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A COMMUTATIVITY THEOREM FOR SEMIPRIME RINGS

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Abstract

It is shown that if R is a semiprime ring with 1 satisfying the property that, for each $x, y \in R$, there exists a positive integer n depending on x and y such that $(xy)^k - x^k y^k$ is central for $k = n, n+1, n+2$, then R is commutative, thus generalizing a result of Kaya.

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Kaya (1976) showed that if R is a primary ring (that is, $R/J(R)$ is simple) or semiprime ring with 1 satisfying the property that, for each $x, y \in R$, there exists a positive integer n depending on x and y such that $(xy)^k = x^k y^k$ for $k = n, n+1, n+2$, then R is commutative, thus generalizing a theorem of Luh (1971), p. 211, who proved the result for a fixed n in the case when R is primary. Ligh and Richoux (1977) has proved the result of Luh, for a fixed n, without assuming that R is primary. Recently Richoux (to appear) has extended the Ligh–Richoux result to arbitrary n. In this note we prove the result stated in the abstract, which generalizes Theorem 2(ii) of Kaya (1976) for the semiprime ring case. However, it is not possible, by Example 2 of Luh (1971), to replace semiprime ring by primary ring in our result.

We use the following notations:

$$Z(R) = \text{the centre of } R,$$

$$J(R) = \text{the Jacobson radical of } R,$$

$$[x, y] = xy - yx.$$

For the sake of convenience, we label some properties of R as follows.

(A) For each $x, y \in R$, there exists a positive integer n depending on x and y such that $(xy)^k - x^k y^k \in Z(R)$ for $k = n, n+1, n+2$.

(B) For each $x, y \in R$, $xy + yx \in Z(R)$.

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(C) For each $x, y \in R$,

$$yx^2 + x^2 y + yx^2 y + 2yxy = xy^2 + y^2 x + xy^2 x + 2xyx.$$

LEMMA 1. *If R is a semisimple ring satisfying* (A), *then R is commutative.*

PROOF. The proof is based on standard technique given by Herstein (1961), p. 29 and Jacobson (1968), p. 220.

First we assume that R is a division ring satisfying (A). Let $[(xy)^k - x^k y^k, z] = 0$ for all $z \in R$. Replacing z by xy and yx, we get respectively,

(1) $$[x^{k-1} y^{k-1}, yx] = 0$$

and

(2) $$[(xy)^k - x^k y^k, yx] = 0.$$

Let $k = n, n+1, n+2$. Then from (1) and (2) we get

(3) $$[(xy)^n, yx] = 0,$$

(4) $$[(xy)^{n+1}, yx] = 0.$$

The last two equations provide us with $xy^2 x = yx^2 y$. Now R is commutative as a part of the proof of Theorem 2.5 of Gupta (1970).

Next we assume R is a primitive ring satisfying (A). If R is not a division ring, then D_2 the ring of 2×2 matrices over some division ring D will be a homomorphic image of subring of R and satisfies (A). But this is impossible as

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

fail to satisfy (A). Hence R must be a division ring and therefore is commutative.

Finally if R is semisimple ring satisfying (A), then R is a subdirect sum of primitive rings R_α each of which, as a homomorphic image of R, satisfies (A) and hence is commutative by the above discussion. Thus R is commutative.

We give the following lemma which will be used frequently in the subsequent study.

LEMMA 2. *Let R be a prime ring and $x \neq 0$, y be elements of R. If x and xy are in $Z(R)$, then y is in $Z(R)$.*

PROOF. Let x, xy be in $Z(R)$. Then $xyz = zxy = xzy$ for all $z \in R$. From this we have $xR(yz - zy) = 0$. Since R is a prime ring and $x \neq 0$ we get $zy = yz$ for all $z \in R$. Thus y is in $Z(R)$.

LEMMA 3. *If R is a semiprime ring of characteristic 2 satisfying* (B), *then R is commutative.*

PROOF. Let us assume that R is a prime ring satisfying (B). Replacing x by xy in (B), we get that R is a commutative ring by an application of Lemma 2.

If R is a semiprime ring satisfying (B), then it is isomorphic to a subdirect sum of prime rings R_α each of which, as a homomorphic image of R, satifies (B) and hence is commutative by the above part. Thus R is commutative.

LEMMA 4. *If R is a semiprime ring satisfying* (C), *then R is commutative.*

PROOF. It suffices to assume that R is a prime ring, using a similar argument as given in the proof of Lemma 3. Replacing x by $x+y$ in (C) and cancelling using (C), we get

$$(y+y^2)[x,y] = [x,y](y+y^2). \tag{5}$$

Replacing x by xy and yx in (1), and adding the results we obtain

$$(y+y^2)[x,y^2] = [x,y^2](y+y^2). \tag{6}$$

Adding (5) and (6), we have

$$(y+y^2)[x,y+y^2] = [x,y+y^2](y+y^2) \tag{7}$$

for all $x, y \in R$.

If the characteristic of R is not 2, then by a sublemma of Herstein (1969), p. 5, we have

$$y+y^2 \in Z(R) \quad \text{for all } y \in R. \tag{8}$$

Replacing y by $x+y$ in (8), we get

$$xy+yx \in Z(R) \quad \text{for all } x, y \in R. \tag{9}$$

Replacing x by xy in (9) and by Lemma 2, we obtain $y \in Z(R)$ unless $xy+yx = 0$ for every x. If $xy+yx = 0$ for every x, then we replace x by y to get $2y^2 = 0$, which will imply that $y^2 = 0$. By (8) $y \in Z(R)$ for all $y \in R$. Hence R is commutative.

If the characteristic of R is 2, then by (7), we have

$$y^2+y^4 \in Z(R) \quad \text{for all } y \in R. \tag{10}$$

Replacing y by y^2 in (10), we have

$$y^4+y^8 \in Z(R) \quad \text{for all } y \in R. \tag{11}$$

Adding (10) and (11), we obtain

$$y^2+y^8 \in Z(R) \quad \text{for all } y \in R. \tag{12}$$

Again replacing y by y^3 in (10), we get

$$(y^2 + y^8)\, y^4 \in Z(R) \quad \text{for all } y \in R. \tag{13}$$

By Lemma 2, $y^4 \in Z(R)$ unless $y^2 + y^8 = 0$. If $y^4 \in Z(R)$, then by (10) $y^2 \in Z(R)$. If $y^2 + y^8 = 0$, then it can be seen that $y^2 = 0$ for all $y \in J(R)$. Hence in either case $y^2 \in Z(J(R))$ for all $y \in J(R)$. Let $x \in J(R)$. Replacing y by $x + y$, we get $xy + yx \in Z(J(R))$. $J(R)$ is commutative by Lemma 3.

Since $\bar{R} = R/J(R)$ is semisimple, it suffices to assume that $\bar{R}$ is a division ring, using a similar argument as given in the proof of Lemma 1. By the argument of the above paragraph, we have $a^2 \in Z(\bar{R})$ unless $a^2 + a^8 = 0$. If $a^2 + a^8 = 0$, then $a^6 = 1 \in Z(\bar{R})$. In either case $a^6 \in Z(\bar{R})$ for all $a \in \bar{R}$. $\bar{R}$ is commutative by Lemma 1 of Belluce and others (1966). Now $J(R)$ is commutative and $xy - yx \in J(R)$ for all $x, y \in R$. By Lemma 1.5 of Herstein (1969) $xy - yx \in Z(R)$ for all $x, y \in R$. R is commutative, again by Lemma 1.5 of Herstein (1969).

THEOREM 1. *If R is a semiprime ring with* 1 *satisfying* (A), *then R is commutative.*

PROOF. Let $x, y \in J(R)$. Then $((1+x)(1+y)) - (1+x)^k(1+y)^k \in Z(R)$, where $k = m, m+1, m+2$. Since $(1+x)$ and $(1+y)$ are invertible, we use the argument of Lemma 1 to obtain

$$(1+x)(1+y)^2(1+x) = (1+y)(1+x)^2(1+y).$$

Thus $J(R)$ satisfies (C). By Lemma 4, $J(R)$ is commutative. $R/J(R)$ is semisimple and satisfies (A), hence is commutative by Lemma 1. Now R is commutative as in the proof of Lemma 4.

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