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# CONTINUITY PROPERTIES OF OPERATOR SPECTRA

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**1. Introduction.** This paper is devoted to the study of convergence and variation of operator spectra with respect to the distance $G$ of Gokhburg and Markus [**5**] for subspaces and linear operators in a Banach space. We use the convention of Kato [**7**] and refer to convergence with respect to $G$ as *generalized convergence*. Letting $T$ denote a linear operator and $\lambda$ a complex number, we prove that the conjugate mapping $c$: $T \to T'$ is continuous, where it is defined, in Theorem 2.6 and that the extended resolvent $\bar{R}$: $(\lambda, T) \to (\lambda - T)^{-1}$ is jointly continuous in Theorem 2.8. Both theorems generalize well-known results (confer [**3**; **7**; **15**]). An example of a sequence of bounded operators which converge to an unbounded operator in the generalized sense is given. We also prove that the spectrum mapping $\sigma_e$ is upper semi-continuous on the set of linear operators in a Banach space in Theorem 3.3 which generalizes results of Newburgh [**11**] and Kato [**7**]. We extend to closed operators with non-void resolvent sets the three sufficient conditions of Newburgh [**11**] for continuity of $\sigma_e$ at an operator $T$ in Theorems 3.6, 3.9 and 3.10.

The terms *subspace* and *operator* mean linear manifold and linear operator, respectively, in this paper. We employ the spectral notation and terminology of Taylor [**15**] in the sequel. We are indebted to Dr. H. A. Gindler for helpful conversations and suggestions in the formative stages of this research.

**2. Generalized convergence.** This section is devoted to investigating some properties of the $G$-topology for subspaces and operators. Let $X$ be a non-trivial Banach space. We denote by $\mathscr{M}(X)$ the class of subspaces of $X$ and $\mathscr{S}(X)$ the class of closed subspaces of $X$. For $Y \in \mathscr{M}(X)$, let $\Sigma(Y)$ denote the set $\{y \in Y; ||y|| = 1\}$ and let $d(x, y) = ||x - y||$ for $x$ and $y$ in $X$.

Let $Y$ and $Z$ be subspaces of $X$. The *opening* of $Y$ and $Z$, $\theta(Y, Z)$, is defined by

$$\theta(Y, Z) = \max \left\{ \sup_{y \in \Sigma(Y)} d(y, Z), \sup_{z \in \Sigma(Z)} d(z, Y) \right\},$$

the definition being completed by setting $\theta(Y, Z) = 1$ if one and only one of $Y$ or $Z$ is $\{0\}$ and $\theta(\{0\}, \{0\}) = 0$. Here $d(x, A) = \inf_{y \in A} ||x - y||$, for $A \subset X$.

This definition of opening is due to Krein, Krasnoselski, and Milman [**8**]. Gokhburg and Markus [**5**] defined the distance $G$ by modifying the definition of opening as follows: let $D$ denote the Hausdorff distance [**6**] induced by $d$ on the class of non-void subsets of $\Sigma(X)$.

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*Definition.* For subspaces $Y$ and $Z$ of $X$, let

$$G(Y,Z) = D(\Sigma(Y),\Sigma(Z)) = \max\left\{\sup_{y\in\Sigma(Y)} d(y,\Sigma(Z)),\ \sup_{z\in\Sigma(Z)} d(z,\Sigma(Y))\right\},$$

if $Y \neq \{0\} \neq Z$, and let $G(Y, Z) = \theta(Y, Z)$, otherwise.

Since $\Sigma(X)$ has diameter $\leqq 2$, $D$ is a pseudo-metric (see [**2**]). It follows that $G$ is a pseudo-metric for $\mathscr{M}(X)$ and a metric for $\mathscr{S}(X)$. Moreover, $G(Y, Z) = G(\bar{Y}, \bar{Z})$ for $Y$ and $Z$ in $\mathscr{M}(X)$ with bar denoting strong closure in $X$. It is proved in [**5**] that if $Y$ and $Z$ belong to $\mathscr{M}(X)$, then $G(Y, Z)/2 \leqq \theta(Y, Z) \leqq G(Y, Z)$, so that $G$ and $\theta$ determine the same uniformity for $\mathscr{M}(X)$.

Let $X'$ denote the conjugate space of $X$. If $A$ is a non-void subset of $X$, let $A^\perp = \{x' \in X'; x'(x) = 0 \text{ for each } x \in A\}$ be the orthogonal complement of $A$ in $X'$. It is known [**12**] that $\theta(Y, Z) = \theta(Y^\perp, Z^\perp)$ for $Y, Z \in \mathscr{M}(X)$. This proves the following lemma.

2.1 Lemma. *Define* $\Gamma$ *on* $\mathscr{M}(X)$ *with values in* $\mathscr{S}(X')$ *by setting* $\Gamma(Y) = Y^\perp$. *Then* $\Gamma$ *is continuous with respect to* $G$.

For non-zero elements $x$ and $y$ in $X$, let

$$n(x, y) = \inf_{\epsilon>0} (\epsilon; ||x - y|| < (e^\epsilon - 1) \min (||x||, ||y||)).$$

Newburgh [**10**] proves that $n$ is a metric for $X - \{0\}$ and uses this function to define a pseudo-metric $\delta$ for the class of non-trivial subspaces of $X$ as follows. Let $\tilde{D}$ be the Hausdorff distance induced by $n$ on the class of non-void subsets of $X - \{0\}$ and for non-trivial subspaces $Y$ and $Z$ in $X$, let $\delta(Y, Z) = \tilde{D}(Y - \{0\}, Z - \{0\})$. Berkson [**1**] proves in Theorem 7.1 that $\delta$ is equivalent to $G$. We can use this fact to deduce further properties of $G$. Note that $\delta(Y, Z) = \delta(\bar{Y}, \bar{Z})$ for non-trivial subspaces $Y$ and $Z$.

Let $X$ and $Y$ be non-trivial Banach spaces and let $E = X \times Y$ be normed into a Banach space by $||(x, y)|| = ||x|| + ||y||$, $x \in X$, $y \in Y$. We need the notion of bounded subspace in $E$ due to Newburgh [**10**].

*Definition.* A subspace $Z$ of $E$ is said to be *bounded over* $X$ (respectively $Y$) if and only if there is a positive constant $K$ depending only on $Z$ such that $||y|| \leqq K||x||$ (respectively, $||x|| \leqq K||y||$) for each $(x, y) \in Z$.

Let $P$ denote the mapping defined on $\mathscr{M}(E)$ with values in $\mathscr{M}(X)$ by $P(Z) = \overline{\{x; (x, y) \in Z\}}$ and let $\mathscr{B}(X, Y) = \{Z \in \mathscr{M}(E); Z \text{ is bounded over } X \text{ and } \overline{P(Z)} = X\}$.

2.2 Theorem. *The class* $\mathscr{B}(X, Y)$ *is open in* $\mathscr{M}(E)$ *with respect to* $G$.

*Proof.* We prove $\mathscr{B}(X, Y)$ is open with respect to $\delta$. Let $0 < \epsilon < \log (3/2)$ and choose $Z_0 \in \mathscr{B}(X, Y)$. From Lemmas 4 and 5 of Newburgh [**10**] there is a $\beta > 0$ such that $Z \in \mathscr{M}(E)$ and $\delta(Z, Z_0) < \beta$ implies $\delta(P(Z), P(Z_0)) = \delta(P(Z), X) < \epsilon$. But this implies $\overline{P(Z)} = X$ from Lemma 3 of [**10**] and the choice of $\epsilon$.

A similar theorem can be deduced with the roles of $X$ and $Y$ interchanged.

Let $\mathscr{T}$ denote the class of linear operators $T$ defined on domains $D(T) \subseteq X$ and having ranges $R(T) \subseteq Y$ and let $\mathscr{T}_c$ denote the subset of closed operators in $\mathscr{T}$. For $S, T \in \mathscr{T}$, we define $G(S, T) = G(\text{graph } S, \text{graph } T)$, where graph $S$ and graph $T$ are considered as subspaces of $E$. $G$ is then a pseudo-metric for $\mathscr{T}$ and a metric for $\mathscr{T}_c$.

For a subset $A$ of $E$, let $A^{-1} = \{(y, x); (x, y) \in A\}$. Then we have $G(Z_1, Z_2) = G(Z_1^{-1}, Z_2^{-1})$ for $Z_1$ and $Z_2$ in $\mathscr{M}(E)$. The next theorem follows from this observation.

2.3 THEOREM. *The mapping $\iota$: $T \to T^{-1}$ defined on the set of invertible operators in $\mathscr{T}$ is continuous with respect to $G$.*

For a closable operator $T \in \mathscr{T}$, let $\bar{T}$ denote the minimal closed linear extension of $T$ i.e. graph $\bar{T} = \overline{\text{graph } T}$. Note that $G(S, T) = G(\bar{S}, \bar{T})$ for closable operators $S$ and $T$ in $\mathscr{T}$. Let $[X, Y] = \{T \in \mathscr{T}_c; D(T) = X\}$. The following theorem is a corollary to Theorem 2.3.

2.4 THEOREM. *The set $\{T \in \mathscr{T}; T$ is closable and $\bar{T} \in [X, Y]\}$ is open in $\mathscr{T}$.*

Let $T'$ denote the conjugate of $T$ if and only if $\overline{D(T)} = X$ and let $\mathscr{D} = \{T \in \mathscr{T}; \overline{D(T)} = X\}$. Part (b) of the following lemma is due to Rota [**13**].

2.5 LEMMA. *For $T \in \mathscr{T}$, let $H(T) = \{(-Tx, x); x \in D(T)\}$. Then*
(a) $G(S, T) = G(H(S), H(T)), \quad S, T \in \mathscr{T}$;
(b) graph $T' = H(T)^\perp$ *for* $T \in \mathscr{D}$.

We denote by $c$ the mapping defined on $\mathscr{D}$ by $c(T) = T'$. The range of $c$ is a subset of the set of closed operators defined on domains in $Y'$ and having ranges in $X'$.

2.6 THEOREM. *The mapping $c$ is continuous.*

*Proof.* Suppose $\{T_n\}$ is a sequence in $\mathscr{D}$ such that $G - \lim T_n = T \in \mathscr{D}$. Then $G - \lim_n H(T_n)^\perp = H(T)^\perp$ from (2.1) and Lemma 2.5(a). Hence, $G - \lim_n T_n' = T'$ in view of the previous lemma.

Since the topologies of $G$ and the operator norm are equivalent for $[X, Y]$ (confer [**1**]), Theorem 2.6 generalized a well-known result (Dunford Schwartz [**3**, p. 478]).

We assume in the sequel that $X = Y$ and $X$ is a complex Banach space. Let $[X, X] = [X]$ and let $C$ denote the space of complex numbers. The proof of the following lemma is taken from Bezak [**2**].

2.7 LEMMA. *Let $\Psi$ denote the mapping defined on $C \times \mathscr{T}$ with values in $\mathscr{T}$ by $\Psi(\lambda, T) = \lambda - T$. Then $\Psi$ is jointly continuous.*

*Proof.* Suppose $\{(\lambda_n, T_n)\}$ is a sequence in $C \times \mathscr{T}$ such that $\lim_n (\lambda_n, T_n) = (\lambda_0, T_0) \in C \times \mathscr{T}$ with respect to the product topology. Let $0 < \epsilon < 1$ and

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choose a positive integer $N$ such that $n \geqq N$ implies

$$G(T_n, T_0) < \frac{\epsilon}{6(1 + |\lambda_0|)^2} \quad \text{and} \quad |\lambda_n - \lambda_0| < \frac{\epsilon}{6}.$$

Fix $n \geqq N$ and choose $x \in D(T_0)$ such that $||(x, (\lambda_n - T_n)x)|| = 1$. This implies $0 < ||x|| \leqq 1$. By hypothesis there is $y \in D(T_0)$ such that

$$||(x, T_n x) - (y, T_0 y)|| < \frac{\epsilon||(x, T_n x)||}{6(1 + |\lambda_0|)^2}.$$

Then

$$\begin{aligned} &||(x, (\lambda_n - T_n)x) - (y, (\lambda_0 - T_0)y|| \\ &\quad \leqq ||x - y|| + ||T_n x - T_0 y|| + ||\lambda_n x - \lambda_0 y|| \\ &\quad \leqq \frac{\epsilon||(x, T_n x)||}{6(1 + |\lambda_0|)^2} + |\lambda_n - \lambda_0| + |\lambda_0|\,||x - y|| \\ &\quad \leqq \frac{\epsilon||(x, T_n x)||}{6(1 + |\lambda_0|)} + \frac{\epsilon}{6}. \end{aligned}$$

Now $||(x, T_n x)|| \leqq ||x|| + ||(\lambda_n - T_n)x|| + |\lambda_n - \lambda_0|\,||x|| + |\lambda_0|\,||x|| \leqq 1 + |\lambda_0| + \epsilon/6$, so that $||(x, (\lambda_n - T_n)x) - (y, (\lambda_0 - T_0)y)|| < \epsilon/2$. This proves

(a) $\sup \{d(\bar{x}, \text{graph } (\lambda_0 - T_0)); \bar{x} \in \Sigma \text{ (graph } (\lambda_n - T_n))\} \leqq \epsilon/2$.

On the other hand, choose $y \in D(T_0)$ such that $||(y, (\lambda_0 - T_0)y)|| = 1$, which implies $0 < ||y|| \leqq 1$. There exists $x \in D(T_n)$ such that

$$||(y, T_0 y) - (x, T_n x)|| < \frac{\epsilon||(y, T_0 y)||}{6(1 + |\lambda_0|)^2}.$$

Then $||(y, T_0 y)|| = ||y|| + ||T_0 y - \lambda_0 y + \lambda_0 y|| \leqq 1 + |\lambda_0|$ and $|\lambda_n| < |\lambda_0| + \epsilon/6$. $||(y, (\lambda_0 - T_0)y) - (x, (\lambda_n - T_n)x)|| \leqq ||x - y|| + ||T_n x - T_0 y|| + ||\lambda_0 y - \lambda_n x|| \leqq \epsilon/6(1 + |\lambda_0|) + |\lambda_0 - \lambda_n| + |\lambda_n|\,||x - y|| < \epsilon/2$, which proves

(b) $\sup \{d(\bar{y}, \text{graph } (\lambda_n - T_n)); \bar{y} \in \Sigma \text{ (graph } (\lambda_0 - T_0))\} \leqq \epsilon/2$.

Combining (a) and (b) we obtain $G(\lambda_n - T_n, \lambda_0 - T_0) \leqq \epsilon$ for $n \geqq N$ which proves the lemma.

We define the extended resolvent $\bar{R}$ on

$$D(\bar{R}) = \{(\lambda, T) \in C \times \mathscr{T}; (\lambda - T)^{-1} \text{ exists}\}$$

by $\bar{R}(\lambda, T) = (\lambda - T)^{-1}$. If $T$ is a fixed operator in $\mathscr{T}$ with non-void resolvent set $\rho(T)$, then $\bar{R}$ restricted to $\rho(T) \times \{T\}$ becomes the usual resolvent $R$ of $T$.

2.8 THEOREM. *The mapping $\bar{R}$ is jointly continuous.*

*Proof.* Let $\iota$ and $\Psi$ be the continuous mappings of (2.3) and (2.7), respectively. Then $R = \phi \circ \psi$, where $\psi = \Psi/D(\bar{R})$ and $\phi = \iota/\psi[D(\bar{R})]$.

This section is concluded with an example of generalized convergence of operators.

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*Example.* Let $X = l^p$, $1 < p < \infty$ and let $T$ be the operator in $[X]$ having the matrix representation $(T_{mn})$ defined by $T_{mn} = 1$ if $m - n = 1$ and $T_{mn} = 0$, otherwise, where $m, n = 1, 2, 3, \ldots$. It is well-known (Taylor [**15**, pp. 266–267]) that $\rho(T) = \{\lambda \in C; |\lambda| > 1\}$, $II_2\sigma(T) = \{\lambda \in C; |\lambda| = 1\}$. Choose a sequence $\{\lambda_n\} \subset \rho(T)$ such that $\lim_n \lambda_n = \lambda$, $|\lambda| = 1$. Then $\{R(\lambda_n, T)\}$ is a sequence in $[X]$ such that $G - \lim_n R(\lambda_n, T) = \bar{R}(\lambda, T)$ from the continuity of $\bar{R}$. Similarly, if we choose $\{\lambda_n\} \subset II_2\sigma(T)$ such that $\lim_n \lambda_n = \lambda$, $|\lambda| = 1$, then $\{\bar{R}(\lambda_n, T)\}$ is a sequence of unbounded operators which converge to the unbounded operator $\bar{R}(\lambda, T)$.

**3. Continuity properties of operator spectra.** In this section we investigate variation in the spectra of operators with respect to $G$. Let $X$ be a complex Banach space and let $C_\infty$ denote the extended complex plane topologized by the chordal metric $\chi$. Let $\mathscr{S}$ denote the class of non-void closed subsets of $C_\infty$. It is proved in Gindler and Taylor [**4**] that $\rho(T)$ and $III_1\sigma(T)$ are open subsets of $C_\infty$ for $T \in \mathscr{T}$. Therefore, the extended spectrum $\sigma_e(T)$ and the set $\sigma_e(T) - III_1\sigma(T)$ are in $\mathscr{S}$ for each $T \in \mathscr{T}$. We denote by $\sigma_e$ the spectrum mapping defined on $\mathscr{T}$ with values in $\mathscr{S}$ by setting $\sigma_e(T)$ to be the extended spectrum of $T$. We also define the spectrum boundary mapping $\delta\sigma_e$ on $\mathscr{T}$ with values in $\mathscr{S}$ by $\delta\sigma_e(T) = \sigma_e(T) - III_1\sigma(T)$.

We need the notions of upper and lower semi-continuity. Let $Y$ be a topological space and let $\mathscr{A}$ be the class of non-void subsets of $Y$ topologized by the upper and lower semi-finite topologies (confer Michael [**9**]). Consider a mapping $f$ of a topological space $X$ into $\mathscr{A}$. Then $f$ is *upper* (respectively, *lower*) *semi-continuous at* $x \in X$ if and only if $f$ is continuous with respect to the upper (respectively, lower) semi-finite topology. We have the following result [**2**].

3.1 THEOREM. *For a topological space $X$ and a metric space $(Y, d)$ of finite diameter, let $D$ be the Hausdorff distance induced by $d$ on the class $\mathscr{A}$ of non-void subsets of $Y$. If $f$ maps $X$ into $\mathscr{A}$, then $f$ is upper and lower semi-continuous at $x \in X$ if $f$ is continuous at $x$ with respect to $D$. Conversely, if $f$ is upper and lower semi-continuous at $x$ and $f(x)$ is a compact subset of $Y$, then $f$ is continuous at $x$ with respect to $D$.*

*Proof.* Suppose $f$ is continuous at $x \in X$ with respect to $D$. Let $U$ be an open subset of $Y$ such that $f(x) \subset U$. We can choose $\epsilon > 0$ so that $N(\epsilon) = \{z \in Y; d(z, y) < \epsilon \text{ for some } y \in f(x)\} \subset U$. Let $\mathscr{W} = \{B \in \mathscr{A}; D(f(x), B) < \epsilon\}$. By hypothesis there is a neighborhood $V$ of $x$ such that $f[V] \in \mathscr{W}$ which implies $f[V] \subset U$ and proves that $f$ is upper semi-continuous at $x$. If $U_0$ is a neighborhood of $y \in f(x)$, then $z \in V$ implies $f(z) \cap U_0 \neq \phi$. This proves $f$ is lower semi-continuous at $x$.

Conversely, suppose $f$ is upper and lower semi-continuous at $x$ and $f(x)$ is compact. We can choose $y_1, y_2, \ldots, y_n$ in $f(x)$ and $\epsilon > 0$ such that $f(x) \subset U = \bigcup_{k=0}^n U(y_i, \epsilon)$, where $U(y_i, \epsilon) = \{z \in Y; d(y_i, z) < \epsilon\}$. From upper semi-continuity there is a neighborhood $V_0$ of $x$ such that $f[V_0] \subset U$. From lower semi-continuity there are neighborhoods $V_i$ of $x$ such that $z \in V_i$ implies $f(z) \cap U(y_i, \epsilon) \neq \emptyset$, $i = 1, 2, \ldots, n$. Let $V = \bigcap_{i=0}^{n} V_i$. If $z \in V$, then $f(z) \subset U$. It follows that $D(f(x), f(z)) \leqq \epsilon$, which completes the proof.

Let $\hat{D}$ denote the Hausdorff distance induced by the chordal metric $\chi$. Since each closed subset of $C_\infty$ is compact, we have the following corollary to the previous theorem.

3.2 Corollary. *$\sigma_e$ (respectively, $\delta\sigma_e$) is continuous at $T \in \mathscr{T}$ with respect to $G$ and $\hat{D}$ if and only if $\sigma_e$ (respectively, $\delta\sigma_e$) is upper and lower semi-continuous at $T$.*

Newburgh [**11**] proved that $\sigma_e$ is upper semi-continuous on $[X]$. Kato [**7**] proved that $\sigma_e$ is upper semi-continuous with respect to $G$ on the set of closed operators with non-void resolvent sets. The following theorem extends these results.

3.3 Theorem. *The mappings $\sigma_e$ and $\delta\sigma_e$ are upper semi-continuous on $\mathscr{T}$.*

*Proof.* We only prove the assertion for $\sigma_e$; the proof for $\delta\sigma_e$ is similar. Let $T \in \mathscr{T}$ and assume without loss of generality that $\sigma_e(T) \neq C_\infty$. Choose a proper open subset $U$ of $C_\infty$ such that $\sigma_e(T) \subset U$ and select $\lambda \notin U$. Then $R(\lambda, T) \in C[X] = \{T \in \mathscr{T}; T \text{ is bounded and } \bar{T} \in [X]\}$. In view of (2.4) and (2.8) we can choose neighborhoods $\mathscr{U}$ of $R(\lambda, T)$ in $C[X]$ and $\mathscr{W}$ of $T$ such that $S \in \mathscr{W}$ implies $R(\lambda, S) \in \mathscr{U}$ and $\lambda \notin \sigma_e(S)$. This proves $\sigma_e(S) \subset U$ for each $S \in \mathscr{W}$ which proves the assertion.

3.4 Theorem. *If $\sigma_e$ is continuous at $T \in \mathscr{D}$, then $\sigma_e$ is continuous at $T'$.*

*Proof.* The assertion is a consequence of Theorem 2.6 and the fact that $\sigma_e(T) = \sigma_e(T')$.

We use the notion of Cauchy domain due to Taylor [**14**]. A *Cauchy domain* $\Delta$ is an open subset of $C_\infty$ which consists of a finite number of components and has a closed rectifiable boundary denoted $b(\Delta)$; moreover, the closures of its components are mutually disjoint. We also employ the operational calculus of Taylor [**14**]. For a closed operator $T$ in $\mathscr{T}$ such that $\rho(T) \neq \emptyset$, let

$$E(\sigma) = \delta I + \frac{1}{2\pi i} \int_{b(\Delta)} R(\lambda, T) d\lambda,$$

where $\sigma$ is a spectral set of $T$, $\Delta$ is a Cauchy domain containing $\sigma_e(T)$ and the integration is performed in the usual counterclockwise sense on $b(\Delta)$. Here $I$ is the identity operator on $X$. $\delta = 1$ if $\infty \in \sigma$ and $\delta = 0$, otherwise (see [**14**]). The following lemma occurs in [**2**].

3.5 Lemma. *Suppose $G - \lim_n T_n = T$, where each $T_n$ and $T$ are either closed operators in $\mathscr{T}$ or operators in $\mathscr{T}$ with strongly dense domains in $X$ and have non-*

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*void resolvent sets. Let $\sigma$ be a non-void spectral set of $T$ and let $\Delta$ be a Cauchy domain such that $\sigma \subset \Delta$ and $\bar{\Delta} \cap (\sigma_e(T) - \sigma) = \emptyset$. Then a positive integer $N$ can be found such that $n \geqq N$ implies $\sigma_e(T) \cap \bar{\Delta} \neq \emptyset$.*

*Proof.* Suppose $\{T_n\}$ and $T$ are closed operators. If $\sigma = \sigma_e(T)$, the assertion is a consequence of the upper semi-continuity of $\sigma_e$. Assume $\sigma \neq \sigma_e(T)$ and the assertion is false. There is a subsequence $\{T_{n(i)}\}$ of $\{T_n\}$ such that $\sigma_e(T_{n(i)}) \cap \bar{\Delta} = \emptyset$, $i = 1, 2, \ldots$. From Theorem 2.8 $\lim_n R(\lambda, T_{n(i)}) = R(\lambda, T)$ uniformily for $\lambda \in b(\Delta)$ because $b(\Delta)$ is compact. Therefore,

$$\lim_i \int_{b(\Delta)} R(\lambda, T_{n(i)}) d\lambda = \int_{b(\Delta)} \lim_i R(\lambda, T_{n(i)}) d\lambda = \int_{b(\Delta)} R(\lambda, T) d\lambda = 0.$$

Hence, $E(\sigma) = \delta I$. If $\infty \in \sigma$, then $E(\sigma) = I$ implying $\sigma = \sigma_e(T)$, a contradiction of hypothesis. If $\infty \notin \sigma$, then $E(\sigma) = 0$ which implies that $\sigma$ is void. In either case a contradiction is obtained which proves the assertion. For the case when each $T_n$ and $T$ have strongly dense domains in $X$, we apply the proof just completed to the conjugates $T_n'$ and $T'$ which completes the proof of the lemma.

The first sufficient condition of Newburgh [**11**] for continuity of $\sigma_e$ is generalized in the following theorem.

3.6 THEOREM. *If $T \in \mathscr{T}$ is either closed or has strongly dense domain in $X$ and $\sigma_e(T)$ is totally disconnected, then $\sigma_e$ is continuous at $T$ with respect to $G$.*

*Proof.* Let $U$ be a neighborhood of $\lambda \in \sigma_e(T)$. By hypothesis there is a spectral set $\sigma$ of $T$ such that $\lambda \in \sigma \subset U$. From a theorem of Taylor [**14**] there is a Cauchy domain $\Delta$ such that $\sigma \subset \Delta \subset \bar{\Delta} \subset U$. If $G - \lim_n T_n = T$, then from the previous lemma there is an integer $N$ such that $\sigma_e(T_n) \cap \bar{\Delta} \neq \emptyset$, $n \geqq N$. Hence, $\sigma_e$ is lower semi-continuous at $T$. The assertion follows from Corollary 3.2 and Theorem 3.3.

We fix $\alpha \in C$ and define the set function $g$ at a non-void subset $A$ of $C_\infty$ by $g(A) = \{\alpha + \mu^{-1}; \mu \in A\}$. Since the mapping $\mu \rightarrow \alpha + \mu^{-1}$ is a homeomorphism of $C_\infty$ onto itself, we have the following lemma.

3.7 LEMMA. *The mapping $g$ is continuous on the class $\mathscr{S}$ of non-void closed subsets of $C_\infty$ with respect to $\hat{D}$, the Hausdorff distance induced by $\chi$ on $\mathscr{S}$.*

We have the following theorem due to Taylor [**14**].

3.8 THEOREM. *Let $T \in \mathscr{T}$ be a closed operator and suppose $\rho(T) \neq \emptyset$. Fix $\alpha \in \rho(T)$ and let $T_\alpha = -R(\alpha, T)$. Then*
(a) *$\alpha + \mu^{-1} \in \rho(T)$ if and only if $\mu \in \rho(T_\alpha)$;*
(b) *$\alpha + \mu^{-1} \in \sigma_e(T)$ if and only if $\mu \in \sigma_e(T_\alpha)$.*

For $S, T \in \mathscr{T}$, we define the product $ST$ on

$$D(ST) = \{x; x \in D(T) \text{ and } Tx \in D(S)\}$$

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by $(ST)x = S(Tx)$. Then $ST \in \mathscr{T}$ for each $S, T \in \mathscr{T}$. The following theorem generalizes the second sufficient condition of Newburgh [**11**] for continuity of $\sigma_e$.

3.9 THEOREM. *Let $T \in \mathscr{T}_c$, the subset of closed operators in $\mathscr{T}$, and suppose $\rho(T) \neq \phi$. If there is a neighborhood $\mathscr{U}$ of $T$ in $\mathscr{T}_c$ such that $S \in \mathscr{U}$ implies $D(ST) = D(TS)$ and $SR = TS$, then $\sigma_e$ is continuous at $T$.*

*Proof.* Choose a proper open subset $U$ of $C_\infty$ such that $\sigma_e(T) \subset U$. Since $\sigma_e$ is upper semi-continuous at $T$, we can assume that $S \in \mathscr{U}$ implies $\sigma_e(S) \subset U$. Choose $\alpha \in C_\infty - U$. If $S \in \mathscr{U}$, then $(\alpha - T)(\alpha - S) = (\alpha - S)(\alpha - T)$ which implies $R(\alpha, T)R(\alpha, S) = R(\alpha, S)R(\alpha, T)$. We also have the fact that $[X]$ is open in $\mathscr{T}_c$ (confer Berkson [**1**]). It follows from these results and Theorem 2.8 that the set $\mathscr{W} = \{R(\alpha, S); S \in \mathscr{U}\}$ is a neighborhood in $[X]$ which satisfies the conditions of Theorem 4 in Newburgh [**11**]. Hence, $\sigma_e$ is continuous at $R(\alpha, T)$. Let $g$ be the function of Lemma 3.7. Then from (3.7) and (3.8) $g$ is continuous at $\sigma_e(R(\alpha, T))$ and $g(\sigma_e(R(\alpha, T))) = \sigma_e(T)$. Let $\mathscr{Y}$ be a neighborhood of $\sigma_e(T)$ in $\mathscr{S}$. We can, therefore, choose a neighborhood $\mathscr{U}_0 \subset \mathscr{U}$ of $T$. such that $S \in \mathscr{U}_0$ implies $\sigma_e(S) \in \mathscr{Y}$. The details are given in [**2**]. This completes the proof.

*Definition.* Let $\mathscr{M}$ be a subset of $\mathscr{T}$ and set

$$\mathscr{M}' = \{T; T \in [X], D(ST) = D(TS) \text{ and } ST = TS \text{ for each } S \in \mathscr{M}\}$$

and $\mathscr{M}'' = (\mathscr{M}')'$. Then $\mathscr{M}$ is said to be *commutative* if and only if $\mathscr{M}''$ is a commutative subset of $[X]$.

Note that if $\mathscr{M} \subset [X]$, then $\mathscr{M} \subset \mathscr{M}''$. The last theorem generalizes the third sufficient condition of Newburgh [**11**] for continuity of $\sigma_e$.

3.10 THEOREM. *Let $T \in \mathscr{T}_c$ and suppose $\rho(T) \neq \emptyset$. If there is a neighborhood $\mathscr{U}$ of $T$ in $\mathscr{T}_c$ such that $\mathscr{U}$ is commutative in the sense of the previous definition, then $\sigma_e$ is continuous at $T$.*

*Proof.* Choose $\alpha \in \rho(T)$. We can assume without loss of generality that $S \in \mathscr{U}$ implies $\alpha \in \rho(S)$. Let $\mathscr{N} = \{R(\alpha, S); S \in \mathscr{U}\}$. From a theorem of Newburgh [**11**], $\mathscr{N}'' = \mathscr{U}''$ which implies $\mathscr{N}$ is a commutative subset of $[X]$. From (2.8), $\mathscr{N}$ is a neighborhood of $R(\alpha, T)$. It follows from Theorem 4 in Newburgh [**11**] that $\sigma_e$ is continuous at $R(\alpha, T)$. The remainder of the proof paraphrases that of the previous theorem.

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