap \{\mathfrak{m}^{\alpha_0(\mathfrak{m})}; \mathfrak{m} \in \mathfrak{S}\}$$

is the normalized representation of $\mathfrak{a} = \bigcap \{\mathfrak{m}; \mathfrak{m} \in \mathfrak{S}\}$. It is well known that the definition of the closure

$$(31) \quad \Gamma\mathfrak{S} = HK\mathfrak{S} \quad \text{for the subsets } \mathfrak{S} \text{ of } \mathfrak{X}$$

gives the set $\mathfrak{X}$ of the maximal ideals of $P$ a compact (not necessarily Hausdorff) topology, called the hull-kernel-topology of $\mathfrak{X}$. The characteristic function $\chi$ of the set $\mathfrak{S}$ is upper semi-continuous with respect to this topology, if and only if $\mathfrak{S}$ is closed (Bourbaki [**6**, Corollary to Proposition 1 in IV.6.2]), and the function $\alpha_0$ in the normalized representation (30) of the ideal $\mathfrak{a}$ is the upper semicontinuous regularization of the characteristic function $\chi$ of $\mathfrak{S}$.

By virtue of the correspondence $\Phi$ in the Theorem 2 between the closed ideals $A$ in $R$ and the ascending chains $\Phi A$ of closed ideals in $P$ the last paragraph implies the following result, observing $\mathfrak{m} = M/\omega R$ and assuming that $R$ is the ring of Theorem 1.1.

THEOREM 1.7. *The arithmetic of the ring $P[[\omega]]$, i.e. the structure of the lattice ordered semigroup $V(R)$ of its closed ideals, can be described by an isomorphism $\sigma$ of $V(R)$ onto the lattice ordered semigroup $S$ of the $\mathbf{N}_{0,\infty}$-valued uppersemicontinuous functions $\alpha$ on the set $\mathbf{M}$ of the maximal ideals in $R$, using the H-K-topology on $\mathbf{M}$ and the operations*

$$(\alpha + \beta)(M) = \alpha(M) + \beta(M), \quad (\alpha \cap \beta)(M) = \sup\{\alpha(M), \beta(M)\},$$
$$(\alpha \cup \beta)(M) = \inf\{\alpha(M), \beta(M)\}$$

*on the lattice ordered semigroup $S$:*

$$(32) \quad \sigma : A \rightarrow \alpha \in S, \quad A = \bigcap \{M^{\alpha(M)}; M \in \mathbf{M}\} \text{ for } A \in V(R).$$

Our investigations have been depending on the hypothesis that $P$ is an $N$-algebra and they motivate our interest in a lattice-theoretical characterization of the (non-necessary commutative) $N$-algebras. It will be given in the next theorem in the case that $P$ is a completely regular, strongly semisimple Banach algebra with identity element, i.e. that the intersection of the maximal ideals in $P$ is equal to (0) and that the hull-kernel topology of $\mathfrak{X}$ is Hausdorff. We need Willcox's ([**12**, Theorem 1.2]) characterization of the smallest closed ideal which has the closed subset $F$ of $\mathfrak{X}$ as its hull: It is the closure $\Gamma\mathfrak{j}(F)$ of the ideal $\mathfrak{j}(F)$ in $P$, consisting of the following set: $a \in \mathfrak{j}(F)$ if and only if there exists an open subset $\mathfrak{O}_a$ of $\mathfrak{X}$ such that $F \subseteq \mathfrak{O}_a$ and $\alpha(\mathfrak{O}_a) = 0$, $\alpha = \hat{a}$.

THEOREM 1.8. *Assume that the completely regular and strongly semi-simple Banach algebra $P$ possesses an identity element $e$. Set*

(33) $\quad \mathfrak{H} = \{\mathfrak{a} \in V(P); \mathfrak{a} = \Gamma\mathfrak{j}(F), F = \Gamma F \subseteq \mathfrak{X}\}$

*and assume*

(34) $\quad \mathfrak{a} \circ \mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}$ *for* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{H}$.

*Then the following statements hold:*

(i) *$\mathfrak{H}$ is a distributive sublattice of $V(P)$.*

(ii) *The relation $\kappa$ of $V(P)$, defined by*

(35) $\quad \mathfrak{a}\kappa\mathfrak{a}^1 = \Gamma\mathfrak{j}(H\mathfrak{a}) = \Gamma\mathfrak{j}(H\mathfrak{a}^1),$

*is a congruence relation on the lattice $V(P)$ and*

(36) $\quad \sigma : \mathfrak{a} \bmod \kappa \rightarrow \Gamma\mathfrak{j}(H\mathfrak{a})$ *for* $\mathfrak{a} \in V(P)$

*is a lattice-isomorphism of $V(P)/\kappa$ onto the sublattice $\mathfrak{H}$ of $V(P)$.*

(37) (iii) *$P$ is an $N$-algebra if and only if $\kappa$ is the diagonal relation on $V(P)$.*

*Remark.* A commutative $N$-algebra satisfies (34) as was remarked at the beginning of this section.

*Proof of Theorem* 1.8. The formulas

(38) $\quad H(\mathfrak{a} \cap \mathfrak{b}) = H(\mathfrak{a}) \cup H(\mathfrak{b})$ and $H(\mathfrak{a} \dotplus \mathfrak{b}) = H\mathfrak{a} \cap H\mathfrak{b}$

follow immediately from the primeness of the maximal ideals in $P$. If $\mathfrak{a} \subseteq \mathfrak{b}$ and $H\mathfrak{a} \subseteq \mathfrak{O}$ then $H\mathfrak{b} \subseteq \mathfrak{O}$ and therefore

(39) $\quad \mathfrak{a} \subseteq \mathfrak{b} \Rightarrow \mathfrak{j}(H\mathfrak{a}) \subseteq \mathfrak{j}(H\mathfrak{b})$ for $\mathfrak{a}, \mathfrak{b} \in V(P)$.

That implies

(40) $\quad \mathfrak{j}(H(\mathfrak{a} \dotplus \mathfrak{b}) \supseteq \mathfrak{j}(H\mathfrak{a}) + \mathfrak{j}(H\mathfrak{b})$ and $j(H(\mathfrak{a} \cap \mathfrak{b})) \subseteq \mathfrak{j}(H\mathfrak{a}) \cap \mathfrak{j}(H\mathfrak{b})$.

Let $\alpha \in \mathfrak{j}(H(\mathfrak{a} \dotplus \mathfrak{b}))$ and therefore $\alpha(\mathfrak{O}) = 0$ on an open subset $\mathfrak{O} \supseteq H(\mathfrak{a} \dotplus \mathfrak{b})$. If $\mathfrak{O}^c$ denotes the complement set $\mathfrak{X}\backslash\mathfrak{O}$, then $(H\mathfrak{a} \cap \mathfrak{O}^c) \cap H\mathfrak{b} = \emptyset$. By virtue of $e \in P$, the Hausdorff space $\mathfrak{X}$ is compact in its $H$-$K$-topology and therefore normal. That warrants the existence of open sets $\mathfrak{O}_i$, $i = 1, 2$, in $\mathfrak{X}$ with

$$H\mathfrak{a} \cap \mathfrak{O}^c \subseteq \mathfrak{O}_1, \quad H\mathfrak{b} \subseteq \mathfrak{O}_2 \quad \text{and} \quad (\Gamma\mathfrak{O}_1) \cap (\Gamma\mathfrak{O}_2) = \emptyset$$

and therefore $(\Gamma\mathfrak{O}_1)^c \cup (\Gamma\mathfrak{O}_2)^c = \mathfrak{X}$. Let $\pi_1, \pi_2 \in P$ be a partition of $e$ subordinated to this open covering of $\mathfrak{X}$ (Willcox [**12**, Corollary 1.1.2]), i.e. $e = \pi_1 + \pi_2$ and $\pi_i((\Gamma\mathfrak{O}_i)^{cc} = 0$, $i = 1, 2$. Then $\alpha\pi_1$ is constant $= 0$ on $\Gamma\mathfrak{O}_1 \cup \mathfrak{O} \supseteq \mathfrak{O}_1 \cup \mathfrak{O} \supseteq H\mathfrak{a}$ and $\alpha\pi_2$ is constant $= 0$ on $\Gamma\mathfrak{O}_2 \supseteq \mathfrak{O}_2 \supseteq H\mathfrak{b}$. By the

https://doi.org/10.4153/CJM-1978-006-1 Published online by Cambridge University Press

very definition of $\mathfrak{j}(F)$ we get $\alpha\pi_1 \in \mathfrak{j}(H\mathfrak{a})$ and $\alpha\pi_2 \in \mathfrak{j}(H\mathfrak{b})$. This implies

(41) $\alpha = \alpha e = \alpha\pi_1 + \alpha\pi_2 \in \mathfrak{j}(H\mathfrak{a}) + \mathfrak{j}(H\mathfrak{b})$

for any $\alpha \in \mathfrak{j}(H(\mathfrak{a} \dot{+} \mathfrak{b}))$. The transition to the closures in $(40_1)$ and (41) proves

(42) $\Gamma\mathfrak{j}(H(\mathfrak{a} \dot{+} \mathfrak{b})) = \Gamma\mathfrak{j}(H\mathfrak{a}) \dot{+} \Gamma\mathfrak{j}(H\mathfrak{b}) \quad \text{for } \mathfrak{a}, \mathfrak{b} \in V(P).$

Let now $\alpha \in \Gamma\mathfrak{j}(H\mathfrak{a}) \cap \Gamma\mathfrak{j}(H\mathfrak{b}))$. By the hypothesis of our Theorem, this intersection is equal to the o-product and therefore to $\Gamma(\mathfrak{j}(H\mathfrak{a}) \cdot \mathfrak{j}(H\mathfrak{b}))$. In other words, we get

$$\alpha = \lim_{n\to\infty} \sum_i (\alpha_{ni}\beta_{ni}), \alpha_{ni}(\mathfrak{O}_{ni}) = 0 = \beta_{ni}(\mathfrak{O}_{ni}'),$$

where the open sets $\mathfrak{O}_{ni}$ and $\mathfrak{O}_{ni}'$ satisfy $H\mathfrak{a} \subseteq \mathfrak{O}_{ni}$ and $H\mathfrak{b} \subseteq \mathfrak{O}_{ni}'$ respectively. Then $\alpha_{ni}\beta_{ni} = 0$ on $\mathfrak{O}_{ni} \cup \mathfrak{O}_{ni}' \supseteq H\mathfrak{a} \cup H\mathfrak{b} = H(\mathfrak{a} \cap \mathfrak{b})$ and therefore $\alpha_{ni}\beta_{ni} \in \mathfrak{j}(H(\mathfrak{a} \cap \mathfrak{b}))$ and $\alpha \in \Gamma\mathfrak{j}(H(\mathfrak{a} \cap \mathfrak{b}))$. This, together with an application of $\Gamma$ to $(40_2)$ proves

(43) $\Gamma\mathfrak{j}(H(\mathfrak{a} \cap \mathfrak{b})) = \Gamma\mathfrak{j}(H\mathfrak{a}) \cap \Gamma\mathfrak{j}(H\mathfrak{b})$

and finishes the proof of the statement that $\mathfrak{H}$ is a sublattice of $V(P)$. Its distributivity follows immediately from (38) together with

(44) $\mathfrak{a} \leftrightarrow H\mathfrak{a} \leftrightarrow \Gamma\mathfrak{j}(H\mathfrak{a}) \quad \text{for } \mathfrak{a} \in \mathfrak{H}$

and the distributivity of the lattice of the closed subsets of a topological space. The formulas (42) and (43) show that the equivalence relation $\kappa$ in the statement (ii) is a congruence relation and therefore $\sigma$ a lattice-isomorphism, remembering (38). For the proof of (iii) we need only to remark that $P$ is an $N$-algebra if and only if

(45) $KH\mathfrak{a} = \Gamma\mathfrak{j}(H\mathfrak{a}) \quad \text{for all } \mathfrak{a} \in V(P)$

is valid.

**2. A characterization of $P[[\omega]]$ as a topologically arithmetical algebra.** Let $P$ be a commutative $N$-algebra with identity element $e$ again. Then for its maximal ideals $\mathfrak{m}$, by § 1, (1), $\mathfrak{m} \circ \mathfrak{m} = \mathfrak{m}$ is valid, where $\mathfrak{m} \circ \mathfrak{m}$ is the closure of the product $\mathfrak{m} \cdot \mathfrak{m}$. The three examples of $N$-algebras, mentioned in the introduction, enjoy the property

(1) $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$ for $\mathfrak{m} \in \mathfrak{X}$,

as will be shown in § 3. This fact is used essentially in the characterization of their **P**-algebras $P[[\omega]]$.

PROPOSITION 2.1. *If the commutative $N$-algebra $P$ satisfies $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$ for its maximal ideal $\mathfrak{m}$ then the powers $M \cdot M \cdots M$ of the maximal ideal $M = \mathfrak{m} + \sum_{j\geq 1} P\omega^j$ in $P[[\omega]]$ are closed also.*

https://doi.org/10.4153/CJM-1978-006-1 Published online by Cambridge University Press

*Proof:* $M$ is closed. Assume $M \cdot M \cdots M = M^n$. Then by Corollary 1.3

$$M^n \cdot M = \left(\sum_{0 \leqq i \leqq n-1} \mathfrak{m}\omega^i + \sum_{j \geqq n} P\omega^j\right) \cdot \left(\mathfrak{m} + \sum_{k \geqq 1} P\omega^k\right)$$
$$= \sum_{0 \leqq i \leqq n-1} \mathfrak{m} \cdot \mathfrak{m}\omega^i + \sum_{j \geqq n} \mathfrak{m}\omega^j + \sum_{h \geqq n+1} P\omega^h = M^{n+1}$$

is valid by virtue of $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$.

THEOREM 2.2. *Let $R$ be a commutative, complete, locally $m$-convex $\mathbf{P}$-algebra, $\mathbf{P} = \mathbf{C}[[\omega]]$, with identity element $e$. Assume that all powers $M$, $M \cdot M$, $M \cdot M \cdot M, \ldots$ of any maximal ideal $M$ in $R$ are closed and $M^n \supsetneqq M^{n+1}$ for $n \in \mathbf{N}$ is valid. Set*

$$N_M = \bigcap \{M^n; n \in \mathbf{N}\}$$

*and assume for the set $\mathbf{M}$ of all maximal ideals*

(2) $\quad \bigcap \{N_M; M \in \mathbf{M}\} = (0).$

*Assume that $\omega R$ is the radical of $R$ and $\bar{M}^2 = \bar{M}$ is valid for $\bar{M} = M/\omega R$ in $R/\omega R$.*

*Conclusions:* (i) *If*

(3) $\quad \dim_{\mathbf{C}} M/M^2 = 1 \quad$ *for* $M \in \mathbf{M}$,

*then to any $a \in R$ and any $M \in \mathbf{M}$ there exists a uniquely determined $\hat{a}(M) \in \mathbf{P}$ such that*

(4) $\quad a \equiv \hat{a}(M)e \pmod{N_M}$.

*In that case*

$$\sigma : a \to \hat{a} \quad \textit{for } a \in R$$

*is a one-to-one algebraic homomorphism of $R$ into the $\mathbf{P}$-algebra $\mathfrak{F}_{\mathbf{P}}(\mathbf{M})$ of all $\mathbf{P}$-valued functions on the set $\mathbf{M}$.*

(ii) *The condition* (3) *is satisfied if $R$ is a topologically arithmetical ring.*

*Proof.* (i) By the Gelfand-Mazur Theorem there exists one and only one $\alpha_0(M) \in \mathbf{C}$ satisfying $a = a_0(M)e \pmod{M}$. By virtue of $\mathfrak{m}^2 = \mathfrak{m}$ in $P = R/\omega R$, where $\mathfrak{m} = M/\omega R$, and of $M \cdot M = M^2$ the equation

(6) $\quad M = \mathbf{C}\omega e + M^2$

is valid and shows $\omega e \in M \setminus M^2$. That proves the case $n = 1$ of the induction assumption

(7) $\quad M^n = \mathbf{C}\omega^n e + M^{n+1}$,

because dim $(M/M^2) = 1$. The sum in (7) is direct, considered as $\mathbf{C}$-space. (6) implies

$$M^{n+1} = (\mathbf{C}\omega^n e + M^{n+1}) \cdot (\mathbf{C}\omega e + M^2)$$
$$\subseteq \mathbf{C}\omega^{n+1} e + M^{n+2} \subseteq M^{n+1}.$$

Then $M^n \supset M^{n+1}$ finishes the proof of the direct $\mathbf{C}$-space-sum-representation (7) for all $n$. Assume that there exists $\alpha_0(M), \ldots, \alpha_{n-1}(M) \in \mathbf{C}$ with

$$(8) \quad a - \sum_{0 \leqq j \leqq n-1} \alpha_j(M)e \in M^n.$$

Then (7) gives the existence of a uniquely determined $\alpha_n(M) \in \mathbf{C}$ such that (8) is valid for $n + 1$ also. That proves (4) considering the definition of $N_M$. The statement about $\sigma$ is clear. (ii) The closed ideals $A$ of $R$ in the interval $[M^2, M]$ of $V(R)$ correspond to the $\mathbf{C}$-subspaces of $M/M^2$ because $M/M^2$ is an $R/M$-module and therefore an $\mathbf{C}$-space. Then $\dim_{\mathbf{C}} M/M \geqq 2$ would imply that $[M^2, M]$ is not a distributive sublattice of $V(R)$.

If the powers $\omega^n R$ of the radical $\omega R$ of $R$ are completely distributive elements in the lattice $V(R)$ then the following theorem proves a property of $R$ which is well known for noetherian and commutative Dedekind-rings.

THEOREM 2.3. *If $\omega^n R$ is a completely distributive element in $V(R)$, $R$ being the ring of Theorem 2.2, then*

$$(9) \quad \omega^n R = \bigcap \{M^n; M \in \mathbf{M}\}$$

*is valid.*

*Proof.* We know

$$(10) \quad M^n = \omega^n R + M^{n+1} \quad \text{for } n \in \mathbf{N}$$

from (7). Then, using $\omega R \subseteq M$, we get

$$\begin{aligned} M^2 = M \cdot M &= \omega^2 R + \omega M^2 + M^4 = \omega^2 R + \omega(\omega^2 R + M^3) + M^4 \\ &= \omega^2 R + M^4. \end{aligned}$$

Analogously $M^3 = M^2 \cdot M = \omega^3 R + M^5$ replacing $M^2$ in $\omega^2 M^2$ by the last formula.

$$\begin{aligned} M^{n+2} &= \omega^{n+2} R + \omega^2 M^{n+2} + \omega^n M^4 + M^{n+6} \\ &\subseteq \omega^{n+2} R + M^{n+4} \subseteq M^{n+2}, \end{aligned}$$

using

$$(11) \quad M^n = \omega^n R + M^{n+2} \quad \text{for } n \geqq 2$$

as induction assumption. That shows

$$\bigcap \{M^n; M \in \mathbf{M}\} = \bigcap (\omega^n R + M^{n+2}) = \omega^n R + \bigcap M^{n+2}$$

by virtue of the complete distributivity of $\omega^n R$. Therefore

$$\begin{aligned} \bigcap \{M^n; M \in \mathbf{M}\} &= \omega^n R + \omega^{n+2} R + \bigcap M^{n+4} = \omega^n R + \bigcap M^{n+4} \\ &= \cdots = \omega^n R + \bigcap \{M^n; M \in \mathbf{M} \text{ and } n \in \mathbf{N}\} \\ &= \omega^n R + \bigcap \{N_M; M \in \mathbf{M}\} = \omega^n R + (0). \end{aligned}$$

It is the aim of the next Theorem 2.5 to show that the algebraic monomorphism $\sigma$ of $R$ into $\mathfrak{C}_{\mathbf{P}}(\mathbf{M})$, introduced in Theorem 2.2, is an algebraic isomorphism of $R$ onto $P[[\omega]]$, $P = R/\omega R$, in the case that all powers $\omega^n R$ are completely distributive elements in the lattice $V(R)$. The isometry-properties of $\sigma$ will be discussed in the Theorem 2.7.

We need the following

LEMMA 2.4: *Let $\hat{P}$ be the Gelfand transform of a commutative semi-simple Banach algebra $P$ with identity element $\epsilon$ and therefore consisting of* $\mathbf{C}$*-valued functions on the set $\mathfrak{X}$ of its maximal ideals $\mathfrak{m}$. Assume*

(12) $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$ *for* $\mathfrak{m} \in \mathfrak{X}$.

*Let $F$ be an $P$-bimodule consisting of residue classes $\bar{t}$ modulo $\hat{P}$ of* $\mathbf{C}$*-valued functions $t$ on $\mathfrak{X}$ and such that*

(13) $\alpha\bar{t} = \bar{t}\alpha$ *for $\alpha \in P$ and $\bar{t} \in F$*

*is valid and $\alpha\bar{t}$ is definable by*

(14) $\alpha\bar{t} : \mathfrak{m} \rightarrow \hat{\alpha}(\mathfrak{m}) \cdot t(\mathfrak{m}) + \hat{P}$ *for* $\mathfrak{m} \in \mathfrak{X}$,

*independently of the choice of $t$ in its residue class $\bar{t}$.*

*Then the zero mapping is the only $F$-derivation on $P$, i.e. the zero mapping is the only* $\mathbf{C}$*-linear mapping $D$ of $P$ into $F$ satisfying*

(15) $D(\alpha \cdot \beta) = \alpha(D\beta) + (D\alpha)\beta$ *for $\alpha, \beta \in P$.*

*Proof.* For $\alpha \in \mathfrak{m}$ and $\beta \in \mathfrak{m}$ we get

(16) $D(\alpha \cdot \beta) : \mathfrak{m} \rightarrow \hat{\alpha}(\mathfrak{m}) \cdot ((D\beta)(\mathfrak{m})) + ((D\alpha)(\mathfrak{m})) \cdot \hat{\beta}(\mathfrak{m}) = 0$

because $\alpha \in \mathfrak{m}$ implies $\hat{a}(\mathfrak{m}) = 0$. To any $\gamma \in P$ and $\mathfrak{m}_0 \in \mathfrak{X}$ there exist an $c \in \mathbf{C}$ and an element $\mu_\gamma \in \mathfrak{m}_0$ such that $\gamma = c\epsilon + \mu_\gamma$. By (12), the element $\mu_\gamma$ is a sum of finitely many products $\alpha \cdot \beta$ where $\alpha \in \mathfrak{m}_0$ and $\beta \in \mathfrak{m}_0$. Then $D\epsilon = D(\epsilon \cdot \epsilon) = 2\hat{\epsilon}(D\epsilon) = 2(D\epsilon)$ and (16) imply together $D_\gamma = c(D\epsilon) + D_{\mu_\gamma} = c \cdot 0 + \sum D(\alpha \cdot \beta) = 0$.

*Remark.* If we replace $F$ by $P = C_{\mathbf{C}}(X)$, $X$ a compact Hausdorff space, then the proof above shows that every (not necessarily continuous) $P$-derivation on $P$ is the zero mapping. This very short proof of the special case $P = C_{\mathbf{C}}(X)$ of Johnson's Theorem, that the only derivation in a semisimple, commutative Banach algebra is 0, the author has learned from W. Żelazko by oral communication.

THEOREM 2.5. *Let $R$ be a commutative complete locally $m$-convex* $\mathbf{P}$*-algebra with identity element and radical $\omega R$, where* $\mathbf{P} = \mathbf{C}[[\omega]]$. *Denote by* $\mathbf{M}$ *the set of the maximal idelas in $R$. Assume that for any $M \in \mathbf{M}$ the $n$-th algebraic powers $M \cdot M \ldots M$, $n = 0, 1, 2, \ldots$, form a strictly descending chain of closed ideals and that the equation*

(17) $\bigcap \{N_M; M \in \mathbf{M}\} = (0)$ *with* $N_M = \bigcap \{M^n; n \in \mathbf{N}\}$

*is valid. Assume that $R$ is topologically arithmetical, i.e. the lattice $V(R)$ of its closed ideals is distributive, and that the powers $\omega^n R$, $n \in \mathbf{N}$, of its radical are completely distributive elements in $V(R)$. Assume $\bar{M}^2 = \bar{M}$ for $\bar{M} = M/\omega R$. Define the mapping*

(18) $\sigma : a \to \hat{a} = \sum_{i \geqq 0} \alpha_i \omega^i$ *for* $a \in R$

*where*

(19) $\hat{a} : M \to \sum_{i \geqq 0} \alpha_i(M) \quad \in \mathbf{P}$ *for* $M \in \mathbf{M}$

*is determined by*

(20) $a = \hat{a}(M)e \pmod{N_M}$.

*Set $P = R/\omega R$, a semisimple Banach algebra. Then $\sigma$ is an algebraic isomorphism of $R$ onto $P[[\omega]]$.*

*Remark.* The maximal ideals $\mathfrak{m}$ in $R/\omega R$ are the images $M/\omega R$ of the maximal ideals $M$ in $R$. Because $M \cdot M$ is closed, the same is true for $\mathfrak{m} \cdot \mathfrak{m}$. Then $\mathfrak{m} = \mathfrak{m} \cdot \mathfrak{m}$ by hypothesis and we can apply the Lemma 4 to $P = R/\omega R$.

The author does not know whether the distributivity of $V(P)$, together with $\mathfrak{m} = \mathfrak{m} \cdot \mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ in $P$, implies that $P$ is an $N$-algebra. Remember Theorem 1.8 in this context.

*Proof of Theorem* 2.5. By Theorem 2.2 we know that the $\alpha_i$, in the representation (18) of $\sigma a = \hat{a}$ are elements of $\mathfrak{F}_{\mathbf{C}}(\mathbf{M})$, the $\mathbf{C}$-space of all $\mathbf{C}$-valued functions on the set $\mathbf{M}$, and that $\alpha_0$ is the Gelfand transform of the element $\bar{a} = a + \omega R$ of the Banach algebra $R/\omega R$. That is the case $n = 1$ of the following induction assumption:

(21) The coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ of $\hat{a} = \sum_{i \geqq 0} \alpha_i \omega^i \in \sigma R$ are elements of $P$, and to given $\phi_0, \phi_1, \ldots, \phi_{n-1} \in P$ there exists an element $\hat{a} = \sum_{i \geqq 0} \alpha_i \omega^i \in \sigma R$ with $\alpha_j = \phi_j$ for $j = 0, 1, \ldots, n - 1$.

1. Let $\hat{a} = \sum \alpha_i \omega^i \in \sigma R$ be given. If there exists an $\hat{a}_n = \alpha_n \omega^n + \alpha_{n+1}' \omega^{n+1} + \ldots \in \sigma R$ then the following argument proves $\alpha_n \in P$: $\hat{a}_n(M) \in \omega^n \mathbf{P}$ for all $M \in \mathbf{M}$ and therefore, by the construction of $\hat{a}_n$ in the Theorem 2.2, $a_n \in M^n$ for all $M \in \mathbf{M}$. Theorem 2.3 shows $a_n \in \omega^n R$. Take $c \in R$ such that $a_n = \omega^n c$. Then $c = \gamma_0 + \gamma_1 \omega + \ldots$ with $\gamma_0 \in P$ by the case $n = 1$ of (21). That implies $\alpha_n = \gamma_0 \in P$.

2. Now assume that $\sigma R$ does not contain an element of the form $\alpha_n \omega^n + \alpha_{n+1}' \omega^{n+1} + \ldots$ to the given $\hat{a} = \sum \alpha_i \omega^i$. By the induction assumption (21) there exists

$$\hat{b} = \alpha_1 + \alpha_2 \omega + \ldots + \alpha_{n-1} \omega^{n-2} + 0\omega^{n-1} + \beta_n \omega^n + \ldots$$

in $\sigma R$. Then

$$\hat{g} = \hat{a} - \hat{b}\omega = \alpha_0 + 0\alpha_1 + \ldots + 0\omega^{n-1} + \alpha_n \omega^n + \beta'_{n+1} \omega^{n+1} + \ldots$$

is an element of $\sigma R$. Let $h$ be any element in $R$ such that

$$\hat{h} = \alpha_0 + \eta_n\omega^n + \eta_{n+1}\omega^{n+1} + \ldots .$$

Then $\sigma R$ contains

$$\hat{g} - \hat{h} = (\alpha_n - \eta_n)\omega^n + \delta_{n+1}\omega^{n+1} + \ldots$$

and therefore, by the section 1 of this proof, $\alpha_n - \eta_n \in P$ is valid. In other words:

(22) $\quad D: \alpha_0 \rightarrow \alpha_n + P \quad \text{for} \quad \alpha_0 \in P$

is a mapping of $P$ into $F = \mathfrak{C}_{\mathbf{C}}(\mathbf{M})/P$, defined by the existence of an element

$$\alpha_0 + \alpha_n\omega^n + \alpha_{n+1}\omega^{n+1} + \ldots \quad \in \sigma R,$$

where $\mathfrak{C}_{\mathbf{C}}(\mathbf{M})$ is the $\mathbf{C}$-space of all $\mathbf{C}$-valued functions on the set $\mathbf{M}$. $F$ can be made a $P$-bimodule, defining, for $\alpha \in P$ and $\bar{t} = t + P \in F$,

$$\alpha\bar{t} : \mathfrak{m} \rightarrow \alpha(\mathfrak{m}) \cdot t(\mathfrak{m}) + P \quad \text{for } \mathfrak{m} \in \mathbf{M},$$

by virtue of $\alpha P \subseteq P$. The mapping $D$ is $\mathbf{C}$-linear and satisfies

$$D(\alpha_0 \cdot \beta_0) = \alpha_0(D\beta_0) + (D\alpha_0)\beta_0$$

because the equation

$$(\alpha_0 + \alpha_n\omega^n + \ldots) \cdot (\beta_0 + \beta_n\omega^n + \ldots) = \alpha_0\beta_0 + (\alpha_0\beta_n + \alpha_n\beta_0)\omega^n + \ldots$$

in $\sigma R$ holds. The application of the Lemma 4 to this $F$-derivation $D$ shows $D = \bar{0}$ and therefore $\alpha_n + P = P$, $\alpha_n \in P$.

3. We have to prove the second statement in (21) for $n$ also: To $\phi_0, \phi_1, \ldots, \phi_n \in P$ there exists an $a \in R$ with

$$\hat{a} = \sum_{0 \leqq j \leqq n-1} \phi_j\omega^j + \alpha_n\omega^n + \ldots$$

by the induction assumption (21). By the first two sections of this proof we know $\alpha_n \in P$. Then, by the case $n = 1$ of (21), there exists a $\hat{d} = (\phi_n - \alpha_n)\omega^0 + \ldots \in \sigma R$. Then $\sigma R$ contains $\hat{d}\omega^n$ and therefore

$$\hat{a} + \hat{d}\omega^n = \sum_{0 \leqq k \leqq n} \phi_k\omega^k + \gamma_{n+1}\omega^{n+1} + \ldots .$$

That finishes the proof of (21) for $n$ in place of $n - 1$. Therefore $\sigma R$ consists of all formal power-series in the indeterminate $\omega$ with coefficients in $P$, and the monomorphism $\sigma$ in the Theorem 2.2 is an algebraic epimorphism of $R$ onto the $\mathbf{P}$-algebra $P[[\omega]]$.

COROLLARY 2.6. *$R$ is segregated, i.e. $R$ is, as a $\mathbf{C}$-space, the direct sum of its subalgebra $Pe$ and its radical $\omega R$.*

Let $P$ be a Banach algebra with identity element under the norm $\|\cdot\|$. Then the $\mathbf{P}$-algebra $R = P[[\omega]]$ of formal power-series in $\omega$ with coefficients in

https://doi.org/10.4153/CJM-1978-006-1 Published online by Cambridge University Press

$P$ is a complete locally $m$-convex algebra under the following sequence $\{q_n; n = 0, 1, 2, \ldots\}$ of seminorms on $R$:

$$q_n\left(\sum_{i\geqq 0} \alpha_i\omega^i\right) = ||\alpha_0|| + ||\alpha_1|| + \cdots + ||\alpha_n||.$$

These seminorms satisfy for $a \in R$

$$q_0 a = ||\alpha_0||, \quad q_n a = q_{n+1}(\omega a), \quad q_n(\alpha_0 e) = ||\alpha_0||.$$

and they are uniquely determined by these properties:

THEOREM 2.7. *Assume $R = P[[\omega]]$, where $P$ is a Banach algebra with the norm $||\cdot||$ and possessing an identity element. Assume $R$ is a locally $m$-convex algebra under a sequence $\{q_n; n = 0, 1, 2, \ldots\}$ of seminorms on $R$, satisfying for $a = \sum \alpha_i\omega^i$ the conditions*

(23) $\quad q_0 a = ||\alpha_0||, \quad q_n a = q_{n+1}(\omega \alpha), \quad q_n(\alpha_0 e) = ||\alpha_0||.$

*Then $q_n = q_n^1$, where*

(24) $\quad q_n^1(a) = ||\alpha_0|| + ||\alpha_1|| + \ldots + ||\alpha_n||.$

*Proof:* Take $0 \leqq \xi \leqq \eta \in \mathbf{R}$ and set

(25) $\quad a(\xi) = \alpha_0 + \sum_{j\geqq 1} \xi\alpha_j\omega^j.$

Then $a(\eta) - a(\xi) \in \omega R$ and therefore, by the triangle inequality and (25),

(26) $\quad q_n(a(\eta)) - q_n(a(\xi)) \leqq (\eta - \xi)q_{n-1}\left(\sum_{j\geqq 1} \alpha_j\omega^{j-1}\right).$

Then $q_0^1 a = ||\alpha_0|| = q_0 a$, and $q_n^1$ satisfies (26) also. The induction assumption $q_{n-1} = q_{n-1}^1$ implies

$$q_n(a(\eta)) - q_n(a(\xi)) \leqq (\eta - \xi)q_{n-1}^1\left(\sum_{j\geqq 1} \alpha_j\omega^j\right) = q_n^1(a(\eta) - q_n^1(a(\xi)),$$

and equivalently,

(27) $\quad q_n(a(\eta)) - q_n^1(a(\eta)) \leqq q_n(a(\xi)) - q_n^1(a(\xi)) \quad$ for $0 \leqq \xi \leqq \eta$.

*First case:* $q_n^1(\sum_{j\geqq 1} \alpha_j\omega^j) = 0$. Then $||\alpha_1|| = \ldots = ||\alpha_n|| = 0$ and therefore $q_n(\sum \alpha_j\omega^j) = 0$ because $q_n(\omega^{n+1}b) = q_0(\omega b) = 0$ for $b \in R$.

*Second case:* $q_n^1(\sum_{j\geqq 1} \alpha_j\omega^j) > 0$ for $a = \sum_{i\geqq 0} \alpha_i\omega^i$. Then $q_n^1(\mathrm{a}(\xi)) > 0$ for $\xi > 0$, and (27), together with $q_n^1(a(\eta)) \geqq q_n^1(a(\xi))$ if $\eta \geqq \xi$, imply that

(28) $\quad t(\xi) = (q_n^1(a(\xi)))^{-1}(q_n(a(\xi)) - q_n^1(a(\xi)))$

is defined and decreasing for $\xi \in [0, \infty]$ and $t(0) = 0$, if $\alpha_0 \neq 0$. (But if $a_0 = 0$ apply $(23_2)$ to $q_n$ and use $q_{n-1} = q_{n-1}^1$.) On the other hand, $t(\infty) = 0$ is a consequence of $a(\xi) = \xi[\xi^{-1}\alpha_0 + \sum_{j\geqq 1} \alpha_j\omega^j]$ where the bracket [ ] goes to

https://doi.org/10.4153/CJM-1978-006-1 Published online by Cambridge University Press

$\sum_{j\geqq 1}\alpha_j\omega^j$ for $\xi \to \infty$. Then the continuity of the seminorms $q_n$ and $q_n{}^1$ together with $(23_2)$ and the induction assumption $q_{n-1}$ prove $t(\xi) \to 0$ for $\xi \to \infty$. Therefore $t(\xi) = \text{const} = 0$ on $[0, \infty]$ and especially $t(1) = 0$, i.e. $q_n a = q_n{}^1 a$.

COROLLARY 2.8. *If the topology of the* $\mathbf{P}$*-algebra* $R$ *in Theorem* 2.5 *is given by a sequence* $\{q_n; n = 0, 1, 2, \}$ *of seminorms such that* $q_0$ *induces the norm* $||\cdot||$ *of the Banach algebra* $R/\omega R \cong P$ *and* $q_n a = q_{n+1}(\omega a)$ *is valid for all* $a \in R$ *and the restriction of* $q_n$ *to the* $\mathbf{C}$*-subalgebra* $Pe$ *of* $R$ *is* $q_n(\alpha e) = ||\alpha||$, *then the algebraic isomorphism* $\sigma$ *of* $R$ *onto* $P[[\omega]]$ *is an "isometric" isomorphism in the sense*

(29) $\quad q_n a = q_n{}^1(\sigma a) \quad$ *for* $a \in R$ *and* $n = 0, 1, 2, \ldots.$

Theorem 2.5 and the Corollary 2.8 are combined in the following Theorem 2.9 to a characterization, by the arithmetic of $R$, of the $\mathbf{P}$-algebras $R = P[[\omega]]$ over those commutative $N$-algebras $P$, which have closed algebraic squares $\mathfrak{m} \cdot \mathfrak{m}$ of its maximal ideals.

THEOREM 2.9. *Let* $P$ *be a commutative* $N$*-algebra satisfying* $\mathfrak{m} \circ \mathfrak{m} = \mathfrak{m} \cdot \mathfrak{m}$ *for its maximal ideals* $\mathfrak{m}$. *Let* $||\cdot||$ *be the norm of* $P$. *Then the following statements hold:*

(i) $R = P[[\omega]]$ *under the set*

$$q_n{}^1\colon \sum\alpha_i\omega \to ||\alpha_0|| + ||\alpha_1|| + \ldots + ||\alpha_n||, \quad n = 0, 1, 2, \ldots$$

*of seminorms can be characterized, up to* $\mathbf{P}$*-algebra isomorphisms* $\sigma$, *by the following properties of* $R$:

1) $R$ *is a topologically arithmetical* $\mathbf{P}$*-algebra, commutative and complete, locally-m-convex with identity element* $e$.

2) *The factor algebra* $R/\omega R$ *is isomorphic as a Banach algebra to* $P$. *The powers* $\omega^n R$ *of the radical* $\omega R$ *of* $R$ *are completely distributive elements of the lattice* $V(R)$ *of the closed ideals in* $R$. *The (algebraic) powers of the maximal ideals* $M$ *in* $R$ *are closed and form, for each* $M$, *a strictly descending chain in* $V(R)$. *The intersection taken over all powers* $M^n$ *and all* $M$ *is equal to the zero ideal in* $R$.

(ii) *If, in addition, the topology on* $R$ *is given by a sequence* $q_n$, $n = 0, 1, 2, \ldots,$ *of seminorms satisfying the following conditions:*

$$q_0 \text{ induces } ||\cdot|| \text{ on } P \cong R/\omega R,$$
$$q_n(a) = q_{n+1}(\omega\alpha) \quad \text{for } a \in R,$$
$$q_n(\alpha e) = ||\alpha|| \quad \text{for } \alpha \in P,$$

*then* $R$ *and* $P[[\omega]]$ *are isometric isomorphic*, i.e. $q_n a = q_n{}^1\sigma a$ *for* $a \in R$ *and* $n \in \mathbf{N}$.

**3. Examples.** We have started § 1 from a commutative $\mathbf{N}$-algebra $P$. If the product $\mathfrak{m} \cdot \mathfrak{m}$ is closed for every maximal ideal in $P$ then the $\mathbf{P}$-algebra $R = P[[\omega]]$ enjoys all properties mentioned in Theorem 2.9, remembering Proposition 2.1. That motivates our interest in examples of commutative $N$-algebras $P$ with all products $\mathfrak{m} \cdot \mathfrak{m}$ being closed for $\mathfrak{m} \in \mathfrak{X}$, the set of the maximal ideals in $P$.

https://doi.org/10.4153/CJM-1978-006-1 Published online by Cambridge University Press

*First Example*: $P = \mathscr{C}_{\mathbf{C}}(X)$, the $B^*$-algebra of all continuous **C**-valued functions on the compact Hausdorff space $X$ under the supremum norm.

It is well known and can be taken as consequences of Behrens [**3**], Theorem 1.2 and 2.4 also that $\mathscr{C}_{\mathbf{C}}(X)$ is an $N$-algebra satisfying $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$ for $\mathfrak{m} \in \mathfrak{X}$, $\mathfrak{X}$ being homoeomorphic to $X$. The arithmetic of $R = \mathscr{C}_{\mathbf{P}}(X)$ has been developed in Behrens [**3**] already, but the importance of the complete distributivity of the powers of the radical in $V(R)$ was first observed in the semigroup theoretic paper of Behrens [**4**].

*Second Example* (Šilov [**10**], discussed also in Rickart [**9**] A.Z.5): $P = BVC(0, 1)$ consisting of all **C**-valued continuous functions on $[0, 1]$ which are of bounded variation using

$$(1) \qquad ||t|| = \sup\{|t(x)|; x \in [0, 1]\} + \operatorname{Var} t \quad \text{for } t \in P$$

as its norm.

The space $\mathfrak{X}$ of the maximal ideals, $\mathfrak{m}$ under the hull-kernel topology is homoeomorphic to the interval $[0, 1]$, and $x_0 \in [0, 1]$ is associated with $\mathfrak{m}_0 = \{t \in P; t(x_0) = 0\}$. It is to show that $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$ for all $\mathfrak{m} \in \mathfrak{X}$.

*Proof.* $t \in \mathfrak{m}_0 = (\operatorname{Re} t)(x_0) = 0 = (\operatorname{Im} t)(x_0)$. Assume $t$ is $R$-valued and set $g = t \cdot \chi_{x_0}$, $h = t \cdot (1 - \chi_{x_0})$, where $\chi_{x_0}$ is the characteristic function of $[x_0, 1]$. Then $g$ and $h$ are continuous and of bounded variation, $t = g + h$ and $g \in \mathfrak{m}_0$, $h \in \mathfrak{m}_0$ because $t(x_0) = 0$. Decompose $g$ to the difference $g = g_1 - g_2$ of the monotonically ascending functions $g_1$ and $g_2$ of bounded variation (see e.g. Rudin [**10**, Theorem 8.13, (a)]). Then $g_i(x_0) = 0$ implies $g_i(x) \geqq 0$ for all $x \in [0, 1]$. Therefore $\sqrt{g_i} \in \mathfrak{m}_0$ also; analogously $h = h_1 - h_2$, $h_i \leqq 0$, in $[0, 1]$. Then

$$t = (\sqrt{g_1})^2 - (\sqrt{g_2})^2 + (\sqrt{-h_1})^2 - (\sqrt{-h_2})^2 \in \mathfrak{m}_0 \cdot \mathfrak{m}_0.$$

*Third Example* (Šilov [**11**], see Rickart [**9**], the second example in A.2.5 for $p = 1$): $P = AC^1(0, 1)$ consisting of all **C**-valued absolutely continuous function on $[0, 1]$, using

$$||t|| = \sup\{|t(x)|; x \in [0, 1]\} + \int_0^1 |t'(x)| dx \quad \text{for } t \in P$$

as its norm.

The absolute continuity of $t \in P$ implies that $t$ is differentiable a.e. (Lebesgue) and absolutely integrable (Rudin [**10**, Theorem 8.17 and 18.18]) and so are $\operatorname{Re} t$ and $\operatorname{Im} t$. They are of bounded variation because $[0, 1]$ is compact. Again $[0, 1]$ is homoeomorphic to $\mathfrak{X}$ under the hull-kernel topology and $\mathfrak{X} = \{\mathfrak{m}_0; \mathfrak{m}_0 = \{t \in P; t(x_0) = 0\}; x_0 \in [0, 1]\}$. The same representation of $\operatorname{Re} t$ (and analogously of $\operatorname{Im} f$) by the $g_i \in \mathfrak{m}_0$ and $h_i \in \mathfrak{m}_0$ as in the second example leads to the question whether the squareroots $\sqrt{g_i}$ and $\sqrt{-h_i}$ belong to $P$. But that is true because $\sqrt{g_1}$ is differentiable at all points in $[0, 1]$ where $g_1$ is so with the possible exception of $x_0$, and $(\sqrt{g_1})'$ is absolutely integrable in

[0, 1] again. That implies the absolute continuity of $\sqrt{g_1}$ and therefore $\sqrt{g_1} \in \mathfrak{m}_0$. This together with the analogous statements for $g_2$ and $h_1$, $h_2$, proves $\mathfrak{m}_0 = \mathfrak{m}_0 \bullet \mathfrak{m}_0$.

*Fourth Example:* It was shown in Behrens [**3**, Theorem 2.2], that the sum $A + B$ of closed ideals $A$ and $B$ in $\mathscr{C}_{\mathbf{P}}(X)$, $X$ a compact Hausdorff space, is closed. That is no longer true in the case $P = BVC(0, 1)$ as the following counterexample proves. For the two closed subsets of [0, 1]:

$$\mathfrak{a} = \{0\} \cup \left\{\frac{1}{2n}\,; n \in \mathbf{N}\right\} \quad \text{and} \quad \mathfrak{B} = \{0\} \cup \left\{\frac{1}{2n-1}\,; n \in \mathbf{N}\right\}$$

set

$$\mathfrak{a} = \{t \in P; t = 0 \text{ on } \mathfrak{A}\} \quad \text{and} \quad \mathfrak{b} = \{t \in P; t = 0 \text{ on } \mathfrak{B}\}.$$

Then the closure $\mathfrak{a} \dotplus \mathfrak{b}$ of the sum $\mathfrak{a} + \mathfrak{b}$ of these two closed ideals in $P$ is the ideal $\mathfrak{m}_0 = \{t \in P; t(0) = 0\}$ because $\mathfrak{A} \cap \mathfrak{B} = \{0\}$. The function $t(x) = x$ belongs to $\mathfrak{m}_0$. Assume, by the way of contradiction, $\mathfrak{m}_0 = \mathfrak{a} + \mathfrak{b}$. Then there would exist $g \in \mathfrak{a}$ and $h \in \mathfrak{b}$ such that $x = g(x) + h(x)$ for $x \in [0, 1]$ and $g(x) = 0$ for all $x \in \mathfrak{A}$ and therefore $h(x) = x$ for $x \in \mathfrak{A}$; analogously $h(x) = 0$ for $x \in \mathfrak{B}$ and $g(x) = x$ for $x \in \mathfrak{B}$. Take $0 < t_0 < t_1 < \ldots < t_{2N-1} \leqq 1$ where $t_i = 1/(2N - i)$ and $N \in \mathbf{N}$. Then $t_i \in \mathfrak{A}$ if $i$ is even and $t_i \in \mathfrak{B}$ if $i$ is odd. That implies

$$\sum_{1 \leqq i \leqq 2N-1} |g(t_i) - g(t_{i-1})| = \frac{1}{2N-1} + \frac{1}{2N-3} + \cdots + \frac{1}{3} + 1$$

and shows that the function $g$ would not be of bounded variation and therefore $g \notin \mathfrak{A}$, q.e.a.

*Remark.* $BVC(0, 1)$ and $AC^1(0, 1)$ are not sub-Banach-algebras of $\mathscr{C}_{\mathbf{C}}[(0, 1])$. Therefore the existence of the square roots of the $g_i$ and $-h_i$, $i = 1, 2$, does not contradict Katznelson's characterization of $\mathscr{C}_{\mathbf{C}}([0, 1])$ in [7].

## References


1. E. A. Behrens, *The arithmetic of the quasi-uniserial semigroups without zero*, Can. J. Math. *23* (1971), 507–516.
2. ——— *Ringtheorie* (Bibliographisches Institut, Zürich 1975).
3. ——— *Topologically arithmetical rings of continuous functions*, Publ. Math. Debrecen *24* (1977), 107–121.
4. ——— *Power series developments in lattice ordered semigroups*, Semigroup Forum, to appear.
5. J. J. Benedetto, *Spectral synthesis* (B. G. Teubner, Stuttgart, 1975).
6. N. Bourbaki, *General topology, Part 2* (Hermann, Paris, 1966).
7. Y. Katznelson, *Sur les algèbres dont les éléments non-négatifs admettent des racines carrées*, Ann. Scient. Ec. Norm. Sup. *77* (1960), 167–174.
8. E. A. Michael, *Locally multiplicatively-convex topological algebras*, Memoirs No. *11* (American Math. Soc., Providence, 1971).

**9.** C. E. Rickart, *General theory of Banach algebras*, (1960), Reprint (R. E. Krieger Publ. Co., Huntington, N.Y., 1974).
**10.** W. Rudin, *Real and complex analysis* (McGraw-Hill, New York, 1974).
**11.** G. Šilov, *On regular normed rings*, Trav. Inst. Math. Stekloff *21*, Moscow (1947).
**12.** A. B. Willcox, *Some structure theorems for a class of Banach algebras*, Pacific J. Math. *6* (1956), 177–192.

*McMaster University,*
*Hamilton, Ontario*