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RIGHT QUOTIENT RINGS OF A RIGHT LCM DOMAIN

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In this paper we continue our investigation of the class of right LCM domains which was introduced in [**2**]. A right LCM domain is an (not necessarily commutative) integral domain with unity in which the intersection of any two principal right ideals is again principal. In this note we study the right quotient rings of such a ring. In Section 1 we describe some of the characteristic properties of right quotient monoids with respect to which quotient rings are formed. Three particular types of quotient rings are described in Section 2. In Section 3 we relate the right ideals of a ring to those of its quotient ring. The right D-chain which is constructed for a principal right ideal domain in [**1**] is described in a more general context in Section 4. Each ring below is assumed to be a ring with unity having no proper divisors of zero.

1. Right quotient monoids. Let S be a submonoid of the monoid R^* of nonzero elements of a ring R. It is well known that if

$$aR \cap bS \neq \emptyset \text{ for each } a \in S, b \in R \tag{1}$$

then $K = RS^{-1} = \{ba^{-1} | b \in R, a \in S\}$ is a ring under operations that extend those of R. Addition and multiplication are carried out in K by using the fact that for each $a \in S$ and $b \in R$, $a^{-1}b = b'(a')^{-1}$ for $ab' = ba' \in aR \cap bS$ (see [**4**]).

A submonoid S of R^* satisfying (1) is a *right Ore system in* R. If R^* is a right Ore system in R then R is a *right Ore domain*. A submonoid S of R^* is *saturated* if for all $a, b \in R$,

$$ab \in S \text{ implies } a \in S \text{ and } b \in S.$$

We shall deal mainly with saturated right Ore systems; these are called *right quotient monoids in* R. If S is a right Ore system in R with $K = RS^{-1}$ then S is contained in the group of units U_K of K; furthermore, $U_K \cap R = S$ if and only if S is saturated. For example, the group of units U_R of R is a right quotient monoid in R. If R is a right Ore domain then R^* is a right quotient monoid in R and $K = R(R^*)^{-1}$ is the right quotient field of R.

For two nonzero elements a, b in a ring R the greatest common left divisor (gcld) of a and b is denoted by $(a, b)_l$ and the least common right multiple (lcrm) is denoted by $[a, b]_r$; $(a, b)_r$ and $[a, b]_l$ have analogous meanings. If two elements $a, b \in R$ have a nonzero common right multiple, say $ab' = ba' \neq 0$, and if $[a, b]_r$ exists, then it is easy to show that $(a', b')_r$ also exists and satisfies

$$ab' = ba' = [a, b]_r(a', b')_r. \tag{2}$$

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This is the case, for example, in a right LCM domain where $[a, b]_r$ is a generator of $aR \cap bR$ if this intersection is nonzero. Similar remarks hold with "right" replaced by "left"; the analogue of (2) is

$$ab' = ba' = (a, b)_l[a', b']_l$$

(see [**2**, Lemma 1]). If S is a right quotient monoid in R and if $a, b \in S$ have a lcrm then it is also a member of S since we have a relation (2) with $a' \in S$. Of course, $(a, b)_l \in S$ if it exists. Similarly the lclm and gcrd of two elements in S are again in S.

For two elements $a, a' \in R$ we define $a \operatorname{tr} a'$ if there is a relation $ab' = ba'$ which is both left and right coprime, that is, for which $(a, b)_l = (a', b')_r = 1$. In view of (2) we have $a \operatorname{tr} a'$ if and only if there exist $a', b' \in R$ such that $ab' = ba' = [a, b]_r$ and $(a, b)_l = 1$; if we require, in addition, that $(a, b)_l$ be a (right) linear combination of a and b then a and a' are similar (see [**4**]).

A submonoid S of R^* is said to be tr-*closed* if whenever $a \operatorname{tr} a'$ and $a \in S$ then $a' \in S$; if $a \in S$ if and only if $a' \in S$ we say that S is rtr-*closed*. The following theorem gives a useful characterization of right quotient monoids for a large class of right LCM domains.

THEOREM 1. *Let R be a right* LCM *domain in which each pair of elements having a nonzero common right multiple has a* hclf. *Let S be a saturated submonoid of R^*. Then S is a right quotient monoid in R if and only if*

(i) *each element a of S is* (*right*) *large, that is, $aR \cap bR \neq 0$ for each $b \in R^*$,*

(ii) *S is* tr-*closed.*

Proof. If S is a right quotient monoid in R then each element of S is large because of (1). Suppose $a \operatorname{tr} a'$ so that $ab' = ba' = [a, b]_r$ with $(a, b)_l = 1$. Choose $a_1 \in S$ and $b_1 \in R$ such that $ab_1 = ba_1$. Thus $a_1 = a'x$ for some $x \in R$, and a' is then in S since S is saturated. Conversely, if S is a saturated submonoid of R^* satisfying (i) and (ii) then for $a \in S$ and $b \in R$ we may choose $0 \neq ab' = ba' \in aR \cap bR$. If $(a, b)_l = d$ and $[a, b]_r = m$ then $a = da_1$, $b = db_1$, $a_1b' = b_1a' = [a_1, b_1]_r$, and $(a_1, b_1)_l = 1$. Consequently $a_1 \operatorname{tr} a'$; moreover $a_1 \in S$ and therefore $a' \in S$ and, this establishes (1).

2. Three particular types of right quotient rings.

We begin this section with a general theorem that applies to all right LCM domains. Recall that an element a of a ring R is (*right*) *large* if aR is a large right ideal, that is, $aR \cap xR \neq 0$ for each $x \in R^*$.

THEOREM 2. *Let R be a right* LCM *domain.*

(i) *The set L_R of all elements of R that are* (*right*) *large is a right quotient monoid in R.*

(ii) *R has a right quotient ring K such that $L_K = U_K$, namely, $K = R(L_R)^{-1}$.*

Proof. The set L_R is not empty since each unit is large. If $ab \in L_R$ then $a \in L_R$ since $abR \subset aR$. Also, for any $x \in R^*$ we have $abR \cap axR \neq 0$ which shows

$b \in L_R$. On the other hand if both a, $b \in L_R$ and $x \in R^*$ then we have $ax' = xa' \neq 0$, $bx'' = x'b' \neq 0$ and therefore $abx'' = x(a'b') \neq 0$ so that $ab \in L_R$. Finally, let $a \in L_R$, $b \in R$ with $[a, b]_r = ab' = ba'$. For any $x \in R^*$, $aR \cap bR \supset aR \cap bxR \neq 0$ so that $aR \cap bR \cap bxR \neq 0$, that is, $ba'R \cap bxR \neq 0$ and this shows $a' \in L_R$. Thus L_R is a right quotient monoid in R. To prove the second part let $z \in L_K$ where $K = R(L_R)^{-1}$; we assume, without loss in generality, that $z \in R$. For any $x \in R^*$ we have $zK \cap xK \neq 0$, say $zb_1a_1^{-1} = xb_2a_2^{-1}$. If $a_1a_2' = a_2a_1'$ (in L_R) then $zb_1a_2' = xb_2a_1'$ so that z is also large in R. Consequently z is a unit in K.

We turn to right LCM domains R that satisfy the following condition:

(3) $\quad [x, yz]_r = [x, y]_r, (x, yx)_l = (x, y)_l$ implies z is a unit.

It is shown in [**2**, Theorem 2] that for such rings, the factorization of an element into primes (nonunits that have no proper factors) is unique up to order of factors and projective factors. In general, two elements a, $a' \in R$ are (*right*) *projective* ($a \operatorname{pr} a'$) if there is a sequence $a = a_1, a_2, \ldots, a_n = a'$ in R such that either $a_i \operatorname{tr} a_{i+1}$ or $a_{i+1} \operatorname{tr} a_i$ for each i. In a ring R, the (*right*) *dimension*, $\dim(a)$, of a may be defined to be the supremum of the lengths of all maximal chains in the poset $\mathscr{L}(aR)$ of all principal right ideals of R containing aR. In a right LCM domain that satisfies (3), $\dim(a)$ is just the number of prime factors in any factorization of a if a is the product of primes, and ∞ otherwise.

If R is a right LCM domain in which each pair of elements having a nonzero common right multiple has a hclf then for each $z \in R^*$ the poset $\mathscr{L}(zR)$ is a lattice; the sup of aR and bR in $\mathscr{L}(zR)$ is $(a, b)_lR$ and the inf of aR and bR is the intersection $[a, b]_rR$ of aR and bR. If, in addition, R satisfies (3) then it is easy to check that each lattice $\mathscr{L}(zR)(z \in R^*)$ is modular; in fact, (3) is equivalent to the modular law [**2**, Theorem 1]. In such a ring, if $a \operatorname{tr} a'$ then the modular lattices $\mathscr{L}(aR)$ and $\mathscr{L}(a'R)$ are transposes and consequently isomorphic; in particular, $\dim(a) = \dim(a')$. Our remarks together with Theorem 2 indicate that the set

$$R' = \{a \in L_R | \dim(a) < \infty\}$$

is a saturated submonoid of L_R which is tr-closed, and rtr-closed if each element of R^* is large. We summarize with the following

THEOREM 3. *Let R be a right* LCM *domain satisfying* (3) *in which each pair of elements having a nonzero common right multiple has a* hclf. *The set R' of all large finite-dimensional elements of R is a right quotient monoid in R which is* rtr-*closed if R is a right Ore domain.*

The right quotient ring associated with R' in Theorem 3 will be considered more closely in Section 4.

Let R be a ring satisfying the hypotheses in Theorem 3. For a set I of primes in R, let S_I be the set of all elements of R' whose prime factors are projective to primes in I. In particular, if p is a prime in R and if I is the set of all primes in R *not* projective to p, we write S_p for S_I. Now S_I is a saturated submonoid of R'

because of the uniqueness of prime factorizations. In addition, if $a \operatorname{tr} a'$ and $a = p_1 \ldots p_n$ is a prime factorization of a then there exist primes $p_1', \ldots, p_n'$ in R such that $a' = p_1' \ldots p_n'$ and $p_i \operatorname{tr} p_i'$; this follows immediately from [**2**, Theorem 5 (part i)]. Therefore S_I is tr-closed. We have established the following

THEOREM 4. *Let R be a right* LCM *domain satisfying* (3) *in which each pair of elements having a nonzero common right multiple has a* hclf. *If I is a set of primes in R then S_I is a right quotient monoid in R. In particular, for any prime p in R, R has a right quotient ring at p, namely $K_p = R(S_p)^{-1}$.*

We remark that our results apply to the free associative algebra $R = D[X]$ over a commutative unique factorization domain D. Each $\mathscr{L}(zR)$ (for $z \in R^*$) is a finite-dimensional modular lattice [**3**, Theorem 8] so that R satisfies the hypotheses of Theorem 4. In this example we have $R' = L_R$.

3. The closure of a right ideal. Let S be a right quotient monoid in a ring R with $K = RS^{-1}$. If A is a right ideal of R then $AS^{-1} = \{as^{-1} | a \in A, s \in S\}$ is a right ideal of K and $AS^{-1} \cap R$ is a right ideal of R containing A. We define the *S-closure* of A by $\operatorname{cl}(A) = AS^{-1} \cap R$. A right ideal A of R is *S-closed* if $\operatorname{cl}(A) = A$. It is not difficult to prove that the poset of closed right ideals of R is a lattice which is isomorphic to the lattice of all right ideals of K; the isomorphism is given by $A \rightarrow AS^{-1}$ with the inverse given by $B \rightarrow B \cap R$.

An element $b \in R$ is *S-closed* if bR is a closed right ideal. In this case b is *right prime to S*, that is, b has no nonunit right factor that belongs to S. For, if $b = b_1a_1$ with $a_1 \in S$ then since a_1 is a unit in K we have $\operatorname{cl}(bR) = bK \cap R = b_1K \cap R = \operatorname{cl}(b_1R)$ from which we deduce $bR = b_1R$. Thus a is a unit in R. The converse, which may be established in a special case, is included in the following.

THEOREM 5. *Let R be a right* LCM *domain in which each pair of elements having a nonzero common right multiple has a* hclf. *Let S be a right quotient monoid in R which is* rtr-*closed and let $K = RS^{-1}$.*

(i) *An element $b \in R$ is S-closed if and only if b is right prime to S. Furthermore, if R has the* acc *for principal right ideals then*

(ii) *the S-closure of each principal right idea of R is also a principal right ideal,*

(iii) *each element $z \in R$ has a unique factorization $z = ba$ where $a \in S$ and b is right prime to S.*

Proof. Let $b \in R$ be right prime to S. Now $bR \subset \operatorname{cl}(bR)$; to establish the reverse inclusion let $z = brs^{-1} \in \operatorname{cl}(bR)$ with $m = [z, b]_r$, $d = (z, b)_l$. Then $z = dz_1, b = db_1, m = dm_1$ and $(z_1, b_1)_l = 1, [z_1, b_1]_r = m_1$. If $m_1 = z_1b_1' = b_1z_1'$ then $b_1 \operatorname{tr} b_1'$. Since $zs = br$ we have $zs = mx$ for some $x \in R$. Thus $s = b_1'x$ and $b_1' \in S$ since S is saturated; consequently $b_1 \in S$ because S is rtr-closed. Since b is right prime to S, b_1 must be a unit. Therefore $b = (z, b)_l$ and $z \in bR$ as desired. This shows that if b is right prime to S then b is S-closed. The converse follows by our remarks preceding the theorem.

For each $z \in R$ let $D_z = \{bR | z = ba \text{ for some } a \in S\}$. Using the acc for principal right ideals we may choose bR maximal in D_z. Thus $z = ba$ where $a \in S$ and b is right prime to S. Since $a \in S$, $zK = bK$ so that $\text{cl}(zR) = \text{cl}(bR) = bR$. This establishes (ii); the third part follows from the first two.

Let R be a right LCM domain such that each pair of elements having a nonzero common right multiple has a hclf. If S is a saturated right quotient monoid in R then the poset $\mathscr{L}(zR)'$ of closed right ideals in the lattice $\mathscr{L}(zR)$ is a sublattice of $\mathscr{L}(zR)$. The mapping $\varphi : \mathscr{L}(zR)' \to \mathscr{L}(zK)$ defined by $\varphi(bR) = bK$ is easily seen to be an order preserving injection. Also, if $bK \in \mathscr{L}(zK)$, then under the hypotheses of Theorem 5 we may assume b is right prime to S so that bR is S-closed; thus φ is also a surjection in this case. We summarize our remarks as follows.

THEOREM 6. *Let R be a right* LCM *domain in which each pair of elements having a nonzero common right multiple has a* hclf, *and assume R has the* acc *for principal right ideals. Let S be a right quotient monoid in R which is* rtr-*closed and let $K = RS^{-1}$. The poset $\mathscr{L}(zK)$ of principal right ideals of K containing zK and the lattice $\mathscr{L}(zR)'$ of closed principal right ideals of R containing zR are lattice isomorphic. In particular, K is also a right* LCM *domain satisfying the* acc *for principal right ideals in which each pair of elements having a nonzero common right multiple has a* hclf.

4. The right D-chain. Throughout this section R will denote a ring satisfying the acc for principal right ideals such that each pair of nonzero elements has a lcrm and hclf satisfying (3). Such a ring may be characterized as a right Ore right LCM domain R such that for each $z \in R^*$, $\mathscr{L}(zR)$ is a modular lattice satisfying the acc. Thus if S is a right quotient monoid in R which is rtr-closed then $K = RS^{-1}$ is a ring of the same kind as R by Theorem 6.

Since each element of R^* is large we see by Theorem 3 that the set R' of finite dimensional elements of R is a right quotient monoid in R which is rtr-closed. Since $K = R(R')^{-1}$ must be a ring of the same kind as R we may consider K' and iterate this procedure. This leads to the right D-chain for R which we shall describe presently. The construction which was first described in [**1**] for the particular case of a principal right ideal domain depends on the following.

LEMMA. *If $K = R(R')^{-1}$ then $K' \cap R$ is a right quotient monoid in R which is* rtr-*closed.*

Proof. Clearly $ab \in K' \cap R$ if and only if $a, b \in K' \cap R$. If $a \text{ tr } a'$ in R then $a \text{ tr } a'$ in K; for if $aR \cap bR = ba'R$ and $(a, b)_l = 1$ then clearly $ba'K = (aR \cap bR)S^{-1} = aK \cap bK$. Suppose $aK \subset dK$ and $bK \subset dK$ where $d \in R$. Since R has the acc for principal right ideals we may assume d is right prime to S. Thus $aR \subset aK \cap R \subset dK \cap R = dR$; similarly $bR \subset dR$ so that d is a unit since $(a, b)_l = 1$ in R. Thus $(a, b)_l = 1$ in K and $a \text{ tr } a'$ in K. Therefore $K' \cap R$ is rtr-closed and the conclusion of the lemma follows from Theorem 1.

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Using transfinite induction we construct the *right D-chain* $\{R^{(\alpha)}|0 \leqq \alpha \leqq \delta\}$ of right quotient monoids in R together with the associated chain $\{K_\alpha = R(R^{(\alpha)})^{-1}|0 \leqq \alpha \leqq \delta\}$ of right quotient rings of R as follows. Let $R^{(0)} = U_R$ and $K_0 = R$. Let $\alpha > 0$ be any ordinal; if $\beta < \alpha$ assume $R^{(\beta)}$ has been defined and is a right quotient monoid in R which is rtr-closed and let $K_\beta = R(R^{(\beta)})^{-1}$. We define $R^{(\alpha)}$ by

$$R^{(\alpha)} = \begin{cases} (K_{\alpha-1})' \cap R, & \text{if } \alpha \text{ is a nonlimit ordinal} \\ \bigcup_{\beta<\alpha} R^{(\beta)}, & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Our induction is valid by the lemma. Clearly $R^{(\beta)} \subset R^{(\alpha)}$ and $K_\beta \subset K_\alpha$ if $\beta \leqq \alpha$. Also $R^{(\alpha)} = R^{(\alpha+1)}$ for some ordinal α; if δ is the least such ordinal then $R^{(\delta)} = R^*$ so that K_δ is the right quotient field of R.

One of the important uses of the right D-chain is in describing the unique factorization property in R to which we now turn. By using Theorem 5 (iii) and finite induction we see that each element $a \in R^*$ may be written uniquely (up to unit factors) as

$$a = z_{\alpha_1} z_{\alpha_2} \ldots z_{\alpha_k} u$$

where α_i are nonlimit ordinals such that $\alpha_0 \geqq \alpha_1 > \ldots > \alpha_k$, $z_{\alpha_i} \in R^{(\alpha_i)}$ and is right prime to $R^{(\alpha_i-1)}$, and u is a unit in R (see [**1**, Theorem 2]). Each z_{α_i} may be characterized as follows. An element $x \in R^*$ is an *α-prime* if xR is maximal in $\{xR|\, x \in R^{(\alpha)}\backslash R^{(\alpha-1)}\}$ ($\alpha > 0$ a nonlimit ordinal). Note that 1-primes are the usual primes in R; if $\alpha > 1$, α-primes have infinite dimension in R and are prime in $K_{\alpha-1}$. Each z_{α_i} above is a product of α_i-primes which is unique up to order of factors and projective factors in K_{α_i-1}. Since the proofs are substantially the same as those in [**1**] we shall not repeat them here. We summarize in the following

THEOREM 7. *Let R be a ring satisfying the* acc *for principal right ideals in which each pair of nonzero elements has a* lcrm *and* hclf *satisfying* (3). *Let $\{R^{(\alpha)}|0 \leqq \alpha \leqq \delta\}$ be the right D-chain in R with $\{K_\alpha|0 \leqq \alpha \leqq \delta\}$ the associated chain of right quotient rings of R. Each element $a \in R^*$ has a unique (up to unit factors) factorization $a = z_{\alpha_1} \ldots z_{\alpha_k} u$ where $\alpha_0 \geqq \alpha_1 > \ldots > \alpha_k$, each z_{α_i} is a product of α_i-primes in R (which is unique up to order of factors and projective factors in K_{α_i-1}) and u is a unit in R.*

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