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UNIPOTENT ORBITAL INTEGRALS OF HECKE FUNCTIONS FOR GL(n)

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ABSTRACT. Let $G = \mathrm{GL}(n,F)$ where F is a p-adic field, and let $\mathcal{H}(G)$ denote the Hecke algebra of spherical functions on G. Let $u_1,\ldots,u_p$ denote a complete set of representatives for the unipotent conjugacy classes in G. For each $1 \le i \le p$, let μ_i be the linear functional on $\mathcal{H}(G)$ such that $\mu_i(f)$ is the orbital integral of f over the orbit of u_i. Waldspurger proved that the μ_i, $1 \le i \le p$, are linearly independent. In this paper we give an elementary proof of Waldspurger's theorem which provides concrete information about the Hecke functions needed to separate orbits. We also prove a twisted version of Waldspurger's theorem and discuss the consequences for $\mathrm{SL}(n,F)$.

1. **Introduction.** Let F be a locally compact, nonarchimedean local field of characteristic zero. Let $G = \mathrm{GL}(n,F)$ and let $K = \mathrm{GL}(n,R)$ where R is the ring of integers of F. Let $C_c^\infty(G)$ denote the set of locally constant, compactly supported, complex-valued functions on G and let $\mathcal{H}(G)$ denote the Hecke algebra of functions in $C_c^\infty(G)$ which are K bi-invariant.

For any $\gamma \in G$ we let G_γ denote the centralizer of $\gamma \in G$ and let

$$\Lambda(f,\gamma) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\,dx, \quad f \in C_c^\infty(G),$$

be the orbital integral of f over the orbit of γ. Thus for each $\gamma \in G$ we have a linear functional

$$f \longmapsto \Lambda(f,\gamma), \quad f \in C_c^\infty(G).$$

Write μ_γ for the restriction of this linear functional to $\mathcal{H}(G)$.

Let $u_1,\ldots,u_p$ be a complete set of representatives for the unipotent conjugacy classes of G. Then it is well known that the linear functionals

$$f \longmapsto \Lambda(f,u_i), \quad f \in C_c^\infty(G),\ 1 \le i \le p,$$

are linearly independent. In [W2], Waldspurger proved that they are still linearly independent when restricted to the Hecke algebra.

THEOREM 1.1 (WALDSPURGER).

$$\{\mu_{u_i}, 1 \le i \le p\}$$

is a linearly independent set of functionals on $\mathcal{H}(G)$.

As a consequence of Waldspurger's result, there exist $\phi_1,\ldots,\phi_p \in \mathcal{H}(G)$ so that $\Lambda(\phi_i,u_j) = \delta_{ij}, 1 \le i,j \le p$. Using the results of [V] we obtain the following germ expansion.

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COROLLARY 1.2. *Let $u_1, \ldots, u_p, \phi_1, \ldots, \phi_p$ be as above. Let $f \in C_c^\infty(G)$. Then there is a neighborhood U of 1 in G so that for all $\gamma \in U$,*

$$\Lambda(f,\gamma) = \sum_{i=1}^{p} \Lambda(f,u_i)\Lambda(\phi_i,\gamma).$$

In this paper we will give an elementary proof of Waldspurger's theorem which provides more concrete information about the Hecke functions needed to separate the unipotent orbits. In particular, we will produce diagonal matrices a_i, $1 \leq i \leq p$, so that if ψ_i is the characteristic function of the double coset Ka_iK, then the matrix with entries $\Lambda(\psi_j, u_i)$ is upper triangular with non-zero diagonal entries. We also prove the following twisted version of Waldspurger's results.

Let κ be an unramified unitary character of $F^\times$ such that $\kappa^n = 1$. Extend κ to a character of G by $\kappa(g) = \kappa(\det g)$ and let $G_0 = \{g \in G : \kappa(g) = 1\}$. Let $\gamma \in G$. Then if $G_\gamma \subset G_0$, let

$$\Lambda_\kappa(f,\gamma) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\kappa(x)\,dx, \quad f \in C_c^\infty(G),$$

be the twisted orbital integral of f over the orbit of γ. If $G_\gamma \not\subset G_0$, set $\Lambda_\kappa(f,\gamma) = 0$ for all $f \in C_c^\infty(G)$.

THEOREM 1.3. *Let $v_1, \ldots, v_q$ be a complete set of representatives for the unipotent conjugacy classes in G such that $G_{v_i} \subset G_0$. Then there are $\phi_1, \ldots, \phi_q \in \mathcal{H}(G)$ such that*

$$\Lambda_\kappa(\phi_i, v_j) = \begin{cases} 1, & \text{if } 1 \leq i = j \leq q; \\ 0, & \text{if } 1 \leq i \neq j \leq q. \end{cases}$$

As in the untwisted case, Theorem 1.3 is proven by constructing characteristic functions ψ_i of double cosets so that the matrix with entries $\Lambda_\kappa(\psi_i, v_j)$ is upper triangular with non-zero diagonal entries. An easy extension of germ expansions to the twisted case yields the following corollary.

COROLLARY 1.4. *Let $v_1, \ldots, v_q, \phi_1, \ldots, \phi_q$ be as above. Let $f \in C_c^\infty(G)$. Then there is a neighborhood U of 1 in G so that for all $\gamma \in U$,*

$$\Lambda_\kappa(f,\gamma) = \sum_{i=1}^{q} \Lambda_\kappa(f,v_i)\Lambda_\kappa(\phi_i,\gamma).$$

In [W1], Waldspurger proved a weaker version of Theorem 1.1, namely that unipotent orbital integrals are linearly independent when restricted to the Iwahori Hecke algebra. Hales then used this result and twisted analogues to produce the following linear independence result for $\mathrm{SL}(n,F)$ [H]. Let $G = \mathrm{GL}(n,F), G_1 = \mathrm{SL}(n,F)$, and $G_u = \{g \in \mathrm{GL}(n,F) : \det g \in R^\times\}$. Let $B \subset K$ be the Iwahori subgroup of G and $K_1 = K \cap G_1, B_1 = B \cap G_1$. Let $\mathcal{H}(G_1, K_1)$ be the Hecke algebra of K_1 bi-invariant functions in $C_c^\infty(G_1)$ and

$\mathcal{H}(G_1, B_1)$ the Iwahori Hecke algebra of B_1 bi-invariant functions. If $n_1, n_2 \in G_1$ are unipotent elements which are conjugate via an element of G_u, it is easy to see that the linear functionals $\mu_{n_i}, i = 1, 2$, coming from restricting the corresponding orbital integrals to $\mathcal{H}(G_1, B_1)$ are equal up to a scalar. Thus it is at most possible to separate G_u conjugacy classes of unipotent elements of $\mathrm{SL}(n, F)$ using functions in $\mathcal{H}(G_1, B_1)$. Hales proves that in fact, if $n_1, \ldots, n_s$ are a complete set of representatives for the G_u-conjugacy classes of unipotent elements of $\mathrm{SL}(n, F)$, then the linear functionals μ_{n_i} are linearly independent on $\mathcal{H}(G_1, B_1)$. However, Hales argument does not extend to the case of $\mathcal{H}(G_1, K_1)$. In the last section we will show that $\{\mu_{n_i}, 1 \leq i \leq s\}$, are not linearly independent when restricted to $\mathcal{H}(G_1, K_1)$ in the case that $G_1 = \mathrm{SL}(3, F)$.

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2. **Unipotent orbital integrals.** Let F be a locally compact, nonarchimedean local field with ring of integers R. Let τ be a generator of the prime ideal P of R. Thus R/P is a finite field with q elements for some prime power q and $|\tau|_F = q^{-1}$. Let $G = \mathrm{GL}(n, F)$ and $K = \mathrm{GL}(n, R)$. Let $C_c^\infty(G)$ denote the set of locally constant, compactly supported, complex-valued functions on G and let $\mathcal{H} = \{\phi \in C_c^\infty(G) : \phi(k_1 g k_2) = \phi(g), g \in G, k_1, k_2 \in K\}$. For any $\gamma \in G$ we let G_γ denote the centralizer of $\gamma \in G$ and let

$$\Lambda(f, \gamma) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\, dx, \quad f \in C_c^\infty(G),\ \gamma \in G$$

be the orbital integral of f over the orbit of γ.

Unipotent conjugacy classes in G can be indexed by partitions of n as follows. Let $\mathbf{P} = \mathbf{P}_n$ be the set of all partitions of n, that is the set of all $\mathcal{P} = (p_1, \ldots, p_n), p_1 \geq p_2 \geq \cdots \geq p_n \geq 0$ with $p_1 + \cdots + p_n = n$. We will write $l(\mathcal{P})$ for the number of non-zero entries in $\mathcal{P}$, and will sometimes also write $\mathcal{P} = (p_1, \ldots, p_t)$ where $t = l(\mathcal{P})$. Given $\mathcal{Q} = (q_1, \ldots, q_t) \in \mathbf{P}, t = l(\mathcal{Q})$, let $N_{\mathcal{Q}}$ be the set of block upper triangular matrices in G of the form

$$Y = \begin{pmatrix} I_{q_1} & Y_{12} & \ldots & Y_{1t} \\ 0 & I_{q_2} & \ldots & Y_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_{q_t} \end{pmatrix}$$

where the blocks are of sizes $q_i \times q_j, 1 \leq i, j \leq t, I_{q_i} \in M(q_i, q_i, F)$ is the identity matrix for all $1 \leq i \leq t$, and $Y_{ij} \in M(q_i, q_j, F)$ has arbitrary entries in F for $i < j$. Now if $\mathcal{Q} \leftrightarrow u_{\mathcal{Q}}$, then $u_{\mathcal{Q}} \in N_{\mathcal{Q}}$ and for all $\phi \in \mathcal{H}$,

$$\Lambda(\phi, u_{\mathcal{Q}}) = \int_{N_{\mathcal{Q}}} \phi(n)\, dn.$$

(See [A-C, 3.10].)

We will also index certain Hecke functions by partitions. For $\mathcal{P} = (p_1, \ldots, p_n) \in \mathbf{P}$ as above, let $a_{\mathcal{P}}$ be the diagonal matix in G with diagonal entries $\tau^{p_1 - 1}, \tau^{p_2 - 1}, \ldots, \tau^{p_n - 1}$. Let $\phi_{\mathcal{P}}$ be the characteristic function of the double coset $K a_{\mathcal{P}} K$. Then $\phi_{\mathcal{P}} \in \mathcal{H}$ and $\Lambda(\phi_{\mathcal{P}}, u_{\mathcal{Q}})$ is equal to the measure of $K a_{\mathcal{P}} K \cap N_{\mathcal{Q}}$.

Let $\mathcal{P} = (p_1, \ldots, p_n)$, $\mathcal{P}' = (p'_1, \ldots, p'_n) \in \mathbf{P}$ where as usual $p_1 \geq p_2 \geq \cdots \geq p_n \geq 0$ and $p'_1 \geq p'_2 \geq \cdots \geq p'_n \geq 0$. Then we say $\mathcal{P} < \mathcal{P}'$ if there is $1 \leq k \leq n$ so that $p_i = p'_i$, $1 \leq i \leq k-1$, and $p_k < p'_k$. We also say $\mathcal{P} \prec \mathcal{P}'$ if there is $1 \leq k \leq n$ so that $p_i = p'_i$, $k+1 \leq i \leq n$ and $p_k < p'_k$.

Finally we define the transpose $\Psi: \mathbf{P} \longrightarrow \mathbf{P}$ as follows. Let $\mathcal{P} \in \mathbf{P}$ and write $\mathcal{P} = (p_1, \ldots, p_n), p_1 \geq p_2 \geq \cdots \geq p_n \geq 0$. For each $1 \leq i \leq n$, write $m(\mathcal{P}, i)$ for the multiplicity of i in $\mathcal{P}$, that is the number of indices $j \in \{1, 2, \ldots, n\}$ such that $p_j = i$. Then we define $\Psi(\mathcal{P}) = \mathcal{Q} = (q_1, \ldots, q_n)$ where for $1 \leq i \leq n, q_i = m(\mathcal{P}, i) + m(\mathcal{P}, i+1) + \cdots + m(\mathcal{P}, n)$. Then $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$ and $\sum_{i=1}^n q_i = \sum_{i=1}^n m(\mathcal{P}, i) i = n$. Thus $\mathcal{Q} \in \mathbf{P}$. Further, Ψ is injective since if $\mathcal{P}, \mathcal{P}' \in \mathbf{P}$ with $\Psi(\mathcal{P}) = \Psi(\mathcal{P}')$ it is easy to see that $m(\mathcal{P}, i) = m(\mathcal{P}', i)$ for all $1 \leq i \leq n$ so that $\mathcal{P} = \mathcal{P}'$. Thus Ψ is a bijection. It is also easy to prove that Ψ^2 is the identity map.

We want to prove the following theorem. Number the elements of $\mathbf{P}$ so that $\mathcal{P}_1 < \mathcal{P}_2 < \cdots < \mathcal{P}_m$, and let $u_i = u_{\Psi(\mathcal{P}_i)} \in N_{\Psi(\mathcal{P}_i)}$, $1 \leq i \leq m$, represent the unipotent conjugacy classes. Let ϕ_i be the characteristic function of $Ka_{\mathcal{P}_i}K$.

THEOREM 2.1. *For all $1 \leq j < i \leq m$ we have $\Lambda(\phi_i, u_j) = 0$. Further, $\Lambda(\phi_i, u_i) \neq 0$ for all $1 \leq i \leq m$.*

LEMMA 2.2. *If $\mathcal{P} < \mathcal{P}'$, then $\Psi(\mathcal{P}) \prec \Psi(\mathcal{P}')$.*

PROOF. Suppose that $\mathcal{P} < \mathcal{P}'$. Write $\Psi(\mathcal{P}) = \mathcal{Q} = (q_1, \ldots, q_n)$ and $\Psi(\mathcal{P}') = \mathcal{Q}' = (q'_1, \ldots, q'_n)$. Then there is $1 \leq k \leq n$ so that $m(\mathcal{P} : i) = m(\mathcal{P}' : i), k+1 \leq i \leq n$, and $m(\mathcal{P} : k) < m(\mathcal{P}' : k)$. Thus $q_i = q'_i$ for $k+1 \leq i \leq n$ and $q_k < q'_k$. Thus $\mathcal{Q} \prec \mathcal{Q}'$. ∎

PROPOSITION 2.3. *Let $\mathcal{P}, \mathcal{Q} \in \mathbf{P}$. Then $Ka_{\mathcal{P}}K \cap N_{\mathcal{Q}} = \emptyset$ unless $\Psi(\mathcal{P}) \preceq \mathcal{Q}$. Further, $Ka_{\mathcal{P}}K \cap N_{\Psi(\mathcal{P})} \neq \emptyset$.*

Proposition 2.3 will be proven in a series of lemmas. However, before beginning the proof of Proposition 2.3, let us see how it implies Theorem 2.1.

PROOF OF THEOREM 2.1. If $j < i$ we have $\mathcal{P}_j < \mathcal{P}_i$ so that by Lemma 2.2 $\Psi(\mathcal{P}_j) \prec \Psi(\mathcal{P}_i)$. Now using Proposition 2.3, $Ka_{\mathcal{P}_i}K \cap N_{\Psi(\mathcal{P}_j)} = \emptyset$, so that $\Lambda(\phi_i, u_j)$ which equals the measure of $Ka_{\mathcal{P}_i}K \cap N_{\Psi(\mathcal{P}_j)}$ is 0. Further, for each i, the measure of $Ka_{\mathcal{P}_i}K \cap N_{\Psi(\mathcal{P}_i)}$ is non-zero since $Ka_{\mathcal{P}_i}K \cap N_{\Psi(\mathcal{P}_i)}$ is a non-empty open subset of $N_{\Psi(\mathcal{P}_i)}$. Thus $\Lambda(\phi_i, u_i) \neq 0$ for all i. ∎

LEMMA 2.4. *Let G_1 be the set of all $g \in G$ such that $|\det g|_F = 1$ and $\|g\|_\infty = \sup_{1 \leq i,j \leq n} |g_{ij}|_F \in \{1, q\}$. Then $G_1 = \cup_{\mathcal{P} \in \mathbf{P}} Ka_{\mathcal{P}}K$.*

PROOF. Note that for every $\mathcal{P} \in \mathbf{P}$, $\det a_{\mathcal{P}} = 1$. Further, $\|a_{\mathcal{P}}\|_\infty = q$ unless $\mathcal{P} = (1, \ldots, 1)$ in which case $\|a_{\mathcal{P}}\|_\infty = 1$. Thus $\cup_{\mathcal{P} \in \mathbf{P}} Ka_{\mathcal{P}}K \subseteq G_1$. Conversely, every $g \in G_1$ is in some double coset KaK where a is a diagonal matrix with diagonal entries $a_i = \tau^{n_i}, 1 \leq i \leq n$, where the n_i are integers. But now $a \in KgK$ implies that $|\det a|_F = |\det g|_F$ and $\|a\|_\infty = \|g\|_\infty \in \{1, q\}$. Thus $\sum n_i = 0$ and $\inf n_i \in \{-1, 0\}$. It is easy to see that up to permutation of the diagonal entries, the $a_{\mathcal{P}}, \mathcal{P} \in \mathbf{P}$ are the only a's with these properties. ∎

We will prove Proposition 2.3 by induction on n. Write

$$n' = n-1, \quad \mathbf{P}' = \mathbf{P}_{n'}, \quad G' = \mathrm{GL}(n', F), \quad K' = \mathrm{GL}(n', R)$$

and $G_1' = \{g' \in G' : |\det g'|_F = 1, \|g'\|_\infty \in \{1, q\}\}$. Fix $\mathcal{Q} = (q_1, q_2, \ldots, q_t) \in \mathbf{P}$ where $t = l(\mathcal{Q})$ and $q_1 \geq q_2 \geq \cdots \geq q_t > 0$ and write $\mathcal{Q}_1 = (q_1, \ldots, q_t - 1) \in \mathbf{P}'$. Suppose that $Y \in N_{\mathcal{Q}}$. Note that $Y \in G_1$ just in case all blocks $Y_{ij} \in M(q_i, q_j, F)$, $1 \leq i < j \leq t$, have entries in $P^{-1} = \{x \in F : |x|_F \leq q\}$. Thus we can write $Y \in N_{\mathcal{Q}} \cap G_1$ in block form as

$$Y = \begin{pmatrix} Y' & Y_{12} \\ 0 & 1 \end{pmatrix}$$

where $Y' \in N_{\mathcal{Q}_1} \cap G_1'$, $Y_{12} \in M(n', 1, P^{-1})$, and $0 \in M(1, n', F)$ denotes the zero matrix.

For $x, y \in G$, we will write $x \sim y$ if $KxK = KyK$.

LEMMA 2.5. *Suppose that $Y \in N_{\mathcal{Q}} \cap G_1$ as above. Then there are $\mathcal{P}' \in \mathbf{P}'$ and $W_{12} \in M(n', 1, P^{-1})$ so that*

$$Y \sim \begin{pmatrix} a_{\mathcal{P}'} & W_{12} \\ 0 & 1 \end{pmatrix}.$$

PROOF. Since $Y' \in G_1'$, by Lemma 2.4 there are $A', B' \in K'$ and $\mathcal{P}' \in \mathbf{P}'$ so that $A'Y'B' = a_{\mathcal{P}'}$. Now

$$\begin{pmatrix} A' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Y' & Y_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B' & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{\mathcal{P}'} & W_{12} \\ 0 & 1 \end{pmatrix}$$

where $W_{12} = A'Y_{12}$. ∎

Let $\mathcal{P}' = (p_1, \ldots, p_r) \in \mathbf{P}'$ as in Lemma 2.5 where $l(\mathcal{P}') = r$ so that $a_{\mathcal{P}'}$ has diagonal entries $a_1 = \tau^{p_1 - 1}, a_2 = \tau^{p_2 - 1}, \ldots, a_r = \tau^{p_r - 1} \in R$ and $a_{r+1} = \tau^{-1}, \ldots, a_{n'} = \tau^{-1}$. Write

$$a_{\mathcal{P}'} = \begin{pmatrix} A & 0 \\ 0 & \tau^{-1} I \end{pmatrix}$$

where $A \in M(r, r, F)$ is the diagonal matrix with entries $a_1, \ldots, a_r$ and $I \in M(n' - r, n' - r, F)$ is the identity matrix.

LEMMA 2.6. *With notation as above, there is $X_{13} \in M(r, 1, P^{-1})$ so that*

$$Y \sim \begin{pmatrix} A & 0 & X_{13} \\ 0 & \tau^{-1} I & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

PROOF. Let $X_{13} \in M(r, 1, P^{-1})$ denote the first r rows of W_{12} and let $X_{23} \in M(n' - r, 1, P^{-1})$ denote the last $n' - r$ rows of W_{12}. Then

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & -\tau X_{23} \\ 0 & 0 & 1 \end{pmatrix} \in K$$

and

$$\begin{pmatrix} A & 0 & X_{13} \\ 0 & \tau^{-1} I & X_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & -\tau X_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 & X_{13} \\ 0 & \tau^{-1} I & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

∎

LEMMA 2.7. *Suppose that the notation is as above. If* $X_{13} \in M(r,1,R)$*, then* $Y \in Ka_{\mathcal{P}}K$ *where* $\mathcal{P} = (p_1,\ldots,p_r,1)$*. If* $X_{13} \notin M(r,1,R)$*, then there is* $1 \leq i \leq r$ *so that* $Y \in Ka_{\mathcal{P}}K$ *where* $\mathcal{P} = (p_1,\ldots,p_{i-1},p_i+1,p_{i+1},\ldots,p_r)$.

PROOF. X_{13} has entries $w_1,\ldots,w_r \in P^{-1}$. By permuting the indices if necessary, we can assume that there is $0 \leq s \leq r$ so that $w_1,\ldots,w_s \notin R$ and $w_{s+1},\ldots,w_r \in R$. Write $X_{14} \in M(s,1,P^{-1})$ for the matrix with entries $w_1,\ldots,w_s$ and $X_{24} \in M(r-s,1,R)$ for the matrix with entries $w_{s+1},\ldots,w_r$. Write $A_1 \in M(s,s,R)$ for the diagonal matrix with entries $a_1,\ldots,a_s$ and $A_2 \in M(r-s,r-s,R)$ for the diagonal matrix with entries $a_{s+1},\ldots,a_r$. Note that since we have permuted the entries we can no longer assume that $p_1 \geq p_2 \geq \cdots \geq p_r$. We can and will assume however that $p_1 \geq p_2,\ldots,p_s$. Now

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & -X_{24} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in K$$

and

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & -X_{24} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 & 0 & 0 & X_{14} \\ 0 & A_2 & 0 & X_{24} \\ 0 & 0 & \tau^{-1}I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1 & 0 & 0 & X_{14} \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \tau^{-1}I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular, we now see that if $s = 0$, then

$$Y \sim \begin{pmatrix} A & 0 & 0 \\ 0 & \tau^{-1}I & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{\mathcal{P}'} & 0 \\ 0 & 1 \end{pmatrix}.$$

Up to a permutation of the diagonal entries, this last matrix is equal to $a_{\mathcal{P}}$ where $\mathcal{P} = (p_1,\ldots,p_r,1)$.

Now assume that $s > 0$. For $1 \leq i \leq s$ we can write $w_i = u_i\tau^{-1}$ where $u_i \in R^\times$ is a unit. Let $U \in M(s,s,F)$ be the diagonal matrix with diagonal entries $u_1,\ldots,u_s$. Then $U^{-1}X_{14} = \tau^{-1} \in M(s,1,P^{-1})$ has every entry equal to τ^{-1}. Since

$$\begin{pmatrix} U & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in K$$

and

$$\begin{pmatrix} U^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 & 0 & 0 & X_{14} \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \tau^{-1}I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} U & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1 & 0 & 0 & U^{-1}X_{14} \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \tau^{-1}I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

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we now have

$$Y \sim \begin{pmatrix} A_1 & 0 & 0 & \tau^{-1} \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \tau^{-1}I & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally, write

$$A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & A_3 \end{pmatrix}$$

where $A_3 \in M(s-1, s-1, R)$ is the diagonal matrix with diagonal entries $a_2, \dots, a_s$. Recall that we are assuming that $a_i = \tau^{p_i - 1}$ where $p_1 \geq p_2, \dots, p_s$. Thus if we write $a_1 \in M(s-1, 1, R)$ for the matrix with all entries equal to a_1, then $A_3^{-1}a_1 \in M(s-1, 1, R)$ since it has entries $a_i^{-1}a_1 = \tau^{p_1 - p_i}, 2 \leq i \leq r$. Thus we have

$$C = \begin{pmatrix} \tau & 0 & 0 & 0 & -1 \\ -1 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ A_3^{-1}a_1 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ -\tau a_1 & 0 & 0 & 0 & 1 \end{pmatrix} \in K$$

and

$$C \begin{pmatrix} a_1 & 0 & 0 & 0 & \tau^{-1} \\ 0 & A_3 & 0 & 0 & \tau^{-1} \\ 0 & 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & \tau^{-1}I & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} D = \begin{pmatrix} \tau a_1 & 0 & 0 & 0 & 0 \\ 0 & A_3 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 \\ 0 & 0 & 0 & \tau^{-1}I & 0 \\ 0 & 0 & 0 & 0 & \tau^{-1} \end{pmatrix}.$$

Thus $Y \sim a_{\mathcal{P}}$ where $\mathcal{P}$ has non-zero entries $p_1 + 1, p_2, \dots, p_s, p_{s+1}, \dots, p_r$. However, we do not know that p_1 is the largest of the p_i's, just that it is the largest among $p_1, \dots, p_s$. Thus we just know that $l(\mathcal{P}) = l(\mathcal{P}')$ and that one non-zero entry p_i in $\mathcal{P}'$ has been replaced by $p_i + 1$. ∎

LEMMA 2.8. *Suppose that $\mathcal{P}, \mathcal{Q} \in \mathbf{P}$ with $Ka_{\mathcal{P}}K \cap N_{\mathcal{Q}} \neq \emptyset$. Then $l(\mathcal{Q}) \geq l\big(\Psi(\mathcal{P})\big)$.*

PROOF. Suppose that $Y \in Ka_{\mathcal{P}}K \cap N_{\mathcal{Q}}$. Then $Y^{-1} \in N_{\mathcal{Q}} \cap Ka_{\mathcal{P}}^{-1}K$. Suppose that p_1 is the largest entry in $\mathcal{P}$. Then $l\big(\Psi(\mathcal{P})\big) = p_1$ and $\|a_{\mathcal{P}}^{-1}\|_\infty = |\tau^{-p_1+1}|_F = q^{p_1-1}$. Thus $\|Y^{-1}\|_\infty = q^{p_1-1}$. But if $l(\mathcal{Q}) = t$, then Y is a $t \times t$ block upper triangular matrix with identity matrices on the diagonal and entries above the diagonal in P^{-1}. Now it is easy to see that $\|Y^{-1}\|_\infty \leq q^{t-1}$. Thus $p_1 \leq t$. ∎

LEMMA 2.9. *Suppose that $\mathcal{P}, \mathcal{Q} \in \mathbf{P}$ with $Ka_{\mathcal{P}}K \cap N_{\mathcal{Q}} \neq \emptyset$. Then $\Psi(\mathcal{P}) \preceq \mathcal{Q}$.*

PROOF. By Lemma 2.8 we know that $l\big(\Psi(\mathcal{P})\big) \leq l(\mathcal{Q})$. Now if $l\big(\Psi(\mathcal{P})\big) < l(\mathcal{Q})$, then $\Psi(\mathcal{P}) \prec \mathcal{Q}$. Thus we may as well assume that $l\big(\Psi(\mathcal{P})\big) = l(\mathcal{Q})$.

The proof of the lemma is by induction on n. It is clearly true for $n = 1$ since there is only one partition. Assume that $n > 1$ and that the lemma is true for $n' = n - 1$. Suppose that $Y \in Ka_{\mathcal{P}}K \cap N_{\mathcal{Q}}$ where $Q = (q_1, \dots, q_t)$ and as before let $\mathcal{Q}_1 = (q_1, \dots, q_t - 1)$. Note that

$$l(\mathcal{Q}_1) = \begin{cases} t, & \text{if } q_t > 1; \\ t - 1, & \text{if } q_t = 1. \end{cases}$$

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Then $Y \in N_{\mathcal{Q}} \cap G_1$ so we can write Y as before as

$$Y = \begin{pmatrix} Y' & Y_{12} \\ 0 & 1 \end{pmatrix}$$

where $Y' \in N_{\mathcal{Q}_1} \cap G_1'$, $Y_{12} \in M(n', 1, P^{-1})$, and $0 \in M(1, n', F)$ denotes the zero matrix. Now there is $\mathcal{P}' \in \mathbf{P}'$ such that $Y' \in K' a_{\mathcal{P}'} K'$. Thus $K' a_{\mathcal{P}'} K' \cap N_{\mathcal{Q}_1} \neq \emptyset$, so by the induction hypothesis, we have $\Psi(\mathcal{P}') \preceq \mathcal{Q}_1$. We will use Lemma 2.7 to see that this implies that $\Psi(\mathcal{P}) \preceq \mathcal{Q}$.

CASE 1. Suppose that we are in the case of Lemma 2.7 that $X_{13} \in M(r, 1, R)$. Thus in this case if $\mathcal{P}' = (p_1, \ldots, p_r)$ we have $\mathcal{P} = (p_1, \ldots, p_r, 1)$. Thus $l\big(\Psi(\mathcal{P})\big) = l\big(\Psi(\mathcal{P}')\big) = p_1$. Now $\Psi(\mathcal{P}') \preceq \mathcal{Q}_1$ implies that $l\big(\Psi(\mathcal{P}')\big) \leq l(\mathcal{Q}_1)$. But then $l\big(\Psi(\mathcal{P})\big) = l\big(\Psi(\mathcal{P}')\big) \leq l(\mathcal{Q}_1) \leq l(\mathcal{Q}) = l\big(\Psi(\mathcal{P})\big)$. Thus $l\big(\Psi(\mathcal{P}')\big) = l(\mathcal{Q}_1)$ and $l(\mathcal{Q}_1) = l(\mathcal{Q})$. Thus $t = p_1$ and $q_t > 1$. Now $m(\mathcal{P}, 1) = m(\mathcal{P}', 1) + 1$ and $m(\mathcal{P}, i) = m(\mathcal{P}', i)$ for $i \neq 1$. Note $m(\mathcal{P}, n) = 0$ since $1 \in \mathcal{P}$. Thus if we write $\Psi(\mathcal{P}') = (m_1, m_2, \ldots, m_t)$, then $\Psi(\mathcal{P}) = (m_1 + 1, m_2, \ldots, m_t)$. Recall that $\mathcal{Q} = (q_1, \ldots, q_t)$, $\mathcal{Q}_1 = (q_1, \ldots, q_t - 1)$. Now $\Psi(\mathcal{P}') \preceq \mathcal{Q}_1$ imples that $m_t \leq q_t - 1 < q_t$. Thus $\Psi(\mathcal{P}) \prec \mathcal{Q}$ unless $t = 1$ in which case $\Psi(\mathcal{P}) = \mathcal{Q} = (n)$.

CASE 2. Suppose that we are in the case of Lemma 2.7 that $X_{13} \notin M(r, 1, R)$. Write $\mathcal{P}' = (p_1, \ldots, p_r)$ as above. Then there is $1 \leq i \leq r$ so that $\mathcal{P}$ is obtained from $\mathcal{P}'$ by replacing p_i with $p_i + 1$. That is there is $1 \leq b = p_i \leq n - 1$ so that $m(\mathcal{P}, b) = m(\mathcal{P}', b) - 1$, $m(\mathcal{P}, b + 1) = m(\mathcal{P}', b + 1) + 1$, and $m(\mathcal{P}, j) = m(\mathcal{P}', j)$, $j \neq b, b + 1$. Thus if we write $\Psi(\mathcal{P}') = (m_1, \ldots, m_k)$, then $\Psi(\mathcal{P}) = (m_1, \ldots, m_b, m_{b+1} + 1, \ldots, m_k)$. Note that $l\big(\Psi(\mathcal{P})\big) = l\big(\Psi(\mathcal{P}')\big) = k$ unless $b = k$, in which case $l\big(\Psi(\mathcal{P})\big) = l\big(\Psi(\mathcal{P}')\big) + 1$. We divide further into cases according to whether $q_t = 1$ or $q_t > 1$ and $b = k$ or $b < k$.

CASE 2A. Suppose that $q_t = 1$ and $b = k$. Then $t = l(\mathcal{Q}) = l(\mathcal{Q}_1) + 1 = l\big(\Psi(\mathcal{P})\big) = l\big(\Psi(\mathcal{P}')\big) + 1 = b + 1$. Thus $\mathcal{Q}, \mathcal{Q}_1, \Psi(\mathcal{P}), \Psi(\mathcal{P}')$ have the forms:

$$\mathcal{Q} = (q_1, \ldots, q_{t-1}, 1), \qquad \mathcal{Q}_1 = (q_1, \ldots, q_{t-1}),$$
$$\Psi(\mathcal{P}) = (m_1, \ldots, m_{t-1}, 1), \qquad \Psi(\mathcal{P}') = (m_1, \ldots, m_{t-1}).$$

Clearly $\Psi(\mathcal{P}') \preceq \mathcal{Q}_1$ implies that $\Psi(\mathcal{P}) \preceq \mathcal{Q}$.

CASE 2B. Suppose that $q_t = 1$ and $b < k$. Then $l\big(\Psi(\mathcal{P})\big) = l\big(\Psi(\mathcal{P}')\big) \leq l(\mathcal{Q}_1) < l(\mathcal{Q})$. This contradicts our assumption that $l\big(\Psi(\mathcal{P})\big) = l(\mathcal{Q})$. Thus this case doesn't occur.

CASE 2C. Suppose that $q_t > 1$ and $b = k$. In this case $\mathcal{Q} = (q_1, \ldots, q_t)$ and $\Psi(\mathcal{P}) = (m_1, \ldots, m_{t-1}, 1)$. Since $1 < q_t$ we have $\Psi(\mathcal{P}) \prec \mathcal{Q}$.

CASE 2D. Suppose that $q_t > 1$ and $b < k$. Now $l(\mathcal{Q}) = l(\mathcal{Q}_1) = l\big(\Psi(\mathcal{P}')\big) = l\big(\Psi(\mathcal{P})\big)$. Thus $t = k$ and $\Psi(\mathcal{P}') \preceq \mathcal{Q}_1$ implies that $m_t \leq q_t - 1 < q_t$. This implies that $\Psi(\mathcal{P}) \prec \mathcal{Q}$ unless $t = k = b + 1$ so that $\mathcal{Q}, \mathcal{Q}_1, \Psi(\mathcal{P}), \Psi(\mathcal{P}')$ are of the form

$$\mathcal{Q} = (q_1, \ldots, q_{t-1}, q_t), \qquad \mathcal{Q}_1 = (q_1, \ldots, q_{t-1}, q_t - 1)$$
$$\Psi(\mathcal{P}) = (m_1, \ldots, m_{t-1}, m_t + 1), \qquad \Psi(\mathcal{P}') = (m_1, \ldots, m_{t-1}, m_t).$$

Again it is clear that $\Psi(\mathcal{P}') \preceq \mathcal{Q}_1$ implies that $\Psi(\mathcal{P}) \preceq \mathcal{Q}$. ∎

LEMMA 2.10. *For any* $\mathcal{P} \in \mathbf{P}, N_{\Psi(\mathcal{P})} \cap Ka_{\mathcal{P}}K \neq \emptyset$.

PROOF. For any $m \geq 1$ let $\mathcal{P}_m = (m) \in \mathbf{P}_m$ and write $a(m) = a_{\mathcal{P}_m} \in \mathrm{GL}(m, F)$. Thus

$$a(m) = \begin{pmatrix} \tau^{m-1} & 0 & \dots & 0 \\ 0 & \tau^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tau^{-1} \end{pmatrix}.$$

Write

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \tau & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & \tau & 1 \\ 1 & 0 & 0 & \dots & 0 & \tau \end{pmatrix}, \quad B = \begin{pmatrix} (-1)^{m-1} & 0 & \dots & 0 & 0 \\ (-1)^{m-2}\tau & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\tau^{m-2} & 0 & \dots & 1 & 0 \\ \tau^{m-1} & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then $A, B \in K_m$ and $Aa(m) = u(m)B$ where

$$u(m) = \begin{pmatrix} 1 & \tau^{-1} & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \tau^{-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

is the unipotent matrix with all superdiagonal entries equal to τ^{-1}.

Now let $\mathcal{P} = (p_1, \dots, p_r)$ where $p_1 \geq \dots \geq p_r$. Then $a_{\mathcal{P}}$ is conjugate via a permutation matrix in K to a block diagonal matrix with diagonal blocks $a(p_1), \dots, a(p_r)$. Thus if we let $u(\mathcal{P})$ be the block diagonal matrix with diagonal blocks $u(p_1), \dots, u(p_r)$, then by the above we have $u(\mathcal{P}) \in Ka_{\mathcal{P}}K$. Finally, we claim that there is a permutation matrix $A \in K$ so that $Au(\mathcal{P})A^{-1} \in N_{\Psi(\mathcal{P})}$. This would show that $Au(\mathcal{P})A^{-1} \in N_{\Psi(\mathcal{P})} \cap Ka_{\mathcal{P}}K$.

We have $\mathcal{P} = (p_1, \dots, p_r), p_1 \geq \dots \geq p_r$ and $\mathcal{Q} = \Psi(\mathcal{P}) = (q_1, \dots, q_t), q_1 \geq \dots \geq q_t$. Write $M_0 = N_0 = 0, M_i = p_1 + \dots + p_i, 1 \leq i \leq r, N_j = q_1 + \dots + q_j, 1 \leq j \leq t$. Now for each $1 \leq m \leq n$ there are unique $1 \leq k \leq r$ and $1 \leq i \leq p_k$ such that $m = M_{k-1} + i$. Similarly there are unique $1 \leq i \leq t$ and $1 \leq k \leq q_i$ so that $m = N_{i-1} + k$. Now we define a permutation σ of $\{1, 2, \dots, n\}$ by setting $\sigma(M_{k-1} + i) = N_{i-1} + k$ where $1 \leq k \leq r$ and $1 \leq i \leq p_k$. Note that since $i \leq p_k$, we have $q_i \geq k$.

Now

$$u(\mathcal{P}) = I + \sum_{k=1}^{r} \sum_{i=1}^{p_k - 1} \tau^{-1} E(M_{k-1} + i, M_{k-1} + i + 1)$$

where I denotes the identity matrix and $E(i,j)$ denotes the matrix with 1 in the (ij) place and zeroes elsewhere. Now if $A \in K$ corresponds to the permutation σ defined above,

$$Au(\mathcal{P})A^{-1} = I + \sum_{k=1}^{r} \sum_{i=1}^{p_k - 1} \tau^{-1} E(N_{i-1} + k, N_i + k).$$

Now $N_{\mathcal{Q}}$ is the set of all block matrices of the form

$$Y = \begin{pmatrix} I_{q_1} & Y_{12} & \dots & Y_{1t} \\ 0 & I_{q_2} & \dots & Y_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{q_t} \end{pmatrix}$$

where the block Y_{ij} is of size $q_i \times q_j$, $1 \leq i,j \leq t$. But for each $1 \leq k \leq r$, $1 \leq i \leq p_k - 1$, we have $k \leq q_{i+1} \leq q_i$. Thus $N_{i-1} + 1 \leq N_{i-1} + k \leq N_i$ and $N_i + 1 \leq N_i + k \leq N_{i+1}$ so that $E(N_{i-1} + k, N_i + k)$ is in the $q_i \times q_{i+1}$ block $Y_{i,i+1}$. Thus $Au(\mathcal{P})A^{-1} \in N_{\mathcal{Q}}$. ∎

This completes the proof of Proposition 2.3, and hence of Theorem 2.1. In order to prove a twisted analogue of Theorem 2.1, we will need the following lemmas.

Let $\mathcal{P}, \mathcal{Q} = (q_1, \dots, q_t) \in \mathbf{P}_n$ and let $Y \in Ka_{\mathcal{P}}K \cap N_{\mathcal{Q}}$. As before we let $n' = n - 1$, $\mathcal{Q}_1 = (q_1, \dots, q_t - 1) \in \mathbf{P}_{n'}$ and write

$$Y = \begin{pmatrix} Y' & Y_{12} \\ 0 & 1 \end{pmatrix}$$

where $Y' \in N_{\mathcal{Q}_1} \cap G_1'$ and $Y_{12} \in M(n', 1, P^{-1})$.

LEMMA 2.11. *Write $Y \in Ka_{\mathcal{P}}K \cap N_{\mathcal{Q}}$ as above and suppose that $Y' \in K'a_{\mathcal{P}'}K'$. If $\mathcal{Q} = \Psi(\mathcal{P})$, then $\mathcal{Q}_1 = \Psi(\mathcal{P}')$.*

PROOF. Using the notation of the proof of Lemma 2.9 we recall that there is $0 \leq b \leq k$ so that $\Psi(\mathcal{P})$, $\Psi(\mathcal{P}')$, $\mathcal{Q}$, $\mathcal{Q}_1$ have the forms

$$\mathcal{Q} = (q_1, \dots, q_t), \quad \mathcal{Q}_1 = (q_1, \dots, q_t - 1),$$
$$\Psi(\mathcal{P}) = (m_1, \dots, m_b, m_{b+1} + 1, \dots, m_k), \quad \Psi(\mathcal{P}') = (m_1, \dots, m_k).$$

Now since we are assuming that $\mathcal{Q} = \Psi(\mathcal{P})$ we have $\mathcal{Q}_1 = (m_1, \dots, m_b, m_{b+1} + 1, \dots, m_k - 1)$. Further, since $Y' \in K'a_{\mathcal{P}'}K' \cap N_{\mathcal{Q}_1}$, we know that $\Psi(\mathcal{P}') \preceq \mathcal{Q}_1$. This is impossible unless $k = b$ or $b + 1$ so that

$$\Psi(\mathcal{P}') = (m_1, \dots, m_b, m_{b+1}), \quad \mathcal{Q}_1 = (m_1, \dots, m_b, m_{b+1} + 1 - 1).$$

∎

Let $\mathcal{P}, \mathcal{Q} \in \mathbf{P}_n$ and write $\mathcal{Q} = (q_1, \dots, q_t), q_1 \geq q_2 \geq \cdots \geq q_t$. Write $Y \in Ka_{\mathcal{P}}K \cap N_{\mathcal{Q}}$ in block form as

$$Y = \begin{pmatrix} I_{q_1} & Y_{12} & \dots & Y_{1t} \\ 0 & I_{q_2} & \dots & Y_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{q_t} \end{pmatrix}$$

where the blocks are of sizes $q_i \times q_j$, $1 \leq i,j \leq t$ and $Y_{ij} \in M(q_i, q_j, P^{-1})$ for $i < j$. Let $1 \leq i \leq t - 1$. Then since $Y_{i,i+1} \in M(q_i, q_{i+1}, P^{-1})$, we have $\tau Y_{i,i+1} \in M(q_i, q_{i+1}, R)$. Write $W_{i,i+1}$ for the image of $\tau Y_{i,i+1}$ in $M(q_i, q_{i+1}, R/P)$.

LEMMA 2.12. *Suppose that* $\mathcal{Q} = \Psi(\mathcal{P})$. *Then with notation as above,*

$$W_{i,i+1} \in M(q_i, q_{i+1}, R/P)$$

has rank q_{i+1} *for all* $1 \le i \le t-1$.

PROOF. The proof is by induction on n. The statement in vacuous when $n = 1$ so we can assume that it is true for $n' = n-1$. Now if we write

$$Y = \begin{pmatrix} Y' & Y_{12} \\ 0 & 1 \end{pmatrix}$$

as before, we know from Lemma 2.11 that $Y' \in N_{\mathcal{Q}_1} \cap K' a_{\mathcal{P}'} K'$ where $\mathcal{Q}_1 = \Psi(\mathcal{P}')$. Write Y' in block form as

$$Y' = \begin{pmatrix} I_{q_1} & Y'_{12} & \dots & Y'_{1t} \\ 0 & I_{q_2} & \dots & Y'_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{q_t-1} \end{pmatrix}$$

and for $1 \le i \le t-2$, let $W'_{i,i+1}$ be the image of $\tau Y'_{i,i+1}$ in $M(q_i, q_{i+1}, R/P)$. Then by the induction hypothesis, $W'_{i,i+1}$ has rank q_{i+1}. But $W_{i,i+1} = W'_{i,i+1}$ for $1 \le i \le t-2$. Thus we need only show that $W_{t-1,t}$ has rank q_t.

Write $\mathcal{P} = (p_1, \dots, p_n), p_1 \ge p_2 \ge \dots \ge p_n \ge 0$. Now since $\mathcal{Q} = \Psi(\mathcal{P})$, we have $t = p_1$ and $q_t = m(\mathcal{P}, p_1)$. Since $Y^{-1} \in K a_{\mathcal{P}}^{-1} K$, there are $A, B \in K$ so that $Y^{-1} = A a_{\mathcal{P}}^{-1} B$. But $a_{\mathcal{P}}^{-1}$ is a diagonal matrix with entries $\tau^{-p_1+1}, \dots, \tau^{-p_n+1}$. Since $t = p_1 \ge p_i$ for all $1 \le i \le n$, we have $\tau^{t-1} a_{\mathcal{P}}^{-1} \in M(n, n, R)$. Further, since exactly q_t of the diagonal entries of $\tau^{t-1} a_{\mathcal{P}}^{-1}$ are equal to 1 while the rest are in P, we see that the image of $\tau^{t-1} a_{\mathcal{P}}^{-1}$ in $M(n, n, R/P)$ has rank q_t. Now since $A, B \in K$, the image of $\tau^{t-1} A a_{\mathcal{P}}^{-1} B$ in $M(n, n, R/P)$ has rank q_t also. Now write Y^{-1} in block form as

$$Y^{-1} = \begin{pmatrix} I_{q_1} & X_{12} & \dots & X_{1t} \\ 0 & I_{q_2} & \dots & X_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{q_t} \end{pmatrix}.$$

It is easy to see that $X_{ij} \in M(q_i, q_j, P^{i-j})$ for all $1 \le i < j \le t$. Thus $\tau^{t-1} Y^{-1} \in M(n, n, R)$ and its image in $M(n, n, R/P)$ has the same rank as the image of $\tau^{t-1} X_{1t}$ in $M(q_1, q_t, R/P)$. Thus since $\tau^{t-1} Y^{-1} = \tau^{t-1} A a_{\mathcal{P}}^{-1} B$, we see that the image of $\tau^{t-1} X_{1t}$ in $M(q_1, q_t, R/P)$ has rank q_t. But $\tau^{t-1} X_{1t} = (-1)^{t-1} (\tau Y_{12})(\tau Y_{23}) \cdots (\tau Y_{t-1,t})$ plus a sum of terms in $M(q_1, q_t, P)$. Thus the image of $\tau Y_{t-1,t}$ in $M(q_{t-1}, q_t, R/P)$ has rank q_t. ∎

Now let κ be an unramified unitary character of $F^\times$ of order d, d a divisor of n. Extend κ to a character of G, and let G_0 be the kernel of κ in G. For any $\gamma \in G$ with $G_\gamma \subset G_0$, let

$$\Lambda_\kappa(f, \gamma) = \int_{G_\gamma \backslash G} f(x^{-1} \gamma x) \kappa(x)\, dx, \quad f \in C_c^\infty(G),$$

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be the twisted orbital integral of f over the orbit of γ. For $\gamma \in G$ with $G_\gamma \not\subset G_0$ we set $\Lambda_\kappa(f,\gamma) = 0$ for all $f \in C_c^\infty(G)$.

As before, for $\mathcal{Q} \in \mathbf{P}_n$, let $u_\mathcal{Q} \in N_\mathcal{Q}$ represent the orbit corresponding to $\mathcal{Q}$. If $\mathcal{Q} = (q_1,\ldots,q_t), q_1 \geq q_2 \geq \cdots \geq q_t$, we can take

$$u_\mathcal{Q} = \begin{pmatrix} I_{q_1} & u_{12} & \ldots & u_{1t} \\ 0 & I_{q_2} & \ldots & u_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_{q_t} \end{pmatrix}$$

where $u_{ij} = 0$ for $j \geq i+2$ and for $1 \leq i \leq t-1$,

$$u_{i,i+1} = \begin{pmatrix} \tau^{-1} I_{q_{i+1}} \\ 0 \end{pmatrix}.$$

Let $\mathbf{P}_n^d = \{\mathcal{Q} \in \mathbf{P}_n : d|m(\mathcal{Q},i),\ 1 \leq i \leq n\}$.

LEMMA 2.13. *Suppose that $\mathcal{Q} \notin \mathbf{P}_n^d$. Then $G_{u_\mathcal{Q}} \not\subset G_0$.*

PROOF. Write $\mathcal{Q} = \Psi(\mathcal{P}), \mathcal{P} = (p_1,\ldots,p_r)$. Now $\mathcal{Q} \notin \mathbf{P}_n^d$ implies that there is at least one i such that d does not divide p_i. Now as in Lemma 2.10, $u_\mathcal{Q}$ is in the same orbit as $u(\mathcal{P})$ where

$$u(\mathcal{P}) = \begin{pmatrix} u(p_1) & 0 & \ldots & 0 \\ 0 & u(p_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u(p_r) \end{pmatrix}.$$

Here as before for $m \geq 1$,

$$u(m) = \begin{pmatrix} 1 & \tau^{-1} & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & \tau^{-1} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

is the $m \times m$ unipotent matrix with all superdiagonal entries equal to τ^{-1}. Now let

$$a = \begin{pmatrix} a_1 I_{p_1} & 0 & \ldots & 0 \\ 0 & a_2 I_{p_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_r I_{p_r} \end{pmatrix}$$

where $a_i \in F^\times, 1 \leq i \leq r$. Then a centralizes $u(\mathcal{P})$ and $\kappa(a) = \prod_{i=1}^r \kappa(a_i)^{p_i}$. Now since there is i such that d does not divide p_i, we can choose a so that $\kappa(a) \neq 1$. Thus $G_{u(\mathcal{P})} \not\subset G_0$, and since $u_\mathcal{Q}$ is conjugate to $u(\mathcal{P})$, we also have $G_{u_\mathcal{Q}} \not\subset G_0$. ■

LEMMA 2.14. *Let $Q \in \mathbf{P}_n^d$ and write $Q = \Psi(\mathcal{P})$. If $x \in G$ with $x^{-1}u_Q x \in Ka_{\mathcal{P}}K$, then $\kappa(x) = 1$.*

PROOF. Let P_Q be the parabolic subgroup of G with unipotent radical N_Q. Let $x \in G$ and write $x = kp, k \in K, p \in P_Q$. Then $x^{-1}u_Q x \in Ka_{\mathcal{P}}K$ if and only if $p^{-1}u_Q p \in Ka_{\mathcal{P}}K$. Further, since we have assumed that κ is unramified, $\kappa(k) = 1$ for all $k \in K$. Thus $\kappa(x) = \kappa(p)$. Thus we may as well assume that $x = p \in P_Q$.

Since $Q \in \mathbf{P}_n^d$, we can write $Q = (m_1, \ldots, m_1, m_2, \ldots, m_2, \ldots, m_k, \ldots, m_k)$ where $m_1 \geq m_2 \geq \cdots \geq m_k \geq 1$ and each m_i appears d times. For any $m \geq 1$ write $u(m) \in GL(md, F)$ for the matrix

$$u(m) = \begin{pmatrix} I_m & \tau^{-1}I_m & 0 & \ldots & 0 \\ 0 & I_m & \tau^{-1}I_m & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \tau^{-1}I_m \\ 0 & 0 & 0 & \ldots & I_m \end{pmatrix}.$$

Now we can write u_Q in block form as

$$u_Q = \begin{pmatrix} u(m_1) & * & \ldots & * \\ 0 & u(m_2) & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u(m_k) \end{pmatrix}.$$

We also write $p \in P_Q$ in block form as

$$p = \begin{pmatrix} p(m_1) & * & \ldots & * \\ 0 & p(m_2) & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p(m_k) \end{pmatrix}.$$

Now

$$p^{-1}u_Q p = \begin{pmatrix} p(m_1)^{-1}u(m_1)p(m_1) & * & \ldots & * \\ 0 & p(m_2)^{-1}u(m_2)p(m_2) & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p(m_k)^{-1}u(m_k)p(m_k) \end{pmatrix}.$$

Assume that $p^{-1}u_Q p \in Ka_{\mathcal{P}}K$. We want to prove that $\kappa(p) = 1$. Since $\det p = \prod_{i=1}^k \det p(m_i)$, it is enough to show that $\kappa\big(\det p(m_i)\big) = 1, 1 \leq i \leq k$.

Fix $m = m_i, 1 \leq i \leq k$, and write

$$p(m) = \begin{pmatrix} p_1 & * & \ldots & * \\ 0 & p_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & p_d \end{pmatrix}.$$

Then

$$p(m)^{-1}u(m)p(m) = \begin{pmatrix} I_m & \tau^{-1}p_1^{-1}p_2 & \dots & * \\ 0 & I_m & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \tau^{-1}p_{d-1}^{-1}p_d \\ 0 & 0 & \dots & I_m \end{pmatrix}.$$

Now we are assuming that $p^{-1}u_Q p \in Ka_{\mathcal{P}}K$. It is also in N_Q, $Q = \Psi(\mathcal{P})$ since P_Q normalizes N_Q. Thus by Lemma 2.12, for $1 \leq i \leq d-1$, $p_i^{-1}p_{i+1} \in M(m,m,R)$ and the image of $p_i^{-1}p_{i+1}$ in $M(m,m,R/P)$ has rank m. Thus $A_{i+1} = p_i^{-1}p_{i+1} \in \mathrm{GL}(m,R)$. Thus $p_{i+1} = p_iA_{i+1} = p_1A_2\cdots A_{i+1} = p_1A'_{i+1}, A'_{i+1} = A_2\cdots A_{i+1} \in \mathrm{GL}(m,R)$. Thus

$$\det p(m) = (\det p_1)^d \det A'_2 \cdots \det A'_d$$

and so $\kappa\big(\det p(m)\big) = 1$. ∎

Write $\mathbf{P}_n^d = \{Q_1,\dots,Q_k\}$, $Q_1 \prec Q_2 \prec \cdots \prec Q_k$. Write $Q_i = \Psi(\mathcal{P}_i)$ and let $u_i = u_{Q_i}$, $\phi_i = \phi_{\mathcal{P}_i}$, $1 \leq i \leq k$.

THEOREM 2.15. *Let $u_1,\dots,u_k,\phi_1,\dots,\phi_k$ be as above. Then*

$$\Lambda_\kappa(\phi_i,u_j) = \begin{cases} 0, & \text{if } j < i; \\ \neq 0, & \text{if } i = j. \end{cases}$$

PROOF. For any i,j,

$$\Lambda_\kappa(\phi_i,u_j) = \int_{G_{u_j}\backslash G} \phi_i(x^{-1}u_jx)\kappa(x)\,dx = 0$$

unless there is $x \in G$ such that $x^{-1}u_jx \in Ka_{\mathcal{P}_i}K$. Write $x = pk, p \in P_{Q_j}, k \in K$. Then $x^{-1}u_jx \in Ka_{\mathcal{P}_i}K$ if and only if $p^{-1}u_jp \in Ka_{\mathcal{P}_i}K \cap N_{Q_j}$.

Suppose that $j < i$. Then $Q_j \prec Q_i = \Psi(\mathcal{P}_i)$ so that by Lemma 2.9, $Ka_{\mathcal{P}_i}K \cap N_{Q_j} = \emptyset$. Thus in this case $\Lambda_\kappa(\phi_i,u_j) = 0$.

Now suppose that $j = i$. Then by Lemma 2.14, $x^{-1}u_ix \in Ka_{\mathcal{P}_i}K$ implies that $\kappa(x) = 1$. Thus $\kappa(x) = 1$ for all $x \in G$ such that $\phi_i(x^{-1}u_ix) \neq 0$, so that

$$\Lambda_\kappa(\phi_i,u_i) = \Lambda(\phi_i,u_i) \neq 0$$

by Lemma 2.10. ∎

3. **Examples for** $\mathrm{SL}(n,F)$. Let $G = \mathrm{GL}(n,F), G_1 = \mathrm{SL}(n,F)$, and $G_u = \{g \in \mathrm{GL}(n,F) : \det g \in R^\times\}$. Let $K_1 = \mathrm{SL}(n,R)$, and let $B_1 = \{b \in K_1 : b_{ij} \in \tau R\ \forall i > j\}$. Let $\mathcal{H}(G_1,K_1)$ be the Hecke algebra of K_1 bi-invariant functions in $C_c^\infty(G_1)$ and let $\mathcal{H}(G_1,B_1)$ be the Iwahori Hecke algebra of B_1 bi-invariant functions in $C_c^\infty(G_1)$. Since $B_1 \subset K_1$, we have $\mathcal{H}(G_1,K_1) \subset \mathcal{H}(G_1,B_1)$. For any unipotent element $n \in G_1$, let μ_n be the distribution on $\mathcal{H}(G_1,B_1)$ defined by

$$\mu_n(f) = \int_{C_{G_1}(n)\backslash G_1} f(g^{-1}ng)\,dg, \quad f \in \mathcal{H}(G_1,B_1).$$

If $n_1, n_2 \in G_1$ are unipotent elements which are conjugate via an element of G_u, it is easy to see [H, 3.1] that the linear functionals $\mu_{n_i}, i = 1,2$, are equal up to a scalar. Thus it is at most possible to separate G_u conjugacy classes of unipotent elements of $\mathrm{SL}(n,F)$ using functions in $\mathcal{H}(G_1, B_1)$. Hales proves that if $n_1, \ldots, n_s$ are a complete set of representatives for the G_u conjugacy classes of unipotent elements of G_1, then $\mu_{n_1}, \ldots, \mu_{n_s}$ are linearly independent on $\mathcal{H}(G_1, B_1)$. His proof involves showing that for each unramified character κ of $F^\times$, the κ-twisted orbital integrals $\Lambda_\kappa(f, u_i)$ defined as in §2 are linearly independent on $\mathcal{H}(G_1, B_1)$ as the u_i run over a complete set of representatives for G-conjugacy classes of unipotent elements of G_1 satisfying $C_G(u_i) \subset G_0 = \{g \in G : \kappa(\det g) = 1\}$. By the results of §2 we know that this is also true for the smaller Hecke algebra $\mathcal{H}(G_1, K_1)$. However, Hales must also show [H, 3.3] that linear independence for each κ implies linear independence for $\mu_{n_1}, \ldots, \mu_{n_s}$. The proof of this result does not generalize to $\mathcal{H}(G_1, K_1)$. In fact the analogue of Hales theorem is not true for $\mathcal{H}(G_1, K_1)$ in the case that $n = 3$.

In the case that $n = 2$, the G_u-conjugacy classes of unipotent elements of $G_1 = \mathrm{SL}(2, F)$ can be represented by the elements

$$n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \{0, \tau^{-1}, \tau^{-2}\}.$$

The $n(x), x \neq 0$, lie in a single G-conjugacy class, but $n(x), n(x')$ lie in the same G_u-conjugacy class just in case $\mathrm{val}(x) \equiv \mathrm{val}(x') (\mathrm{mod}\, 2)$. Here for $x \in F^\times, \mathrm{val}(x) = m \in \mathbf{Z}$ if $x = u\tau^m, u \in R^\times$. For $m \in \mathbf{Z}, m \geq 0$, let $a(m)$ denote the diagonal matrix in G_1 with entries τ^m, τ^{-m}. Then for $m = 0, n(x) \in K_1 a(0) K_1 = K_1$ just in case $x \in R$. For $m > 0, n(x) \in K_1 a(m) K_1$ just in case $\mathrm{val}(x) = -m$. Let ϕ_m be the characteristic function of $K_1 a(m) K_1$. Then we can separate the G_u -conjugacy classes using $\phi_m, m = 0, 1, 2$. First, $n_0 = n(0) \in K_1 a(0) K_1, n_1 = n(\tau^{-1}) \in K_1 a(1) K_1$, and $n_2 = n(\tau^{-2}) \in K_1 a(2) K_1$. Thus $\mu_{n_i}(\phi_i) \neq 0, 0 \leq i \leq 2$. But $\mu_{n_0}(\phi_i) = 0, i = 1, 2$, and $\mu_{n_i}(\phi_j) = 0, 1 \leq i \neq j \leq 2$. Thus we have independence for $\mathrm{SL}(2, F)$.

Now suppose that $n = 3$ so that $G_1 = \mathrm{SL}(3, F)$. For $x, y, z \in F$, write

$$n(x, y, z) = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the regular G-orbit of elements with $x \neq 0, z \neq 0$ splits into three G_u-orbits determined by the image of $\mathrm{val}(xz^{-1})$ in $\mathbf{Z}/3\mathbf{Z}$. Thus the three orbits can be represented by $n_i = n(\tau^i, 0, 1), i = 0, 1, 2$. Write $\mu_m = \mu_{n_m}, m = 1, 2$.

PROPOSITION 3.1. *There is a constant $c \neq 0$ so that $\mu_1(\phi) = c\mu_2(\phi)$ for all $\phi \in \mathcal{H}(G_1, K_1)$.*

PROOF. Let $N = \{n(x, y, z) : x, y, z \in F\}$ and fix a Haar measure ν on N. For $m = 1, 2$, let $N_m = \{n(x, y, z) : \mathrm{val}(xz^{-1}) = m (\mathrm{mod}\, 3)\}$. For any diagonal matrix a in G_1, let

ϕ_a be the characteristic function of K_1aK_1. Then for $m = 1, 2$ there is a positive constant c_m such that $\mu_m(\phi_a) = c_m\nu(N_m \cap K_1aK_1)$. Define $\psi\colon G_1 \longrightarrow G_1$ by $\psi(g) = (wgw^{-1})^t$ where

$$w = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

represents the longest element of the Weyl group and the superscript t denotes transpose. Then $\psi\big(n(x,y,z)\big) = n(z,y,x)$ so that ψ is a measure preserving transformation of N with $\psi(N_1) = N_2, \psi(N_2) = N_1$. Further, $\psi(K_1aK_1) = K_1aK_1$ for every diagonal matrix $a \in G_1$. Thus

$$\nu(N_2 \cap K_1aK_1) = \nu\big(\psi(N_1 \cap K_1aK_1)\big) = \nu(N_1 \cap K_1aK_1)$$

for all a. ∎

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