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Nagoya Math. J.
Vol. 86 (1982), 173-201

COUNTING THE NUMBER OF BASIC INVARIANTS FOR $G \subset GL(2, k)$ ACTING ON $k[X, Y]$

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List of notation

The notations used in this paper without explicit mention are listed below. Here R is a positively graded Noetherian ring, $\mathfrak{a}$ a homogeneous ideal of R, and $f, g, \cdots, h$ are homogeneous elements of R.

$\dim R$ = Krull dimension of R
emb. $\dim R$ = embedding dimension of R
$\operatorname{ht} \mathfrak{a}$ = height of $\mathfrak{a}$
$\mu(\mathfrak{a})$ = minimal number of generators of $\mathfrak{a}$
$\operatorname{hd} \mathfrak{a} = \operatorname{hd} R/\mathfrak{a} - 1$ = homological dimension of $\mathfrak{a}$
$(f, g, \cdots, h)$ = ideal generated by $f, g, \cdots, h$
$[f\ g\ \cdots\ h]$ = row vector
$o(G)$ = order of a finite group G

$\mu(\mathfrak{a})$ and $\operatorname{hd} \mathfrak{a}$ are also written $\mu_R(\mathfrak{a})$ and $\operatorname{hd}_R(\mathfrak{a})$ when R needs to be mentioned. Polynomial rings are always regarded as graded rings with natural gradation.

Introduction

In this paper we consider a certain group representation ρ that is defined for each finite subgroup G of $GL(2, k)$. ρ is explained as follows: Let G act linearly on the polynomial ring $R = k[x, y]$, and let $\mathfrak{a} = (R^G_+)R$ be the ideal of R generated by all the non-constant invariant forms. Then the representation module V of ρ is the space spanned by a set of basic relations (syzygies) of $\mathfrak{a}$ over R. Since $\operatorname{hd} R/\mathfrak{a} = 2$, we have that $\mu(\mathfrak{a}) = \dim V + 1$. When $\operatorname{ch} k = 0$ or otherwise $\operatorname{ch} k$ does not divide $o(G)$, a set of generators of the ideal $\mathfrak{a}$ chosen from among invariant forms generate the ring of invariants R^G as an algebra over k. Consequently we also

Receiver May 6, 1980.

have emb. $\dim R^G = \dim V + 1$; for example if $\dim V = 2$, R^G is a 'hypersurface'. The study of ρ has its origin in an attempt to answer the question raised by S. Goto which asks when R^G is a hypersurface, R being a polynomial ring of any dimension acted on linearly by a finite group G. It had been an empirically established fact that for every finite group G in $SL(2, C)$, R^G is a hypersurface, and any answer to Goto's problem should explain it. It is in fact proved here by showing $\dim V = 2$ for $G \subset SL(2, C)$; although it does not generalize to answer Goto's problem, it leads to the question what ρ is for $G \subset GL(2, k)$ in general.

The main results of this paper are Theorem 3.6 and its proof, where ρ is determined for subgroups in $SL(2, C)$, and what is stated in § 7, where ρ is determined for abelian groups in $GL(2, C)$. It should be emphasized that the primary interest of Theorem 3.6 lies not in the statement itself but in the method to prove it. In fact the invariant theory for finite groups of $GL(2, C)$ has been studied in its full detail, and for each $G \subset GL(2, C)$, a set of basic invariants has been (and can be) computed. (See, for example, [7] or [8].) Thus upon looking at it we at once have the statement of Theorem 3.6 and then we can determine ρ, but this is not our intention.

When G is abelian, R^G is isomorphic to a two dimensional normal semigroup ring $K[M]$, and the generators of M can best be dealt with by means of continued fractions. Let M be a normal semigroup in Z_+^2 (expressed additively), and assume $\{(a_i b_i) \mid i = 1, 2, \cdots, m\}$ is the basis of M with the order that $a_1 > a_2 > \cdots > a_m$. Here we can assume a_i's (and b_i's) do not have a common divisor. Then any successive three terms a_i, a_{i+1}, a_{i+2} are related by the formula $a_i/a_{i+1} = B_i - a_{i+2}/a_{i+1}$, where B_i is the smallest positive integer such that $B_i a_{i+1} \geq a_i$. In this paper we do not assume this knowledge but we prove what is equivalent to it in a form suitable to our purpose (Proposition 6.2 and Proposition 6.4). This is partly for the sake of self-containedness and partly for that, in order to write $\rho(G)$ for an abelian group, we need the numbers $r_i = a_i - a_{i+1}$ instead of a_i, and it seems that the numbers r_i are occasionally more properly dealt with than a_i themselves. For example, from the fact that the sequence r_i is monotone decreasing follows an unexpected result (Theorem 8.9).

The definition of ρ is given in § 2. Actually ρ is defined for any group acting on a polynomial ring R over a field, but only when $\dim R = 2$ and

G is finite, we know 'a priori' the properties (i), (ii) of Theorem 2.4 are the case. These properties are a direct consequence of the so called structure theorem of homologically one dimensional ideals, and are quite helpful to determine the degree of ρ ($= \dim V$). This might interest one to know other cases in which $\operatorname{hd} \mathfrak{a} = 1$ even when $\dim R \neq 2$. We have an example of such cases when $k^* = GL(1, k)$ acts linearly on $R = k[X, Y, Z]$ with $\dim R^G = 2$. A proof for this based on the fact r_i is monotone decreasing, as mentioned above, is given in § 8.

The structure theorem of homologically one dimensional ideals is stated in § 1. Theorem 2.1 in § 2 is a very basic fact in invariant theory (originally due to Hilbert) that makes it possible to replace generators of algebras by generators of ideals. Besides these two well known theorems little is presupposed in this paper.

§§ 4, 5, as well as § 1, are of preliminary nature.

The author wishes to express his deepest thanks to Prof. H. Yamada, Prof. H. Morikawa, and especially to Prof. H. Matsumura for their patience and constant encouragement over the long period of his sickness. Special thanks are also due to K. Watanabe, S. Goto and Y. Namikawa for their valuable suggestions.

§ 1. The structure theorem of homologically one dimensional ideals

In this section R denotes a polynomial ring over an arbitrary field unless otherwise specified. R_+ denotes the homogeneous maximal ideal.

Assume a homogeneous ideal $\mathfrak{a} \subset R$ is minimally generated by $f_1, f_2, \cdots, f_{n+1}$, and $\operatorname{hd} \mathfrak{a} = 1$ (or we may also say $\operatorname{hd} R/\mathfrak{a} = 2$). Let

$$0 \longrightarrow R^n \xrightarrow[M]{} R^{n+1} \xrightarrow[F]{} R$$

be a minimal free resolution of $R/\mathfrak{a}$. We shall write an element of a free module as a row vector and represent a homomorphism between free modules as a matrix in such a way that, in the notation $M : R^n \to R^{n+1}$ for example, if $v \in R^n$, then its image by M is vM, the usual matrix product. Under this convention, F is a column matrix with entries a minimal set of generators of $\mathfrak{a}$, so that we may assume ${}^tF = [f_1\, f_2 \cdots f_n\, f_{n+1}]$. (${}^tF$ is the transpose of F. Throughout, we will write a column matrix in this way in order to save space.) Since $\mathfrak{a}$ is homogeneous, the entries of M may, as well as f_i's, be taken homogeneous. We recall that an ideal I (in any ring R) is called perfect if $\operatorname{hd} R/I = \operatorname{ht} I$. Now let us state the structure theorem of homologically one dimensional ideals in the homogeneous case.

THEOREM 1.1. *With the notation above, let M_i be the matrix obtained from M by deleting the i-th column, and let $D_i = \det M_i$. Then there is a homogeneous element $h \in R$ such that $f_i = (-1)^i h D_i$. h is a greatest common divisor of f_i's, and h is a unit if and only if $\mathfrak{a}$ is perfect* (*i.e.*, $\operatorname{ht} \mathfrak{a} = 2$).

For proof, see Peskine-Szpiro [5], Chaptre I, Theorem 3.3, where this is proved over a local ring. Homogeneous translation is immediate.

Let M be an n by $n+1$ matrix over R, and let $D_i = \det M_i$ be as in Theorem 1.1 above. We shall refer to D_i as the i-th maximal minor of M, and denote by $I(M)$ the ideal generated by all the D_i. We have

PROPOSITION 1.2. *Suppose*

$$0 \longrightarrow R^n \xrightarrow[M]{} R^{n+1} \xrightarrow[F]{} R$$

is a complex, and $F \neq 0$ and F has no units as entries. Then the complex is exact if and only if $\operatorname{ht} I(M) \geq 2$.

Proof. This is a special case of Buchsbaum-Eisenbud [1], Theorem.

Remark 1.3. Either of Theorem 1.1 and Proposition 1.2 can be regarded as a corollary of the other, but we treated them independently, as they are of different nature. In the sequel, Proposition 1.2 will be referred to as Buchsbaum-Eisenbud criterion.

DEFINITION 1.4. Suppose that

$$R^\nu \xrightarrow[M]{} R^\mu \xrightarrow[F]{} R$$

is exact, $M \otimes R/R_+ = 0$, and the entries of M and F are homogeneous. A row vector $v \in R^\mu$ is called a *relation* of F if it is in $\operatorname{Ker} F$. Thus, for example, every row of M is a relation of F. M is called a *relation matrix* of F (or of $\operatorname{Im} F$). Note if M' is another relation matrix of F, then there is an invertible matrix $U = [u_{ij}]$ such that $u_{ij} \in R$ are homogeneous and $M' = UM$. A relation of F is said to be *basic* if it can be a row of a relation matrix of F. Thus, $v \in R^\mu$ is a basic relation if and only if $v \in \operatorname{Ker} F$ and $v \notin (R_+) \operatorname{Ker} F$. Assume $[a_1\ a_2 \cdots a_\mu]$ is a relation of F (and all a_i are homogeneous). Then we have: $\deg a_1 f_1 = \deg a_2 f_2 = \cdots = \deg a_\mu f_\mu$, whenever $a_i \neq 0$. Say this number is equal to p. Then p is called the *degree* of the relation.

In Theorem 1.1, assume $M = [a_{ij}]$ and the i-th row of M is a relation of degree p_i. Then by definition we have: $\deg a_{ij} = p_i - d_j$ with $d_j = \deg f_j$. Put $d = \deg h$. Then the theorem, in particular, says:

COROLLARY 1.5. $p_1 + p_2 + \cdots + p_n = d_1 + d_2 + \cdots + d_{n+1} - d$. (*cf.* Peskine and Szpiro [5] § 3.)

Remark 1.6. The numbers p_i may also be explained as follows: If $R(-p)$ denotes a free module on one generator having degree p, then p_i are such that M is a degree 0 map in the sequence

$$0 \longrightarrow \bigoplus_{i=1}^{n} R(-p_i) \xrightarrow[M]{} \bigoplus_{i=1}^{n+1} R(-d_i) \xrightarrow[F]{} R(0)\,.$$

The following two lemmas are application of Proposition 1.2 and Corollary 1.5 to be used in § 3.

LEMMA 1.7. *Let $R = k[x, y]$, and $\mathfrak{a}$ a homogeneous ideal of R minimally generated by f_1, f_2, and f_3 such that* $\operatorname{ht}(f_1, f_2) = 2$. *With $d_1 = \deg f_1$, assume $d_1 + d_2 = d_3 + 2$. Then we have $(f_1, f_2) : f_3 = (x, y)$.*

Proof. Note, in R, any ideal of height 2 is perfect. Let $M = [a_{ij}]$ and F be as in Theorem 1.1 with $n + 1 = 3$. Then it is easy to see that $(f_1, f_2) : f_3 = (a_{13}, a_{23})$. Since $\operatorname{ht}(f_1, f_2) = 2$, $a_{13} \neq 0$ and $a_{23} \neq 0$. This means $\deg a_{i3} > 0$ for $i = 1, 2$, because they cannot be units. Thus, if p_1 and p_2 are the degrees of the first and the second rows (relations) of M, then $p_i \geq d_3 + 1$ for $i = 1, 2$. This says that $2(d_3 + 1) \leq p_1 + p_2 = d_1 + d_2 + d_3$. (Note we can use Corollary 1.5 with $d = 0$, since $\mathfrak{a}$ is perfect.) Because of the condition posed on the degree of the generators, the only possibility is that $p_1 = d_3 + 1$, which implies $\deg a_{13} = \deg a_{23} = 1$. These two elements generate an ideal of height 2, hence $(a_{13}, a_{23}) = (x, y)$ as desired.

Remarks 1.8. (i) For any two elements f_1 and f_2 in $R = k[x, y]$ with $\operatorname{ch} k = 0$, if f_3 is the Jacobian of f_1 and f_2, then the condition of the lemma concerning degrees is satisfied.

(ii) Assume M is a relation matrix of ${}^tF = [f_1\, f_2 \cdots f_\mu]$. Then, as was said in the proof of the lemma, it generally holds that the ideal generated by all the elements that appear in the last column of M is the ideal $(f_1, f_2, \cdots, f_{\mu-1}) : f_\mu$.

LEMMA 1.9. *Let $R = k[x, y]$, where k is a field of characteristic* 0.

https://doi.org/10.1017/S002776300001984X Published online by Cambridge University Press

Assume $f, h \in R$ *are homogeneous elements such that* $\mathrm{ht}(f, h) = 2$ *and* $\deg f \geq 2$, $\deg h \geq 2$. *Let* δ *be the Jacobian determinant of* f *and* h. *Then we have that* $\mu(f, h, \delta) = 3$; *in particular* $\delta \notin (f, h)$.

Proof. We write $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$ for any $f \in R$. Then by definition $\delta = \det\begin{pmatrix} f_x & f_y \\ h_x & h_y \end{pmatrix}$. Let us consider the matrix $M = \begin{pmatrix} -h_x & f_x & y \\ h_y & -f_y & x \end{pmatrix}$. If D_i denotes the i-th maximal minor of M, we obtain the complex

$$0 \longrightarrow R^2 \xrightarrow[M]{} R^3 \xrightarrow[F]{} R, \qquad \text{where } {}^tF = [D_1 \ -D_2 \ D_3]\,.$$

(Note $MF = 0$ holds generally.) Notice that

$$\begin{aligned} D_1 &= xf_x + yf_y = (\deg f)f \\ D_2 &= -xh_x - yh_y = -(\deg h)h \\ D_3 &= h_x f_y - f_x h_y = -\delta\,. \end{aligned}$$

Thus $I(M)$ contains f and h, and since $\mathrm{ht}(f, h) = 2$ by assumption, Buchsbaum-Eisenbud criterion proves that the complex is exact, and in particular it is a minimal free resolution of $R/\mathrm{Im}\,F$ (for otherwise a unit would appear in the matrix M). Thus the ideal $I(M) = (f, h, \delta)$ is minimally generated by three elements.

Remark 1.10. The lemma above holds more generally: let $R = k[x_1, x_2, \cdots, x_n]$, $n \geq 2$, and assume $\mathfrak{f} = (f_1, f_2, \cdots, f_n)$ is a homogeneous system of parameters of R such that $\deg f_i \geq 2$ for all i. Then $\mu(f_1, f_2, \cdots, f_n, \delta) = n + 1$, where δ is the Jacobian determinant of $\mathfrak{f}$.

This can be proved by showing δ is a generator of the socle of the Gorenstein ring $R/\mathfrak{f}$, as was pointed out by S. Goto. The method here seems more appropriate for our purpose to prove Theorem 3.6.

§2. The representation ρ

We want to fix some notations and terminology as we review basic definitions and facts of invariant theory.

Let k be an algebraically closed field. When a linear algebraic group G over k acts on a k-algebra by k-automorphisms, we denote by a^g ($a \in R$, $g \in G$) the image of a by the automorphism g. If $a = a^g$ for all $g \in G$, then a is an invariant. If a and a^g differ only by unit multiple for all $g \in G$, a is a semi-invariant. By R^G will be denoted the ring of invariants, i.e., the subring of R consisting of all the invariants. If $M = [a_{ij}]$ is a matrix over R, M^g will denote the matrix $[a_{ij}^g]$. When R^G is finitely generated over k, a set $\{f_1, f_2, \cdots, f_\mu\}$ of invariants is called a system of basic invariants if they generate the ring R^G over k and they are irredundant.

Now assume G is a linearly reductive linear algebraic group (i.e., every rational G-module, not necessarily finite dimensional, is completely reducible), and R is a finitely generated k-algebra. Then it is well known that R^G is finitely generated over k (when G acts k-rationally on R, of course).

The following fact on which the proof of finite generation of R^G is based is very important in this paper.

THEOREM 2.1. *Let G be linearly reductive, and R finitely generated over k. Assume R is positively graded and the action preserves grading. Let $I = (R_+^G)R$ be the ideal of R generated by all the invariants without constant terms. Then an ideal basis of I chosen from among invariant forms* is *an algebra basis of R^G, i.e., if $I = (f_1, f_2, \cdots, f_\mu)$, $f_i \in R^G$, then $R^G = k[f_1, f_2, \cdots, f_\mu]$.*

Proof can be found wherever finite generation of R^G is proved, e.g. Mumford [ ] or Fogarty [ ].

When a ring is positively graded, the minimal number of generators of a homogeneous ideal and the embedding dimension of the ring have definite meaning. The theorem implies:

COROLLARY 2.2. $\mu(I) = \text{emb.}\dim R^G$.

Remark 2.3. Later we concern ourselves only with (i) torus groups and (ii) finite groups when ch $k = 0$, both of which are well known to be linearly reductive.

Now suppose R is a polynomial ring (and G linearly reductive). Let $R^G = k[f_1, f_2, \cdots, f_\mu]$ with $\mu = \text{emb.}\dim R^G$. Then f_i are a minimal basis of the ideal $I = (R_+^G)R$. Put ${}^{\tau}F = [f_1\, f_2 \cdots f_\mu]$, and let M be a relation matrix of F, so that

$$R^\nu \xrightarrow[M]{} R^\mu \xrightarrow[F]{} R \qquad \text{is exact with minimal } \nu\,. \quad (\text{See § 1.})$$

Since f_i are invariant, we see that $R^\nu \xrightarrow[M^g]{} R^\mu \xrightarrow[F]{} R$ is also exact, hence there is an invertible matrix $U = [u_{ij}]$ over R such that $M^g = UM$. If $\bar{u}_{ij} \in k$ denotes the residue class of u_{ij} module R_+ (= the homogeneous maximal ideal of R), we see $[\bar{u}_{ij}]$ is uniquely determined by g, hence we may write $[\bar{u}_{ij}] = \rho(g)$. Clearly ρ is a homomorphism, and thus we have obtained a representation. Since $(R_+)\mathrm{Ker}\,F$ and $\mathrm{Ker}\,F$ are both G-modules, G acts on $\mathrm{Ker}\,F/(R_+)\mathrm{Ker}\,F = V$, which is nothing but the representation module of ρ. Since G is linearly reductive, we may assume $\mathrm{Ker}\,F = V \oplus (R_+)\,\mathrm{Ker}\,F$ as G-modules. If the rows of M have been chosen to be a basis of such V, it holds that $M^g = \rho(g)M$.

We will call $\rho : G \to GL_k(V)$ the representation of G to the syzygy space of F.

If $\mathrm{hd}\,I = 1$, ρ has the following property.

THEOREM 2.4. *In the situation above assume* $\mathrm{hd}\,I = 1$. *If I is perfect, then,*

(i) $\rho(G) \subset SL_k(V)$.

(ii) *If* $V = V_1 \oplus V_2$ *is a proper decomposition of V as G-modules and* $\rho_i : G \to GL_k(V_i)$ *are the corresponding representations, then* $\rho_i(G) \not\subset SL_k(V_i)$, $i = 1, 2$.

Proof. (i) We may assume a relation matrix M of F is such that $M^g = \rho(g)M$ for any $g \in G$. It follows that if D is any maximal minor of M, then $D^g = \det \rho(g)D$. But according to Theorem 1.1, D is an invariant, which implies $\det(g) = 1$.

(ii) M being as above, we may further assume that the first n_1 rows of M span V_1. Let M_1 be the submatrix consisting of these rows. If $\det \rho_1(g) = 1$ for all $g \in G$, then all the maximal minors of M_1 are invariants. Now compare the following two numbers:

$$N_1 = \mathrm{Min}\{\deg D \,|\, D \text{ is a maximal minor of } M\}$$
$$N_2 = \mathrm{Min}\{\deg \Delta \,|\, \Delta \text{ is a maximal minor of } M_1\}.$$

We immediately see that $N_2 < N_1$ since M_1 is a proper submatrix of M. On the other hand since the maximal minors of M, being precisely f_i's (modulo unit multiple), generate the ring of invariants, we have $N_2 \geq N_1$. This contradiction proves (ii).

Remark 2.5. In Theorem 2.4 replace "I perfect" by "I not perfect". Then a greatest common divisor of f_i's, say h, is a semi-invariant with character $\det \rho^{-1}$. In fact if D is any maximal minor of M, D is a semi-invariant with character $\det \rho$. Since hD, being one of f_i's, is an invariant, the assertion follows. It is easy to see that in this case, too, (ii) holds without any modification.

Remark 2.6. When a finite group G acts linearly on $R = k[x, y]$, we are in the situation of Theorem 2.4, provided $\operatorname{ch} k = 0$ or $(\operatorname{ch} k, o(G)) = 1$. Let H be the subgroup of $G \subset GL(2, k)$ generated by all the reflexions. Then G/H acts on R^H, which is a polynomial ring (in two variables). Thus we may talk about ρ for G/H. It is easy to see that $\rho(G)$ and $\rho(G/H)$ are equivalent and, in particular, $\operatorname{Ker} \rho \supset H$. Accordingly we may assume, to obtain $\rho(G)$, G does not contain any reflexions. (cf. § 7)

The next proposition applies to any finite $G \subset GL(2, C)$ not containing reflexions.

PROPOSITION 2.7. *Assume* $\operatorname{ch} k = 0$. *Let* $\rho : G \to GL_k(V)$ *be a representation of* (*any*) *group* G, *having the property* (i) *and* (ii) *in Theorem* 2.4. *Assume* ρ *is completely reducible. If* $G/[G, G]$ *is cyclic of order* p (*say*), *then* ρ *decomposes into at most* p *irreducible factors.*

Proof. Let $\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_n$ be a decomposition of ρ, and let $g \in G$ be a generator of $G/[G, G]$. Notice that $\det \rho_i(G)$ is a cyclic group generated by $\det \rho_i(g)$, since $\det \rho_i([G, G]) = \{1\}$. Hence we can write det $\rho_i(g) = \omega^{a_i}$, where ω is a primitive p-th root of 1. If $n > p$, we get a contradiction to Lemma 2.8 below.

LEMMA 2.8. *Assume there are given* n *integers* a_i. *If* p *is a positive integer such that* $p < n$, *then there is a proper subset* $I \subsetneq \{1, 2, \cdots, n\}$ *such that* $\sum_{i \in I} a_i \equiv 0 \bmod p$.

Proof is left to the reader.

§ 3. The number of basic invariants for $G \subset SL(2, C)$

In this section we assume k is an algebraically closed field of characteristic 0. G always denotes a finite subgroup of $SL(2, k)$. R is $k[x, y]$ on which G acts by linear transformation of x and y. We will always be assuming G is non-trivial, in which case G cannot leave a linear form invariant.

Notation 3.1. For $f_1, f_2 \in R$, $J(f_1, f_2)$ denotes the Jacobian of f_1 and f_2, i.e., $J(f_1, f_2) = \det [\partial f_i / \partial x_j]$, where $x_1 = x$ and $x_2 = y$.

LEMMA 3.2. *If* $f, h \in R^G$, *then* $J(f, h) \in R^G$. *More generally, if* f *and* h *are semi-invariants such that* $fh \in R^G$, *then* $J(f, h) \in R^G$.

Proof is easy by direct computation.

PROPOSITION 3.3. *Let* $R^G = k[f_1, f_2, \cdots, f_{n+1}]$ *with* $\deg f_1 \le \deg f_2 \le \deg f_i$, $i \ge 3$. *Assume* $\deg f_1 > 2$. *Then* $\mathrm{ht}\,(f_1, f_2) = 2$.

Proof. Everything will be considered in R. Assume $\mathrm{ht}\,(f_1, f_2) = 1$. Then f_1 and f_2 have a greatest common divisor. Let it be f and write $f_i = fh_i$, $i = 1, 2$. Then one easily sees h_1 is not a constant, and hence $\mathrm{ht}\,(h_1, h_2) = 2$. Since f_1 is an invariant, G permutes the divisors of f_1, and the same is true for f_2. From the fact that h_1 and h_2 have no common divisor, it follows that f and h_1 are semi-invariants. By the preceding lemma, $J(f, h_1)$ is an invariant whose degree is equal to $\deg f + \deg h_1 - 2 = \deg f_1 - 2$. This contradicts the assumption of $\deg f_1$ to be minimal and > 2.

PROPOSITION 3.4. *Let* $R^G = k[f_1, f_2, \cdots, f_{n+1}]$, f_i *being a system of basic invariants. Assume* $2 < \deg f_1 \le \deg f_2 \le \deg f_i$ *for* $i \ge 3$. *Set* $\delta = J(f_1, f_2)$. *Then* f_1, f_2, δ *can be a part of a system of basic invariants.*

Proof. First of all δ is an invariant by Lemma 3.2. Let $\mathfrak{m} = (f_1, f_2, \cdots, f_{n+1})R^G$ be the homogeneous maximal ideal of R^G. Then by comparing the degrees of the generators of $\mathfrak{m}^2$ and the degree of δ, we see that if $\delta \in \mathfrak{m}^2$, then the only possibility is $\delta = f_1^2$ (mod unit multiple), which cannot be the case by Lemma 1.9. Thus $\delta \notin \mathfrak{m}^2$. Now it suffices only to show that f_1, f_2 and δ are linearly independent over k, which is true again by Lemma 1.9.

PROPOSITION 3.5. Let $R^G = k[f_1, f_2, \cdots, f_{n+1}]$ with basic invariants f_i. Assume $2 < \deg f_1 \le \deg f_2 \le \deg f_i$ for $i \ge 3$, and $f_{n+1} = J(f_1, f_2)$. Then $(f_1, f_2, \cdots, f_n) : f_{n+1} = (x, y)$.

Proof. $(f_1, f_2, \cdots, f_n); f_{n+1} \supset (f_1, f_2) : f_{n+1} = (x, y)$ by Lemma 1.7. (cf. Remark 1.8.)

THEOREM 3.6. Write $R^G = k[f_1, f_2, \cdots, f_{n+1}]$ with minimal n. Then $n = 2$. Moreover we can choose f_i so that $f_3 = J(f_1, f_2)$.

Proof. Case I. Assume $\deg f_i > 2$, for all i. Then, by Proposition 3.4, we may further assume $\deg f_1 \le \deg f_2 \le \deg f_i$ for $i \ge 3$, and $f_{n+1} = J(f_1, f_2)$. Put ${}^tF = [f_1\, f_2 \cdots f_{n+1}]$ and let M be a relation matrix of F over R, so that we have the exact sequence

$$0 \longrightarrow R^n \xrightarrow[M]{} R^{n+1} \xrightarrow[F]{} R\,.$$

Let $\rho : G \to GL_k(V)$ be the representation of G to the syzygy space of F. We may assume M has been chosen so that $M^g = \rho(g)M$ for $g \in G$. (See § 2.) By Proposition 3.5 and Remark 1.8 (ii), there are at least two basic relations of degree equal to $\deg f_{n+1} + 1$, which, say, is equal to p. Let M_1 be the matrix consisting of all the rows of M which are relations of degree p (see Definition 1.4), and M_2 the matrix consisting of the other rows of M. Further let V_1 and V_2 be the vector spaces spanned by the rows of M_1 and of M_2 respectively. It is clear that V decomposes: $V = V_1 \oplus V_2$ as G-modules, and correspondingly $\rho = \rho_1 \oplus \rho_2$.

Let us restrict our attension to M_1, V_1 and ρ_1. Because all the rows of M_1 are relations of degree p, all the elements in the last column of M_1 are either linear forms or 0. Thus, in view of Proposition 3.5, we can assume that the last column of M_1 is ${}^{\tau}[x\ y\ 0\ 0 \cdots 0]$. Now consider the effect of $g \in G$ to the last column of M_1. g transforms ${}^{\tau}[x\ y\ 0 \cdots 0]$ to ${}^{\tau}[x^g\ y^g\ 0 \cdots 0]$, hence $\rho_1(g)$ takes the form

$$\rho_1(g) = \left(\begin{array}{c|c} \rho_{11}(g) & * \\ \hline 0 & \rho_{12}(g) \end{array}\right),$$

where $\rho_{11}(g)$ is a 2×2 matrix satisfying

$$\rho_{11}(g)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^g \\ y^g \end{pmatrix},$$

so that $\det \rho_{11}(g) = 1$. Now by the complete reducibility of ρ and by Theorem 2.4 we are forced to conclude that there have been only two basic relations and therefore $n = 2$.

Case II. Assume $\deg f_1 = 2$. Then f_1 is either a product of two independent linear forms or a square of a linear form. Hence we may assume that, by change of variables, f_1 is either xy or x^2. Keeping in mind the fact $G \subset SL(2, k)$, one easily sees that in the first case ($f_1 = xy$), G is generated by

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \qquad \text{with } \omega^s = 1,$$

and in the second case, G is generated by $-E_2$. In either case $R^G = k[xy, x^s, y^s]$ for some s, hence $n = 2$. Rewrite $R^G = k[xy, x^s + y^s, x^s - y^s]$. Then the last generator is the Jacobian of the first two. Q.E.D.

§ 4. Monomial ideals in $k[x, y]$ and their syzygy

The first lemma and the example following it are easy and proof is omitted.

LEMMA 4.1. *Let R be a graded UFD, and let $\mathfrak{a}$ be a homogeneous ideal (minimally) generated by two elements: (f, g). Then we have:*

(i) $\operatorname{hd} R/\mathfrak{a} = 2$. *To be precise, if $f = f_1 d$ and $g = -g_1 d$ with d a greatest common divisor of f and g, then*

$$0 \longrightarrow R \xrightarrow{[g_1\ f_1]} R^2 \xrightarrow{{}^t[f\ g]} R$$

is a minimal free resolution of $R/\mathfrak{a}$.

(ii) *Assume $[b\ a]$ is a relation of ${}^t[f\ g]$ (i.e., $[b\ a]{}^t[f\ g] = 0$). Then a basic relation of ${}^t[f\ g]$ is obtained from $[b\ a]$ by dividing out a greatest common divisor of b and a, i.e., if $d = GCD(b, a)$, $1/d[b\ a]$ is it.*

(iii) *Assume $[b\ a]$ is a relation of ${}^t[f\ g]$, and b and a are homogeneous. Then* $\deg b \geq \deg a \Leftrightarrow \deg f \leq \deg g$.

EXAMPLE 4.2. Let $R = k[x, y]$ and let $f = x^b y^b$, $g = x^{a'} y^{b'}$ with $a > a'$ and $b < b'$. Set $a - a' = r$ and $b' - b = s$. Then a minimal free resolution of $R/\mathfrak{a}$ is

$$0 \longrightarrow R \xrightarrow{[-y^s\ x^r]^t} R^2 \xrightarrow{[f\ g]} R\,.$$

Next we consider a relation matrix of a general monomial ideal in $R = k[x, y]$. Let $\mathfrak{a}$ be the ideal generated by the monomials

$$f_i = x^{a_i} y^{b_i}\,,\ i = 1, 2, \cdots, n + 1,\ n \geq 1\,.$$

If the generators have been chosen minimal, we may assume, with suitable numbering of the generators, that $a_1 > a_2 > \cdots > a_{n+1}$ and $b_1 < b_2 < \cdots < b_{n+1}$. With the positive integers $r_i = a_i - a_{i+1}$ and $s_i = b_{i+1} - b_i$, $i = 1, 2, \cdots, n$, we define the matrix M to be

$$\begin{bmatrix} -y^{s_1} & x^{r_1} & 0 & 0 & \cdots & & \\ 0 & -y^{s_2} & x^{r_2} & 0 & 0 \cdots\cdots & & \\ & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & & \\ & & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \\ & & & \cdots\cdots & 0 & -y^{s_n} & x^{r_n} \end{bmatrix}.$$

Now we have

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PROPOSITION 4.3. *The matrix M above is a relation matrix of* ${}^{\tau}F = [f_1 f_2 \cdots f_{n+1}]$. *That is,*

$$0 \longrightarrow R^n \xrightarrow[M]{} R^{n+1} \xrightarrow[F]{} R$$

is a minimal free resolution of $R/\mathfrak{a}$.

Proof. That $MF = 0$ is obvious. Among the maximal minors of M are a power of x and a power of y. Hence ht $I(M) = 2$. This shows the complex is exact by Buchsbaum-Eisenbud criterion.

In the next remark, we want to show the maximal minors of M explicitly, but we treat M a little more generally for a later purpose.

Remark 4.8. Let

$$M = \begin{bmatrix} B_1 & A_1 & & & & \\ & B_2 & A_2 & & & \\ & & \cdots\cdots\cdots\cdot & & & \\ & & & \cdots\cdots\cdots\cdots & & \\ & & & & B_n & A_n \end{bmatrix}.$$

Then the i-th minor of M is, disregarding the signs,

$$D_i = \prod_{j=1}^{i-1} B_j \prod_{j=i}^{n} A_j .$$

In particular, $D_1 = \prod_{j=1}^{n} A_j$, and $D_{n+1} = \prod_{j=1}^{n} B_j$.

Remark 4.6. Let $\mathfrak{a}$ and M be as in Proposition 4.3. Then the maximal minors of M coincide with the generators of $\mathfrak{a}$ if and only if $\mathfrak{a}$ is perfect, which is equivalent to $a_{n+1} = b_1 = 0$.

The following lemma is used in § 6.

Lemma 4.7. *Let M and F be the matrices given in Proposition* 4.3. *Let M' be matrix of the following type:*

$$\begin{bmatrix} * & x^{r_1} & & & & \\ & * & x^{r_2} & & & \\ & & \cdots\cdots & & & \\ & & & \cdots\cdots & & \\ & & & & * & x^{r_n} \end{bmatrix},$$

where the elements in the $*$-ed positions are known only in the quotient field of R (and might not be in R). Even then $M'F = 0$ implies $M' = M$.

Proof. This is clear because the condition $M'F = 0$ determines the elements in the $*$-ed positions uniquely.

§ 5. Certain monomial ideals in $k[x, y]$ and $k[X, Y, Z]$

We denote by Z_+^2 and Z_+^3 the additive semigroups of non-negative intergers of rank 2 and rank 3.

In this and the next sections, we are primarily concerned with the subsemigroups $M_{p,r} \subset Z_+^2$ and $S_{p,r} \subset Z_+^3$ defined as follows:

Definition 5.1. For any pair of positive integers p, r, we define

$$M_{p,r} = \{[a\ b] \in Z_+^2 \mid a + rb \in (p)\} \quad \text{and}$$
$$S_{p,r} = \{[a\ b\ \ell] \in Z_+^3 \mid a + rb - p\ell = 0\}.$$

Here $(p) = pZ$, the set of multiples of p.

Remark 5.2. (i) Note that the projection $[a\ b\ \ell] \to [a\ b]$ gives an isomorphism from $S_{p,r}$ onto $M_{p,r}$ as semigroups.

(ii) When $r \equiv r'(p)$, $M_{p,r}$ and $M_{p,r'}$ are the same subset of Z_+^2, but $S_{p,r}$ and $S_{p,r'}$ may be different as sets, although they are isomorphic as semigroups, since they are isomorphic to $M_{p,r} = M_{p,r'}$.

Now assume, temporarily, k is an algebraically closed field of characteristic 0, and let $\omega \in k$ be a primitive p-th root of 1. Let us consider the automorphism g of the polynomial ring $R = k[x, y]$ that takes x to ωx and y to $\omega^r y$. Then $x^a y^b$ is left unchanged by g if and only if $[a\ b] \in M_{p,r}$. Hence if G is the cyclic group of automorphisms of R generated by g, then the ring of invariants R^G is the semigroup ring $k[M_{p,r}]$. If $\mathfrak{a} = (R_+^G)R$, then by Theorem 2.1 and Remark 2.3 the minimal ideal basis for $\mathfrak{a}$ consisting of monomials is a minimal set of generators of R^G as a k-algebra, which is precisely the minimal basis of the semigroup $M_{p,r}$. (Note that a basis for $M_{p,r}$ is unique.) The situation for $S_{p,r}$ is the same as for $M_{p,r}$ if G is replaced by a certain action of a 1-dimensional torus. In fact consider the action of $T = GL(1, k)$ on the polynomial ring $A = k[X, Y, Z]$ defined by

$$X \longrightarrow tX, \quad Y \longrightarrow t^r Y, \quad Z \longrightarrow t^{-p} Z, \qquad \text{for } t \in T.$$

Then the ring of invariants A^T is the semigroup ring $k[S_{p,r}]$, and the minimal generating set of the semigroup is a minimal set of generators of the ideal $(A_+^T)A$.

Let us fix our notations for the rings and the ideals corresponding to $M_{p,r}$ and $S_{p,r}$ as follows:

Notation 5.3.

$$\begin{array}{ll} R = k[x, y] = k[Z_+^2] & A = k[X, Y, Z] = k[Z_+^3] \\ R_{p,r} = k[M_{p,r}] & A_{p,r} = k[S_{p,r}] \\ \mathfrak{a}_{p,r} = (M_{p,r})R & I_{p,r} = (S_{p,r})A \end{array}$$

To be precise, $R_{p,r}$ (resp. $A_{p,r}$) is the subring of $k[x, y]$ (resp. $k[X, Y, Z]$) generated by those monomials whose exponents are in $M_{p,r}$ (resp. $S_{p,r}$), and $\mathfrak{a}_{p,r}$ (resp. $I_{p,r}$) is the ideal of R (resp. A) generated by the monomials $\neq 1$ in $R_{p,r}$ (resp. $A_{p,r}$).

Remark 5.4. In the notations above, k is an arbitrary field. Since the semigroups are defined certainly independent of the field, the other things are as well defined, although, for example, $R_{p,r}$ may not appear as the ring of invariants for some k. Even then it is true that, as long as the generators are concerned, the ideal, the algebra and the semigroup in the correspondence are regarded as the same. This is easy to see once it is known for k such that $k = \bar{k}$ and $\mathrm{ch}\, k = 0$.

Remark 5.5. We have the following commutative diagram of rings:

$$\begin{array}{ccc} A & \xrightarrow{\psi} & R \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ A_{p,r} & \longrightarrow & R_{p,r} \end{array},$$

where i are the inclusions and ψ is the projection $X \to x$, $Y \to y$, $Z \to 1$. ψ induces the isomorphism $A_{p,r} \xrightarrow{\sim} R_{p,r}$ which corresponds to the isomorphism of the semigroups $S_{p,r} \xrightarrow{\sim} M_{p,r}$. Note also that $\mathrm{Ker}\, \psi = (Z - 1)A$ and $A/I_{p,r} \otimes_A A/(Z - 1) \xrightarrow{\sim} R/\mathfrak{a}_{p,r}$.

Remark 5.6. (i) In the sequel whenever we say that

$$[a_i\ b_i], \qquad i = 1, 2, \cdots, n + 1$$

is the minimal basis of the semigroup $M_{p,r}$, it will be tacitly assumed that they are arranged in the order that $a_1 \geq a_2 \geq \cdots \geq a_{n+1}$, so that the fact is $a_1 > a_2 > \cdots > a_{n+1}$ and $b_1 < b_2 < \cdots < b_{n+1}$. (cf. the paragraph preceding Proposition 4.3.) The same will be applied to $S_{p,r}$, so if we say

$$[a_i\, b_i\, \ell_i], \qquad i = 1, 2, \cdots, n+1$$

is the minimal basis of $S_{p,r}$, then $a_1 > a_2 > \cdots, > a_{n+1}$, and $b_1 < b_2 < \cdots < b_{n+1}$. As to the sequence ℓ_i see Lemma 5.8 below.

(ii) Note, with the convention made above, that the first term of the minimal basis of $M_{p,r}$ is $[p\ 0]$ and that of $S_{p,r}$ is $[p\ 0\ 1]$.

Remark 5.7. When $p = r$, the minimal basis of $M_{p,p}$ is $[p\ 0]$ and $[0\ 1]$ with $n = 1$. This corresponds to the ring of invariants of $P = k[x, y]$ under the action of the automorphism

$$\begin{pmatrix} \omega & \\ & 1 \end{pmatrix},$$

where ω is a primitive p-th root of 1. In fact $R^G = k[x^p, y]$. Although one might expect the notation '$M_{p,0}$' in this case, we do not let $r = 0$.

LEMMA 5.8. *Suppose* $[a_i\ b_i\ \ell_i]$, $i = 1, 2, \cdots, n+1$ *is the minimal basis of* $S_{p,r}$. *Then we have that* $\ell_1 \leq \ell_2 \leq \ell_3 \leq \cdots \leq \ell_{n+1}$.

Proof. By the definition of $S_{p,r}$

$$a_i = p\ell_i - rb_i \quad \text{and} \quad a_{i+1} = p\ell_{i+1} - rb_{i+1}\,.$$

Hence $a_i - a_{i+1} = p(\ell_i - \ell_{i+1}) + r(b_{i+1} - b_i)$. Note we have that $a_i - a_{i+1} > 0$ and $b_{i+1} - b_i > 0$. Then, if $(\ell_i - \ell_{i+1})$ were positive, we would have $a_i - a_{i+1} > p$, a contradiction to $a_1 = p \geq a_i$.

PROPOSITION 5.9. *Put* $\mathfrak{a} = \mathfrak{a}_{p,r}$ *and* $I = I_{p,r}$. *Then,*

(i) $\mathrm{hd}_R\, R/\mathfrak{a} = 2$, *and* $\mathfrak{a}$ *is perfect.*

(ii) $\mathrm{hd}_A\, A/I = 2$, *I is imperfect and Z is the greatest common divisor of (the generators of) I.*

(iii) *If L is a minimal free resolution of A/I over A, $L \otimes_A A/(Z-1)$ is a minimal free resolution of $R/\mathfrak{a}$ over R, with the identification $A/(Z-1) \xrightarrow{\sim} R$.*

Proof. (i) Since $\mathfrak{a}$ contains a power of x and a power of y, $\mathrm{ht}\,\mathfrak{a} = 2 = \mathrm{hd}\, R/\mathfrak{a}$.

(ii) Set $r_i = a_i - a_{i+1}$, $s_i = b_{i+1} - b_i$ and $k_i = \ell_{i+1} - \ell_i$, $i = 1, 2, \cdots, n$. (Note $k_i \geq 0$ by the preceding lemma.) Define the matrices M_1 and F_1 as follows:

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$$M_1 = \begin{bmatrix} -Y^{s_1}Z^{k_1} & X^{r_1} & 0 & \cdots & & & \\ 0 & -Y^{s_2}Z^{k_2} & X^{r_2} & 0 & \cdots & & \\ & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & & \\ & & & \cdots\cdots & \cdots\cdots & \cdots\cdots & \\ & & & \cdots & 0 & -Y^{s_n}Z^{k_n} & X^{r_n} \end{bmatrix},$$

$${}^{\tau}F_1 = [X^{a_1}Y^{b_1}Z^{\ell_1} \quad X^{a_2}Y^{b_2}Z^{\ell_2} \quad \cdots\cdots \quad X^{a_{n+1}}Y^{b_{n+1}}Z^{\ell_{n+1}}]\,.$$

Then we claim that

$$L : 0 \longrightarrow A^n \xrightarrow[M_1]{} A^{n+1} \xrightarrow[F_1]{} A$$

is a minimal free resolution of A/I. In fact it is obvious that $M_1F_1 = 0$. To prove the complex is exact, it suffices to show $\operatorname{ht} I(M_1) \geq 2$ by Buchsbaum-Eisenbud criterion. This indeed is true for $I(M_1)$ contains a power of X and a monomial in Y and Z. (Consider the first and the last minors of M_1.) For the second assertion, note that the first of the minimal generators of I is X^pZ. On the other hand, the first maximal minor of M_1 is X^α, where $\alpha = \sum s_i$, and α is equal to p. This shows $I(M_1)Z = I$.

(iii) We can assume L is the complex given in the proof of (ii). In L substitute Z by 1, and one obtains the minimal free resolution of $R/\mathfrak{a}$ given in § 4. (One may also use the argument of Lemma 8.3.)

§ 6. The finite sequence $\Phi(p, r)$

DEFINITION 6.1. Let p, r be positive integers, and let $[a_i\ b_i\ \ell_i]$, $i = 1, 2, \cdots, n+1$ be the minimal generators of $S_{p,r}$. With the assumption of Remark 5.6, let $a_i - a_{i+1} = r_i, i = 1, 2, \cdots, n$. Then we denote the sequence r_j symbolically by $\Phi(p, r) = r_1 \oplus r_2 \oplus \cdots \oplus r_n$. If it happens that $r_2 \oplus r_3 \oplus \cdots \oplus r_n = \Phi(p', r')$ for some p' and r', we shall write $\Phi(p, r) = r_1 \oplus \Phi(p', r')$. Note $\Phi(p, r)$ depends only on the residue class of r modulo p.

The following proposition, of which the proof is carried out elementarily, enables us to compute $\Phi(p, r)$ actually.

PROPOSITION 6.2. *Assume $p > r > 0$. Let $p' = p - r$ and let r' be the integer such that $r \equiv r'$ (p') and $p' \geq r' > 0$. Suppose $[a_i\ b_i\ \ell_i]$, $i = 1, 2, \cdots, n+1$ are the minimal generators of $S_{p,r}$ indexed as in Remark* 5.6. *Then*:

(i) *$[a_i\ b_i]$, $i = 1, 2, \cdots, n+1$ are the minimal generators of $M_{p,r}$.*

(ii) *$[a_i\ b_i - \ell_i]$, $i = 2, 3, \cdots, n+1$ are the minimal generators of $M_{p',r'}$.*

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Proof. (i) See Remark 5.2 (i).

(ii) Set $S = S_{p,r}$, $M = M_{p,r}$ and $M' = M_{p',r'}$. Let S' be the semigroup in Z_+^3 generated by $[a_i\ b_i\ \ell_i]$, $i = 2, 3, \cdots, n+1$. Consider the linear map $f: Z^3 \to Z^2$ that sends $[a\ b\ \ell]$ to $[a\ b-\ell]$. We are going to show that f induces an isomorphism of S' onto M', which proves the assertion, for certainly generators are mapped to generators by an isomorphism of semigroups.

Step I. First let us note that $b_i - \ell_i \geq 0$ for $i \neq 1$. In fact if $\ell_i - b_i > 0$, then it follows that $a_i = p\ell_i - rb_i = p(\ell_i - b_i) + (p-r)b_i > p$ because of $b_i \geq 0$. This contradicts the fact $p = a_1 > a_i$ for $i \neq 1$.

Step II. We show that $[a_i\ b_i - \ell_i] \in M'$ provided that $i \neq 1$, i.e., $f(S') \subset M'$. By definition $a_i + rb_i = p\ell_i = (r + p')\ell_i$. This, together with the fact that $r \equiv r'(p)$, implies $a_i + r'(b_i - \ell_i) \equiv 0\ (p')$, i.e., $[a_i\ b_i - \ell_i] \in M'$.

Step III. We prove that, for any $[a\ b'] \in M'$, there is $[a\ b\ \ell] \in S = S_{p,r}$ such that $b - \ell = b'$. (Note, presently, we do not claim $[a\ b\ \ell]$ is in S'.) For this consider the system of linear equations

$$(*) \qquad \begin{aligned} a + rb &= p\ell & (1)\\ b - \ell &= b', & (2) \end{aligned}$$

where b and ℓ are regarded as unknowns. We want to find a solution $[b\ \ell]$ in Z_+^2 (then $[a\ b\ \ell]$ is the required element in S).

Since $p = r + p'$, (1) is equivalent to

$$a + r(b - \ell) = p'\ell\ . \qquad (1)'$$

Since $r \equiv r'\ (p')$, we have the number $B \in Z$ that satisfies $r = r' + Bp'$. It can easily be proved that $r \geq r'$, i.e., $B \geq 0$. (See the proof of Proposition 6.5 below.) With the integer B and with (2), (1)$'$ is equivalently transformed to

$$a + r'b' = p'\ell - Bp'b'\ . \qquad (1)''$$

That $[a\ b'] \in M'$ implies the existence of ℓ' such that

$$a + r'b' = p'\ell'\ .$$

Therefore (1)$''$ is equivalent to

$$\ell' = \ell - Bb', \quad \text{i.e.,} \quad \ell = \ell' + Bb'\ . \qquad (3)$$

For b, we have $\qquad b = b' + \ell = (B+1)b' + \ell'\ . \qquad (4)$

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(3) and (4) is the required solution of (*); in the matrix notation

$$\begin{pmatrix} b \\ \ell \end{pmatrix} = \begin{pmatrix} B+1 & 1 \\ B & 1 \end{pmatrix} \begin{pmatrix} b' \\ \ell' \end{pmatrix},$$

where, we repeat, B is the integer satisfying $r = r' + Bp'$.

Step IV. We prove $f|_{S'} : S' \to M'$ is surjective. Let $[a'\ b']$ be a member of the minimal basis of M'. Then $p' \geq a'$. In Step III we showed there was $[a\ b\ \ell] \in S$ such that $a = a'$ and $b - \ell = b'$. Since S is generated by S' and $[a_1\ b_1\ \ell_1] = [p\ 0\ 1]$, and since $p > p'$, $[a\ b\ \ell]$ above has to be contained in S'. Because $f|_{S'}$ is a homomorphism of semigroups, this proves it is surjective.

That $f|_{S'}$ is injective is in fact trivial; an argument, for example, is to consider the ring homomorphism $k[S'] \to k[M']$ which f induces. The rings are both 2-dimensional domains, and it can have no kernel. Q.E.D.

Remark 6.3. In the course of proof we actually proved that the linear map

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & B+1 & 1 \\ 0 & B & 1 \end{pmatrix} : Z^3 \to Z^3$$

induces an isomorphism between S' and $S_{p',r'}$.

PROPOSITION 6.4. *Let $p > r > 0$ be positive integers. Then we have*

$$\Phi(p, r) = r \oplus \Phi(p', r'), \quad \textit{where}$$
$$p' = p - r, \quad \textit{and} \quad r' \equiv r(p).$$

Proof. Immediate by Proposition 6.2 and the definition of $\Phi(p, r)$.

Since $\Phi(p, r)$ depends only on the residue class of r mod p, we may always be assuming $p \geq r > 0$. By Remark 5.7, $\Phi(p, p) = p$. Thus the sequence $\Phi(p, r)$ ends when it has reduced to $\Phi(r_n, r_n) = r_n$. Here are some properties of $\Phi(p, r)$ that follow immediately from Proposition 6.4.

PROPOSITION 6.5. *Let $\Phi(p, r) = r_1 \oplus r_2 \oplus \cdots \oplus r_n$. Then*: (i) $r_1 \geq r_2 \geq \cdots \geq r_n$. (ii) *If c is a positive integer, $\Phi(cp, cr) = cr_1 \oplus cr_2 \oplus \cdots \oplus cr_n$.* (iii) $r_n = (p, r)$, *the greatest common divisor of p and r.*

Proof. (i) Let p' and r' be as in Proposition (6.4). By induction it suffices to show $r \geq r'$. Assume $\frac{1}{2}p \geq r$. Then $p' = p - r \geq r$, hence $r = r'$. Assume $\frac{1}{2}p < r$. Then $p' = p - r < r$, so $r' \leq p' < r$. (ii) Easy by induction. (iii) It is easy to see $(p, r) = (p', r')$, hence the assertion is immediate by induction.

The next theorem shows not only the generators but the syzygies of $I_{p,r}$ are related to those of $\mathfrak{a}_{p',r'}$.

THEOREM 6.6. *As in Proposition 6.2, assume $p > r > 0$, and $p' = p - r$ and $r' \equiv r(p')$. Let M_1 be the matrix defined in the proof of Proposition 5.9:*

$$M_1 = \begin{pmatrix} -Y^{s_1}Z^{k_1} & X^{r_1} & & \\ & \cdots\cdots\cdots\cdots & & \\ & & \cdots\cdots\cdots\cdots & \\ & & -Y^{s_n}Z^{k_n} & X^{r_n} \end{pmatrix},$$

so that

$$0 \longrightarrow A^n \xrightarrow[M_1]{} A^{n+1} \xrightarrow[F_1]{} A$$

is a minimal free resolution of $A/I_{p,r}$. Define the matrix M' by

$$M' = \begin{bmatrix} -y^{s_2-k_2} & x^{r_2} & 0 & \cdots\cdots & & \\ 0 & -y^{s_3-r_3} & x^{r_3} & 0 & \cdots\cdots & \\ & & \cdots\cdots\cdots\cdots\cdots\cdots & & & \\ & & & \cdots\cdots\cdots\cdots\cdots\cdots & & \\ & & & 0 & -y^{s_n-k_n} & x^{r_n} \end{bmatrix}.$$

Then M' is a relation matrix of $\mathfrak{a}_{p,r}$ over $R = k[x, y]$.

Proof. Let M_1' be the matrix obtained from M_1 by omitting the first row and the first column:

$$M_1 = \left(\begin{array}{c|c} * & **** \\ \hline * & \\ * & \\ * & M_1' \\ * & \\ * & \end{array}\right).$$

Let $e_i = X^{a_i}Y^{b_i}Z^{\ell_i}$, $i = 1, 2, \cdots, n + 1$ be the minimal generators of $I_{p,r}$, and define the matrix F_1' by $F_1' = [e_2\, e_3 \cdots e_{n+1}]$. Then we have the exact sequence

$$L_1' : 0 \longrightarrow A^{n-1} \xrightarrow[M_1']{} A^n \xrightarrow[F_1']{} A.$$

(To prove this is exact, use Buchsbaum-Eisenbud criterion.) In the complex L_1' substitute X by x, Y by y, and Z by y^{-1}, and one obtains a complex of free modules over $\bar{R} = k[x, y, y^{-1}]$:

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$$L' : 0 \longrightarrow \bar{R}^{n-1} \xrightarrow[M']{} \bar{R}^{n} \xrightarrow[F']{} \bar{R},$$

where M' is the matrix in the statement of our present proposition. The complex L' is exact, for it is nothing else than $L'_1 \otimes_A A/(YZ - 1)$, and $YZ - 1$, being inhomogeneous, cannot be a zerodivisor on the cokernel of F'_1. Proposition 6.2 says precisely that the entries of F' are the minimal generators of $\mathfrak{a}_{p',r'}$, and we are in the situation where Lemma 4.7 applies, so that the fact is that the complex L' is defined over $R = k[x, y]$ and the maps restrict to the submodules $R^{\nu} \subset \bar{R}^{\nu}$ to give a minimal free resolution of $R/\mathfrak{a}_{p',r'}$ over $R : 0 \longrightarrow R^{n-1} \xrightarrow[M']{} R^{n} \xrightarrow[F']{} R.$ Q.E.D.

Assume k is an infinite field, and let $t \in T = GL(1, k)$ act on $A = k[X, Y, Z]$ by $X \to tX$, $Y \to t^r Y$, $Z \to t^{-p} Z$, where p and r are positive integers. Then, as we saw in the last section, the ring of invariants A^T is $A_{p,r} = k[S_{p,r}]$, and the ideal $(A^T_+)A = I_{p,r}$ is homologically 1-dimensional (Proposition 5.8). Let us denote by $P_{p,r}$ the representation of R to the syzygy space of $I_{p,r}$, which was defined in § 2. Then we have:

PROPOSITION 6.7. *According to Proposition* 6.4 *write* $\Phi(p, r) = r_1 \oplus \Phi(p', r')$. *(*$r_1$ *is such that* $r_1 \equiv r(p), 0 < r_1 \leq p$.*) Then as a representation of* $T = GL(1, k)$, *we have*:

$$P_{p,r}(t) = t^{r_1} \oplus P_{p',r'}(t), \qquad t \in T.$$

Proof. As we have shown in the proof of Proposition 5.9, a relation matrix of $I_{p,r}$ has the form

$$\begin{bmatrix} * & X^{r_1} & & & \\ & * & X^{r_2} & & \\ & & \cdots\cdots & & \\ & & & \cdots\cdots & \\ & & & * & X^{r_n} \end{bmatrix},$$

where $\Phi(p, r) = r_1 \oplus r_2 \oplus \cdots \oplus r_n$. It follows at once that $P_{p,r}$ is given by

$$t \longrightarrow \begin{bmatrix} t^{r_1} & & & \\ & t^{r_2} & & \\ & & \ddots & \\ & & & t^{r_n} \end{bmatrix}.$$

(Consider how $t \in T$ multiplies X^{r_i}'s and disregard about it for the monomials in the $*$-ed positions, as they should be the same.)

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§ 7. The syzygy of $(R_+^G)R$ for finite abelian groups in $GL(2, k)$

Throughout this section, $k = \bar{k}$, $\operatorname{ch} k = 0$.

Let $G \subset GL(2, k)$ be a finite abelian group put in the diagonal form. Assume G does not contain any reflexions. (Generally, an invertible matrix of finite order is called a pseudo-reflexion if all but one of the eigenvalues are equal to 1. In this paper we say reflexion for pseudo-reflexion.) Then it is easy to see that G is cyclic. (In fact, consider the projections $\begin{pmatrix} \omega_1 & \\ & \omega_2 \end{pmatrix} \in G \to \omega_i \in k^*$. Any element in the kernel of either of them would be a reflexion, hence G is mapped injectively to k^*. And a finite subgroup of k^* is cyclic.) Let $g = \begin{pmatrix} \omega_1 & \\ & \omega_2 \end{pmatrix}$ be a generator of G. Then, because G contains no reflexions, if $o(G) = p$, both ω_1 and ω_2 are primitive p-th roots of 1. Thus there is r such that $\omega_1 = \omega_2^r$. Note that the residue class of $r \bmod p$ is uniquely determined by G, and also that r is relatively prime to p. Rewrite $\omega = \omega_1$, $g = \begin{pmatrix} \omega & \\ & \omega^r \end{pmatrix}$. Now let G act on $R = k[x, y]$ by $x^g = \omega x$, $y^g = \omega^r y$. Then the ring of invariants R^G is, using the notation of § 5, $k[M_{p,r}]$. Let $\mathfrak{a} = (R_+^G)R$ and let M be the relation matrix of $\mathfrak{a}$ given in § 4:

$$M = \begin{bmatrix} -y^{s_1} & x^{r_1} & & & \\ & -y^{s_2} & x^{r_2} & & \\ & & \cdots\cdots & & \\ & & & \cdots\cdots & \\ & & & -y^{s_n} & x^{r_n} \end{bmatrix}.$$

Then by definition of $\Phi(p, r)$ and by Proposition (4.9) (iii), we have that $\Phi(p, r) = r_1 \oplus r_2 \oplus \cdots \oplus r_n$. Let s be positive integer such that $rs \equiv 1(p)$. Then G may as well be generated by $g^s = \begin{pmatrix} \omega^s & \\ & \omega \end{pmatrix}$, and by interchanging the roles of x and y it immediately follows that $\Phi(p, s) = s_n \oplus s_{n-1} \oplus \cdots \oplus s_1$. (Note the reversed order of indices. Also note it is implied that $\Phi(p, r)$ and $\Phi(p, s)$ have the same length.) Since $\mathfrak{a}$ is perfect, the generators of $\mathfrak{a}$ coincide with the maximal minors of M;

$$D_\nu = \prod_{i=\nu}^{n} x^{r_i} \prod_{i=1}^{\nu-1} y^{s_i}, \qquad \nu = 1, 2, \cdots, n+1.$$

These are also an algebra basis of R^G.

Let $\rho : G \to GL_k(V)$ be the representation of G to the syzygy space of $\mathfrak{a}$. We can think V is the space spanned by the rows of M and ρ is such that $\rho(g)M = M^g$. Then we at once have:

$$\rho(g) = \begin{bmatrix} \omega^{r_1} & & & \\ & \omega^{r_2} & & \\ & & \ddots & \\ & & & \omega^{r_n} \end{bmatrix}.$$

In view of Proposition 6.4 it might be in some number theoretic sense interesting to note:

$$\rho(g^s) = \begin{bmatrix} \omega^{s_n} & & & \\ & \omega^{s_{n-1}} & & \\ & & \ddots & \\ & & & \omega^{s_1} \end{bmatrix}.$$

Note $\operatorname{Ker}\rho$ is trivial since $r_n = 1$.

Next we consider generally a finite abelian group $G \subset GL(2, k)$. We may assume G has been diagonalized. Let H be the group generated by all the reflexions in G. Then it is easy to see that $R^H = k[x^\alpha, y^\beta]$, for some α and β. The induced action of G on R^H, regarding x^α and y^β as new variables, contains no longer reflexions, and we can apply the preceding consideration to G/H and $R^H = k[x^\alpha, y^\beta]$. Certainly R^G is the ring of invariants of R^H under the action of G/H. Thus, with certain $\Phi(p, r) = \oplus r_i$ and $\Phi(p, s) = \oplus s_{n-i}$ such that $rs \equiv 1\ (p)$,

$$M = \begin{bmatrix} -y^{\beta s_1} & x^{\alpha r_1} & & & \\ & -y^{\beta s_2} & x^{\alpha r_2} & & \\ & & \cdots\cdots\cdots & & \\ & & & \cdots\cdots\cdots & \\ & & & -y^{\beta s_n} & x^{\alpha r_n} \end{bmatrix}$$

is a relation matrix of $(R_+^G)R^H$ over R^H. Since the inclusion $R^H \to R$ is faithfully flat, the matrix M above serves as a relation matrix of $(R_+^G)R$ over R. Let ρ be the representation of G to the syzygy space of $(R_+^G)R$. Then one sees easily that $\operatorname{Ker}\rho = H$, and $\rho(G)$ can be thought of as the representation of G/H to the syzygy space of $(R_+^G)R^H$ over R^H. (cf. Remark (2.6).)

§ 8. Homological dimension of certain monomial ideals in $k[X, Y, Z]$

Let $R = k[X, Y]$ and $A = k[X, Y, Z]$. Define the morphism $\phi : R \to A$ by $X \to XZ$ and $Y \to YZ$. Denote by $(\ )_{Z=1}$: $A = R[Z] \to R$ the morphism of R-algebras that sends Z to 1. Clearly $(\ )_{Z=1}$ is a ring retract of ϕ.

If $M = [a_{ij}]$ is a matrix over A, we will write $M_{Z=1}$ for the matrix $[(a_{ij})_{Z=1}]$, which is a matrix defined over R. Conversely, if $M = [a_{ij}]$ is a matrix with $a_{ij} \in R$, we write $\phi(M)$ for $[(a_{ij})]$. Note that for a homogeneous polynomial $f \in R$, we have that $\phi(f) = fZ^{\deg(f)}$. If $\mathfrak{a} \subset R$ is an ideal, the ideal $\phi(\mathfrak{a})A$ will be denoted simply by $\phi(\mathfrak{a})$.

LEMMA 8.1. (i) *For a homogeneous ideal* $\mathfrak{a} \subset R$, $\mu_R(\mathfrak{a}) = \mu_A(\phi(\mathfrak{a}))$. (ii) *If* $f \in A$ *is homogeneous, then* $(f)_{Z=1} = 0$ *implies* $f = 0$.

Proof. (i) Since ϕ has a ring retract, it holds that $\phi(\mathfrak{a}) \cap R = \mathfrak{a}$ for any ideal $\mathfrak{a} \subset R$, from which the assertion follows easily. (ii) Clear.

As in the preceding sections we are concerned with monomial ideals in R, but this time we start with a matrix of the following form:

$$M = \begin{bmatrix} Y^{s_1} & X^{r_1} & & & & \\ & Y^{s_2} & X^{r_2} & & & \\ & & \cdots\cdots & & & \\ & & & \cdots\cdots & & \\ & & & & \cdots\cdots & \\ & & & & Y^{s_n} & X^{r_n} \end{bmatrix},$$

where r_i and s_i are positive integers. The purpose of this section is to consider for what monomial ideal $\mathfrak{a} \subset R$ it holds that $\mathrm{hd}_A A/\phi(\mathfrak{a}) = 2$. For this we need consider the matrix M^* derived from M as follows:

$$M^* = U^{-1}\phi(M),$$

where U is the $n \times n$ diagonal matrix

$$\begin{bmatrix} Z^{m_1} & & & \\ & Z^{m_2} & & \\ & & \ddots & \\ & & & Z^{m_n} \end{bmatrix},$$

with $m_i = \mathrm{Min}\{r_i, s_i\}$. Note that the i-th row of M is

$$[\cdots \quad Y^{s_i} \quad X^{r_i} \quad \cdots], \text{ and hence that of } \phi(M) \text{ is}$$
$$[\cdots \quad Y^{s_i}Z^{s_i} \quad X^{r_i}Z^{r_i} \quad \cdots].$$

Therefore the i-th row of M is, letting $\ell_i = |r_i - s_i|$,

$$[\cdots \quad Y^{s_i} \quad X^{r_i}Z^{\ell_i} \quad \cdots] \text{ if } s_i \leq r_i, \text{ and}$$
$$[\cdots \quad Y^{s_i}Z^{\ell_i} \quad X^{r_i} \quad \cdots] \text{ if } s_i \geq r_i.$$

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Throughout this section the above notations ϕ, R, A, M, M^* etc. are kept fixed.

PROPOSITION 8.2. *Let $\mathfrak{a} \subset R$ be a homogeneous ideal and suppose, with an F, $0 \longrightarrow R^n \underset{M}{\longrightarrow} R^{n+1} \underset{F}{\longrightarrow} R$ is a minimal free resolution of $R/\mathfrak{a}$. Then if* $\mathrm{hd}_A A/\phi(\mathfrak{a}) = 2$, $0 \longrightarrow A^n \underset{M^*}{\longrightarrow} A^{u+1} \underset{\phi(F)}{\longrightarrow} A$ *is a minimal free resolution of $A/\phi(\mathfrak{a})$ over A.*

For proof we need the following

LEMMA 8.3. *Let $\mathfrak{a} \subset R$ be a homogeneous ideal, and let L be a minimal free resolution of $A/\phi(\mathfrak{a})$ over A. If* $\mathrm{hd}_A A/\phi(\mathfrak{a}) = 2$, *then $L \otimes_A A/(Z-1)$ is, identifying $A/(Z-1)$ with R, a minimal free resolution of $R/\mathfrak{a}$ over R.*

Proof. Note $\Phi(\mathfrak{a})$ is homogeneous and $Z-1$ is not a zero divisor on $A/\phi(\mathfrak{a})$, from which it follows that $L \otimes_A A/(Z-1)$ is exact. To say it is minimal is that the ranks of the free modules in the complex are the betti numbers of $R/\mathfrak{a}$. But this is clear for the first betti number is $\mu_R(\mathfrak{a})$ $\mu_A(\phi(\mathfrak{a}))$ and the second betti number is one less than that.

Proof of Proposition 8.2. Let M' be the kernel of $\phi(F)$ so that the complex $0 \longrightarrow A^n \underset{M'}{\longrightarrow} A^{n+1} \underset{\phi(F)}{\longrightarrow} A$ is a minimal free resolution of $A/\phi(\mathfrak{a})$. Certainly the entries of M' can be assumed homogeneous. By the lemma above we may assume $M'_{Z=1} = M$. Then, in view of Lemma 8.1 (ii), we see that M' and M have 0's in the same positions. This is to say that, if ${}^{\tau}F = [f_1 f_2 \cdots f_{n+1}]$, then the i-th row of M' is essentially a basic relation of $\phi(f_i)$ and $\phi(f_{i-1})$. On the other hand, the i-th row of $\phi(M)$ is a (may-not-be-basic) relation of $\phi(f_i)$ and $\phi(f_{i+1})$. (In fact $\phi(M)\phi(F) = \phi(MF) = 0$.) Thus it turns out that M' is the matrix obtained from $\phi(M)$ by dividing out greatest common divisors from all the rows, which is precisely M^*. (cf. Lemma 4.1 (ii).)

LEMMA 8.4. *The following conditions are equivalent.*

(i) $\mathrm{ht}\, I(M) \geq 2$.

(ii) *If $s_j > r_j$ for some j, then $s_i \geq r_i$ for all $i > j$.*

Proof. Recall that M^* is the matrix

$$\begin{bmatrix} B_1 & A_1 & & & \\ & B_2 & A_2 & & \\ & & \cdots\cdots & & \\ & & & \cdots\cdots & \\ & & & B_n & A_n \end{bmatrix},$$

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where $\qquad B_i = Y^{s_i}Z^{\ell_i}$ and $A_i = X^{r_i}$, if $s_i \geq r_i$,

and $\quad B_i = Y^{s_i}$ and $A_i = X^{r_i}Z^{\ell_i}$, if $s_i \leq r_i$.

We recall also $D_\nu = \prod_{i=1}^{\nu-1} B_i \prod_{i=\nu}^{n} A_i$, $\nu = 1, 2, \cdots, n+1$ are the maximal minors of M^*, and $I(M^*)$ is the ideal they generate.

Let us prove (ii) implies (i) first. Assume $s_i \leq r_i$ for all i. Then D_{n+1} is a power of Y and D_1 is a monomial in X and Y. This proves ht $I(M^*)$ ≤ 2. In the case $s_i \geq r_i$ for all i, the symmetry in X and Y shows ht $I(M^*) \geq 2$ as well. Assume we have $s_i \leq r_i$ for $i = 1, 2, \cdots, j$, and $s_i \geq r_i$ for $i = j+1, j+2, \cdots, n$ for some $j \neq 1, n$. In this case D_j is a monimial in X and Y, and does not contain Z as a divisor. (In any case) D_1 is a monomial in X and Z, and D in Y and Z. Thus we have ht $I(M^*)$ ≥ 2. We have proved (ii) $\Rightarrow$ (i) completely.

The negation of the condition (ii) is: There are indices $j < k$ such that $s_j > r_j$ and $s_k < r_k$. When this is the case, it is true that both B_j and A_k have Z as a divisor. Because $j < k$, D_ν has either B_j or A_k as a factor for ν whatever. This shows $I(M^*) \subset (Z)$ and ht $I(M^*) = 1$. We have proved (i) $\Rightarrow$ (ii).

THEOREM 8.5. *Let $\mathfrak{a}$ be an ideal in $R = k[X, Y]$ generated by the monomials $f_i = X^{a_i}Y^{b_i}$, $i = 1, 2, \cdots, n+1$, where we assume without loss of generality $a_1 > a_2 > \cdots > a_{n+1}$ and $b_1 < b_2 < \cdots < b_{n+1}$. Then the following conditions are equivalent.*

(i) $\mathrm{hd}_A A/\phi(\mathfrak{a}) = 2$.

(ii) *If* $\deg f_j < \deg f_{j+1}$ *for some* j, *then* $\deg f_i \leq \deg f_{i+1}$ *for all* $i > j$.

Proof. Set $r_i = a_i - a_{i+1}$ and $s_i = b_{i+1} - b_1$. Let ${}^{\tau}F = [f_1 \ -f_2\ f_3 \cdots (-1)^n f_{n+1}]$. Then

$$0 \longrightarrow R^n \xrightarrow[M]{} R^{n+1} \xrightarrow[F]{} R$$

is a minimal free resolution of $R/\mathfrak{a}$. (M is the matrix fixed in the beginning.) Consider the complex

$$(*) \qquad 0 \longrightarrow A^n \xrightarrow[M^*]{} A^{n+1} \xrightarrow[\phi(F)]{} A\,.$$

If $\mathrm{hd}_A A/\mathfrak{a} = 2$, then, by Proposition 8.2, the complex (*) is exact. Hence by Buchsbaum-Eisenbud criterion we have ht $I(M^*) \geq 2$. Conversely, too, ht $I(M^*) \geq 2$ implies (*) is exact by Buchsbaum-Eisenbud criterion. As we have seen in Lemma 4.1 (iii) $s_i \geq r_i \Leftrightarrow \deg f_1 \leq \deg f_{i+1}$. Now the proof is complete by Proposition 8.4.

Remark 8.6. Borrowing a term from elementary calculus, the condition (ii) of the theorem may be described by saying that there are no 'maxima' in the graph of the map $i \to \deg f_i$. The two ends of the graph are not counted as maxima whatever values they take. In this terminology it can be conceived that the theorem is generalized to: For any monomial ideal $\mathfrak{a}$ of R, if $0 \to A^{\tau} \to A^{\nu} \to A^{\mu} \to A$ is a minimal free resolution of $A/\phi(\mathfrak{a})$, τ is the number of maxima in the graph of $i \to \operatorname{def} f_i$. The theorem is to be the special case when $\tau = 0$.

COROLLARY 8.7. *In the same notations of Theorem* 8.5 *and its proof, a sufficient condition for* $\operatorname{hd}_A A/\phi(\mathfrak{a}) = 2$ *is that* $s_1 \leq s_2 \leq \cdots \leq s_n$ *and* $r_1 \geq r_2 \geq \cdots \geq r_n$.

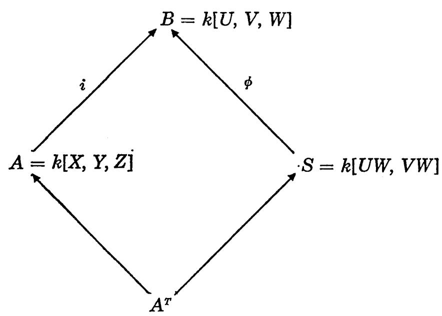
Proof. This is clear from Lemma 8.4, for, as was said in the proof of Theorem 8.5, $\operatorname{hd}_A A/\phi(\mathfrak{a}) = 2$ if and only if $\operatorname{ht} I(M^*) \geq 2$.

PROPOSITION 8.8. *Assume k is algebraically closed and* $\operatorname{ch} k = 0$. *Let* $\phi : R = k[X, Y] \to A = k[X, Y, Z]$ *be the map, as before, defined by* $\phi(X) = XZ$ *and* $\phi(Y) = YZ$. *Let G be a finite abelian group acting linearly on R, and let* $\mathfrak{a} = (R_+^G)R$. *Then we have that* $\operatorname{hd}_A A/\phi(\mathfrak{a}) = 2$.

Proof. We may assume G has been diagonalized. Then the assertion follows immediately from Corollary 8.7, Proposition 6.5 (i) and the results in § 6.

THEOREM 8.9. *Assume* $k = \bar{k}$, $\operatorname{ch} k = 0$. *Let* $T = GL(1, k)$ *be a one dimensional torus acting on* $A = k[X, Y, Z]$ *by linear transformation of the variables. Suppose* $\dim A^T = 2$. *Then we have* $\operatorname{hd}_A A/(A_+^T)A = 2$.

Proof. We can assume the action is such that $X \to t^a X$, $Y \to t^b Y$, $Z \to t^{-p} Z$, for $t \in T$, with a, b, p all positive. (In fact if 0 appears among the exponents, the assertion is easy. If there are two negatives and one positive, consider t^{-1} instead of t.) Consider the diagram of rings



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where i is the map defined by $i(X) = U^a$, $i(Y) = V^b$, $i(Z) = W^p$, and ϕ is the natural inclusion. Observe: (1) S is the ring of invariants of B under the action of $T^* = GL(1, k)$ that sends $U \to tU$, $V \to tV$, $W \to t^{-1}W$ for $t \in T^*$. (2) The action of T on A in our present consideration is precisely that which is induced by the action of T^* on B. (3) There is a finite abelian group G^* that acts diagonally on B such that the ring of invariants B^{G^*} is the image of A by i. (4) If G is the group of automorphisms of S that G^* induces, then $S^G = A^T$.

Now let $\mathfrak{m} = A^T_+$, the maximal ideal of A^T. Instead of $\mathrm{hd}_A A/\mathfrak{m}A = 2$, we may prove $\mathrm{hd}_B B/\mathfrak{m}B = 2$, since $A \to B$ is faithfully flat. Consider $\mathfrak{m}B$ as an ideal that comes from A^T via S; then $\mathfrak{m}B$ may be written $\phi(\mathfrak{a})B$, with $\mathfrak{a} = (S^G_+)S$, S^G_+ being $\mathfrak{m}$. Thus it turns out $\mathrm{hd}_B B/\mathfrak{m}B = 2$ is nothing but was proved in Proposition 8.8.

Remark 8.10. In the proof of the theorem above, only the exponents of monomials are actually encountered; consequently the assumption that $k = \bar{k}$, $\mathrm{ch}\, k = 0$ is inessential. Indeed S^G (in the proof) is expressed as $k[M]$ for a certain semigroup M, but M in turn defines a semigroup ring $k[M]$ over any field. Hence Theorem 8.9 is valid for an arbitrary field k (with a suitable interpretation of a torus action in the case k is a finite field).

Remark 8.11. The first part of Proposition 5.9 (ii) is a special case of Theorem 8.9, where $a = 1$. The proof does not work for the general case because Lemma 5.8 fails to hold.

In the situation of the proof of Theorem 8.9, let $\begin{pmatrix} \omega^a & \\ & \omega^b \end{pmatrix}$ act on $R = k[X, Y]$. Then the projection $Z \to 1$ induces the isomorphism $A^T \xrightarrow{\sim} R^G$. Write

$$A^T = k[X^{\lambda_i} Y^{\mu_i} Z^{\ell_i} \mid i = 1, 2, \cdots, n + 1]\,,$$

where $$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1}\,, \qquad \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n+1}\,.$$

Then the re-examination of the proof of Theorem 8.5 shows that there is j such that $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_j \leq \ell_{j+1} \leq \cdots \leq \ell_n \leq \ell_{n+1}$. Lemma 5.8 says $j = 1$ in the case $a = 1$, but in the general case it can happen $j \neq 1, n + 1$.

Now let ρ be the representation of T to the syzygy space of $I = (A^T_+)A$. Then ρ is given by

$$\rho(t) = t^{bs_1} \oplus t^{bs_2} \oplus \cdots \oplus t^{bs_{j-1}} \oplus t^{ar_j} \oplus t^{ar_{j+1}} \oplus \cdots \oplus t^{ar_n}\,,$$

where $r_i = \lambda_i - \lambda_{i+1}$, and $s_i = \mu_{i+1} - \mu_i$, $i = 1, 2, \cdots, n$, and j is, as above, an index at which ℓ takes the minimum value. (Note j may not be unique. If not, we may take any such j.) This can be seen by considering the syzygy matrix of I, as in the proof of Proposition 6.7. The details are left to the reader.

Remark 8.12. In [9], H. Tanimoto proved, among other things, the following.

Let $A = k[X_1, X_2, \cdots, X_n, Z]$ be the polynomial ring in the variables $X_1, X_2, \cdots, X_n, Z$, and let $T = GL(1, k)$ act on A by $X_i \to t^{q_i}X_i$, and $Z \to t^{-p}Z$, where $q_i \geq 0$ and $p > 0$. Suppose the integers q_i and p satisfy the condition

(*) there are two integers a and b such that $\{\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_n\} \subset \{0, \bar{a}, \bar{b}, -\bar{a}, -\bar{b}\}$, where $\bar{\ }$ denotes residue class modulo p. Then it holds that $\text{hd}\, A/(A_+^T)A = n$.

Note the condition (*) is automatically satisfied if $n \leq 2$ (or $p \leq 5$), hence this can be regarded as a generalization of Theorem 8.9.

Without the condition (*), although it holds that $\text{hd}\, A/I \geq n$ (hence it is either n or $n + 1$), Tanimoto [9] also gives a counter-example to the equality $\text{hd}\, A/I = n$. (To see $\text{hd}\, A/I \geq n$, one can use Theorem 7.1 of Hochster [3].)

References

[1] D. A. Buchsbaum and D. Eisenbud, What makes a complex exact?, J. of Alg., **25**, No. 2 (1973), 259–268.
[2] J. Fogarty, Invariant theory, Benjamin, New York, 1969.
[3] M. Hochster, Cohen-Macaulay rings, combinatorics, and simplicial complexes, Ring theory II, Lecture notes in pure and app. math., **26**, Dekke (1977), 171–223.
[4] D. Mumford, Geometric invariant theory, Springer, New York, 1965.
[5] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie local, Publ. Math. I.H.E.S. No. 42, Paris, 1973.
[6] ——, Liaison des variétés algébriques I, Invent. Math., **26** (1974), 271–302.
[7] O. Riemenschneider, Die Invarianten der endlichen Untergruppen von $GL(2, C)$, Math. Z., **153** (1977), 37–50.
[8] T. A. Springer, Invariant theory, Springer Lect. Notes 585, 1977.
[9] H. Tanimoto, Monomial ideals and invariant subrings, Master thesis, Nagoya Univ. 1980 (in Japanese, unpublished).

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