ANZIAM J. 50(2008), 258–265
doi:10.1017/S144618110900008X

ON PSEUDOMONOTONE SET-VALUED MAPPINGS IN TOPOLOGICAL VECTOR SPACES

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(Received 21 August, 2008; revised 15 November, 2008)

Abstract

In this paper we extend results of Inoan and Kolumban on pseudomonotone set-valued mappings to topological vector spaces. An application is made to a variational inequality problem.

2000 *Mathematics subject classification*: primary 49J40; secondary 90C33.

Keywords and phrases: upper semi-continuity, C-pseudomonotonicity, Knaster–Kuratowski–Mazurkiewicz mapping.

1. Introduction

Different notions of pseudomonotonicity are used in functional analysis in the context of nonlinear operators. They are important in the theory of variational inequalities, in optimization, and equilibrium problems (see [1, 2, 6]). In many applications concerning elastic–plastic torsion problems, contact problems, heat conduction, thermoelasticity and economic theory there appear nonmonotone, possibly multivalued maps (see for instance [9, 10]).

In 1968, Brezis [2] introduced one type of pseudomonotone operator which was named by many authors as a topological pseudomonotone operator. Another type of pseudomonotone operator was introduced by Karamardian [8] in 1976 in the single-valued case. This pseudomonotonicity notion is sometimes called algebraic. These two pseudomonotonicity concepts are different (see [7]). Inoan and Kolumban [7] studied three types of pseudomonotone set-valued mappings in a topological vector spaces setting. Two of them are generalizations of the classical notions mentioned above. The third generalization, which they called C-pseudomonotonicity, is a weaker notion and common generalization of the algebraic and topological pseudomonotonicity for set-valued maps. This paper is inspired and

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motivated by [7]. In this paper we generalize the main results of [7] to the topological vector spaces.

2. Pseudomonotone set-valued mappings

Throughout the paper, let X and Y be two real Hausdorff topological vector spaces, $T : X \to 2^Y$ be a set-valued mapping, and $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$ be a bilinear continuous function.

DEFINITION 2.1 (Inoan and Kolumban [7]). The mapping T is called A-pseudomonotone if, for every $x, x' \in X$,

$$\sup_{f \in T(x)} \langle x' - x, f \rangle \geq 0 \quad \text{implies} \quad \sup_{f \in T(x')} \langle x' - x, f \rangle \geq 0.$$

Definition 2.1 generalizes the notion of algebraic pseudomonotonicity, which was introduced by Karamardian [8], for set-valued mappings. The following definition generalizes the notion of topological pseudomonotonicity introduced by Brezis.

DEFINITION 2.2 (Inoan and Kolumban [7]). The mapping T is called B-*pseudomonotone* if, for every $x \in X$ and for every net $\{x_i\}_{i \in I}$ in X with $x_i \to x$, $f_i \in T(x_i)$, and $\liminf_i \langle x - x_i, f_i \rangle \geq 0$, implies that for every $y \in X$ there exists $f(y) \in T(x)$ such that

$$\limsup_i \langle y - x_i, f_i \rangle \leq \langle y - x, f(y) \rangle.$$

DEFINITION 2.3 (Inoan and Kolumban [7]). We say that T is C-*pseudomonotone* if, for every $x, y \in X$ and every net $\{x_i\}$ in X with $x_i \to x$,

$$\sup_{f \in T(x_i)} \langle (1-t)x + ty - x_i, f \rangle \geq 0, \quad \forall t \in [0, 1], \ \forall i \in I,$$
$$\Rightarrow \sup_{f \in T(x)} \langle y - x, f \rangle \geq 0.$$

REMARK 2.4. We make the following remarks.

(a) There are some examples of applications which are C-pseudomonotone but are neither A-pseudomonotone nor B-pseudomonotone (see [7, Example 5]).
(b) We can define the above definitions on a nonempty convex subset of X.
(c) The expression C-psedudomonotone set-valued mapping in the framework of variational inequalities is due to Inoan and Kolumban [7].

Let X be a topological vector space, K be a nonempty convex subset of X, and X^* the topological dual of X. A set-valued mapping $T : K \to 2^{X^*}$ is called 0-segmentary closed (C-pseudomonotone in the sense of Definition 2.3) if the function $h : K \times K \to \mathbb{R}$ defined by

$$h(x, y) = - \sup_{y^* \in T(y)} \langle y^*, x - y \rangle$$

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is 0-segmentary closed (see [5, 7]). In [7], the authors obtained some examples of this kind of mapping.

DEFINITION 2.5. The set-valued mapping T is called *upper semi-continuous* at $x \in X$ if for each open set V containing $T(x)$ there is an open set U containing x such that for each $t \in U$, $T(t) \subseteq V$. We say that T is upper semi-continuous on X if it is upper semi-continuous at all $x \in X$.

LEMMA 2.6 (Tan [12]). *Assume that for any $x \in X$, $T(x)$ is compact. Then T is upper semi-continuous on X if and only if for any net $\{x_\alpha\} \subset X$ such that $x_\alpha \to x$ and for every $y_\alpha \in T(x_\alpha)$, there exist $y \in T(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$.*

DEFINITION 2.7 (Zhou and Tian [13]). Let X be a nonempty set, Y a topological space, and $T : X \to 2^Y$ a set-valued map. Then, T is called *transfer closed-valued* if, for every $x, y \in K$ and $y \notin T(x)$, there exists $x' \in X$ such that $y \notin \operatorname{cl} T(x')$, where $\operatorname{cl} T(x')$ denotes the topological closure of $T(x')$.

REMARK 2.8. It is clear that, $T : X \to 2^Y$ is transfer closed-valued if and only if

$$\bigcap_{x\in X} T(x) = \bigcap_{x\in X} \operatorname{cl} T(x).$$

Theorem 7 of [7] shows that the mapping $T : X \to 2^Y$ is C-pseudomonotone if and only if, for each $x, y \in X$,

$$\operatorname{cl}\Bigg(\bigcap_{z\in[x,y]} F(z)\Bigg) \cap [x, y] = \Bigg(\bigcap_{z\in[x,y]} F(z)\Bigg) \cap [x, y], \tag{2.1}$$

where $F(z) = \{x \in X \mid \sup_{f\in T(x)} \langle z - x, f\rangle \geq 0\}$.

From the inclusion

$$\operatorname{cl}_X\Bigg(\bigcap_{z\in[x,y]} F(z)\Bigg) \cap [x, y] \subseteq \bigcap_{z\in[x,y]} \operatorname{cl}_X(F(z)) \cap [x, y],$$

and the relation (2.1) through Remark 2.8 we infer that if for each $x, y \in K$ the restriction F to the set $[x, y]$, that is $F|_{[x,y]} : [x, y] \to 2^X$ defined by

$$F|_{[x,y]}(z) = \Big\{w \in X : \sup_{f\in T(w)} \langle z - w, f\rangle \geq 0\Big\}$$

is a transfer-closed valued mapping, then T is C-pseudomonotone.

The following theorem improves [7, Theorem 9].

THEOREM 2.9. *Let $S : X \times X \to 2^Y$ be a set-valued mapping. Let $T : X \to 2^Y$ be defined by $T(x) = S(x, x)$. If $S|_{\triangle\times\triangle}$ (the restriction of S to the set $\triangle \times \triangle = \{(x, x) : x \in X\}$) is upper semi-continuous with nonempty compact values, then T is C-pseudomonotone.*

PROOF. Let $x, y \in X$, $\{x_i\}$ in X be such that $x_i \to x$ and

$$\sup_{f\in S(x_i,x_i)} \langle (1 - t)x + ty - x_i, f\rangle \geq 0 \quad \forall t \in [0, 1], \ \forall i \in I. \tag{2.2}$$

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Since $S|_{\triangle\times\triangle}$ has nonempty compact values and the bilinear mapping $\langle\cdot,\cdot\rangle$ is continuous then, for each i and for $t=1$, there is $f_i \in S(x_i, x_i)$ such that

$$\langle y - x_i, f_i\rangle = \sup_{f\in S(x_i,x_i)} \langle y - x_i, f\rangle. \tag{2.3}$$

Now by applying Lemma 2.6 there exist a subnet $\{f_j\}$ of $\{f_i\}$ and $g \in T(x) = S(x, x)$ such that $f_j \to g$. Hence, the continuity of the bilinear mapping $\langle\cdot,\cdot\rangle$, (2.3), and (2.2) imply that $\langle y - x, g\rangle \geq 0$. Consequently $\sup_{f\in T(x)}\langle y - x, f\rangle \geq 0$. This completes the proof. □

The next result improves [7, Corollary 11].

COROLLARY 2.10. *If $T : X \to 2^Y$ is an upper semi-continuous set-valued mapping with nonempty compact values, then T is C-pseudomonotone.*

PROOF. Let $S(x, y) = T(x)$ and apply Theorem 2.9. □

For the next result we need the following lemma which is due to Blum and Oettli [1].

LEMMA 2.11. *Let D be a convex, compact set and let K be a convex set. Let $f : D \times K \to \mathbb{R}$ be concave and upper semi-continuous in the first variable, and convex in the second variable. Assume that*

$$\max_{x\in D} f(x, y) \geq 0 \quad \forall y \in K.$$

Then there exists $\bar{x} \in D$ such that $f(\bar{x}, y) \geq 0$ for all $y \in K$.

Now we have the following generalization of [7, Theorem 13] in the setting of topological vector spaces.

THEOREM 2.12. *If:*

(a) *T is B-pseudomonotone;*

(b) *$T(x)$ is nonempty convex and compact for each $x \in K$;*

then T is C-pseudomonotone.

PROOF. Let $x, y \in X$ and $\{x_i\}$ be a net in X with $x_i \to x$. Assume that

$$\sup_{f\in T(x_i)} \langle (1-t)x + ty - x_i, f\rangle \geq 0 \quad \forall t \in [0, 1],\ i \in I. \tag{2.4}$$

Since the values of T are compact, then for $t = 0$ and for any $i \in I$, there exists $f_i \in T(x_i)$ such that

$$\langle y - x_i, f_i\rangle = \sup_{f\in T(x_i)} \langle y - x_i, f\rangle \geq 0.$$

Hence, $\liminf_i\langle x - x_i, f_i\rangle \geq 0$, and so by condition (a), there exists $f(y) \in T(x)$ so that

$$0 \leq \limsup\langle y - x_i, f_i\rangle \leq \langle y - x, f(y)\rangle \tag{2.5}$$

(note that the inequality follows from (2.4) when $t = 1$).

Therefore, the conclusion follows from (2.5). □

3. Application

Our aim in this section is to establish an existence result for the following problem which is called a variational inequality problem in the setting of topological vector spaces.

Let X and Y be two Hausdorff topological vector spaces, let K be a nonempty, convex subset of X, and $T : K \to 2^Y$ be a set-valued mapping.

$$\text{Find } x \in K \text{ such that } \sup_{f \in T(x)} \langle y - x, f \rangle \geq 0 \quad \forall y \in K. \tag{VI}$$

Such variational inequalities occur, for example, in the study of optimality for parametric variational problems of the form

$$\min_{u \in K} \max_{\lambda \in \Lambda} \int_\Omega G(\lambda, t, u(t), \nabla u(t))\, dt,$$

where K is a closed and convex set of the Sobolev space $H^1(\Omega)$, Λ is a set of parameters and Ω is a bounded subset of $\mathbb{R}^n$ (see for instance [7, 11]).

We denote the solution set of (VI) by S. The next theorem is needed later.

By checking the proof of Theorem 2.1 in [3, Page 113, lines 15–19], one can realize that the authors, in fact, obtained the following generalization of the Fan–Knaster–Kuratowski–Mazurkiewicz lemma [4].

THEOREM 3.1. *Let X be a topological vector space and K be a nonempty convex subset of X. Suppose that $\Gamma, \widehat{\Gamma} : K \to 2^K$ are two multivalued mappings such that:*

(i) *$\widehat{\Gamma}(x) \subseteq \Gamma(x)$, for all $x \in K$;*
(ii) *$\widehat{\Gamma}$ is a Knaster–Kuratowski–Mazurkiewicz (KKM) map;*
(iii) *for each finite subset A of K, Γ is transfer closed-valued on* co *A;*
(iv) *for each $x, y \in K$, $\mathrm{cl}_K(\bigcap_{z \in [x,y]} \Gamma(z)) \cap [x, y] = (\bigcap_{z \in [x,y]} \Gamma(z)) \cap [x, y]$;*
(v) *there is a nonempty compact convex set $B \subseteq K$ such that $\mathrm{cl}_K(\bigcap_{x \in B} \Gamma(x))$ is compact.*

Then, $\bigcap_{x \in K} \Gamma(x) \neq \emptyset$.

We remark that condition (iv) of Theorem 3.1 is equivalent to the C-pseudomonotonicity of the set-valued mapping $T : X \to 2^Y$ (see [7, Theorem 7]).

THEOREM 3.2. *Let X and Y be two Hausdorff topological vector spaces, K a nonempty convex subset of X, and $T : K \to 2^Y$ be a set-valued mapping. Assume that:*

(a) *T is C-pseudomonotone;*
(b) *for each finite subset A of K, T is upper semi-continuous on* co*A;*
(c) *there exist a compact, convex subset $B \subseteq K$ and a compact set $D \subseteq K$ such that*

$$\forall x \in K \setminus D \quad \textit{and} \quad \exists y \in B \quad \textit{such that} \quad \sup_{f \in T(x)} \langle y - x, f \rangle < 0; \quad \textit{and}$$

(d) *$T(x)$ is nonempty compact, for all $x \in K$.*

Then the solution set of (VI) is a nonempty, compact subset of B.

PROOF. Define $\Gamma = \hat{\Gamma} : K \to 2^K$ by

$$\Gamma(y) = \Big\{x \in K : \sup_{f \in T(x)} \langle y - x, f\rangle \geq 0\Big\}.$$

We now show that Γ fulfils all of the assumptions of Theorem 3.1. Note that for each $y \in K$, $\Gamma(y) \neq \emptyset$ since $y \in \Gamma(y)$. We first show that Γ is a KKM map. Assume on the contrary that there exist $A = \{x_1, x_2, \ldots, x_n\} \subseteq K$, $z \in \mathrm{co}A$ and $z \notin \bigcup_{i\in\{1,2,\ldots,n\}} \Gamma(x_i)$. Then, by the definition of Γ and choosing a fixed element $f \in T(z)$, we have

$$\langle x_i - z, f\rangle < 0 \quad \forall i = 1, \ldots, n.$$

This and $z \in \mathrm{co}\ A$ imply that $0 = \langle z - z, f\rangle < 0$ which is a contradiction. Now we prove that for each finite subset A of K, Γ is transfer closed-valued on co A. To see this, applying Remark 2.8, we must show the following equality

$$\bigcap_{y\in \mathrm{co}\ A} \mathrm{cl}(\Gamma(y) \cap \mathrm{co}\ A) = \bigcap_{y\in \mathrm{co}\ A} (\Gamma(y) \cap \mathrm{co}\ A),$$

for each finite subset A of K.

Let $z \in \bigcap_{y\in \mathrm{co}\ A} \mathrm{cl}(\Gamma(y) \cap \mathrm{co}\ A)$. Hence, for each $y \in \mathrm{co}\ A$, there exists a net $\{x_i\}$ in $\Gamma(y) \cap \mathrm{co}\ A$ which converges to z. Thus, for all i,

$$\sup_{f\in T(x_i)} \langle y - x_i, f\rangle \geq 0.$$

Since, by condition (d), $T(x_i)$ is compact, for all i, and the bilinear mapping $\langle \cdot, \cdot\rangle$ is continuous, there exists $f_i \in T(x_i)$ such that

$$\langle y - x_i, f_i\rangle = \sup_{f\in T(x_i)} \langle y - x_i, f\rangle \geq 0. \tag{3.1}$$

Now by Lemma 2.6, there exist a subnet $\{f_j\}$ of $\{f_i\}$ and a $g \in T(z)$ such that $f_j \to g$. Hence, from (3.1) we obtain $\langle y - z, g\rangle \geq 0$, and so $\sup_{f\in T(x)}\langle y - z, f\rangle \geq 0$. This shows that $z \in \Gamma(y)$. Since $x_i \to z$, $x_i \in \mathrm{co}\ A$, and co A is compact (note that A is a finite subset of K) we have $z \in \mathrm{co}\ A$. Consequently, $z \in \Gamma(y) \cap \mathrm{co}\ A$. Therefore,

$$\bigcap_{y\in \mathrm{co}\ A} \mathrm{cl}(\Gamma(y) \cap \mathrm{co}\ A) \subseteq \bigcap_{y\in \mathrm{co}\ A} (\Gamma(y) \cap \mathrm{co}\ A).$$

The reverse of the above inclusion is obvious. From condition (a) and [7, Theorem 7] we obtain the following expression for each $x, y \in K$:

$$\mathrm{cl}_K\Big(\bigcap_{z\in[x,y]} \Gamma(z)\Big) \cap [x, y] = \Big(\bigcap_{z\in[x,y]} \Gamma(z)\Big) \cap [x, y].$$

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From condition (d) we have that $\mathrm{cl}_K(\bigcap_{x\in B}\Gamma(x))$ is compact. Therefore, Γ satisfies conditions (i)–(v) of Theorem 3.1 and so there exists $\bar{x}\in K$ such that $\bar{x}\in\bigcap_{y\in K}\Gamma(y)$. Hence, we have

$$\sup_{f\in T(\bar{x})}\langle y-\bar{x}, f\rangle \geq 0 \quad \forall y\in K.$$

Thus, $\bar{x}$ is a solution of (VI), that is $\bar{x}\in S$, where S denotes the solution set of (VI), and hence S is nonempty. From condition (c) we deduce that S is a subset of B and so $S=S\cap B$. Now since B is compact and $S=S\cap B$ we obtain that S is a compact subset of B with respect to the induced topology from X on S. This completes the proof. □

REMARK 3.3. Let X be a normed space and let X^* be the topological dual of X. Then the coercivity condition (c) in Theorem 3.2 is weaker than the following condition of [7, Theorem 15]:

there exists a weakly compact subset $A\subseteq K$ and $z_0\in K$ such that

$$\sup_{f\in T(x)}\langle z_0-x, f\rangle < 0 \quad \text{for every } y\in K\setminus A.$$

Indeed, let A be a weakly compact subset $A\subseteq K$. Then by the Mazur theorem, $\overline{\mathrm{co}\,A}$ is compact and convex. Let $B=\overline{\mathrm{co}\,A}$ and $D=\{z_0\}$. Since $K\setminus\overline{\mathrm{co}\,A}\subseteq K\setminus A$, then from the above condition we obtain condition (c) in Theorem 3.2.

One can see that condition (c) of [7, Theorem 15] is a particular case of condition (b) of Theorem 3.2. Hence, Theorem 3.2 generalizes and improves [7, Theorem 15].

The following corollary improves [7, Corollary 16].

COROLLARY 3.4. *Let X and Y be two Hausdorff topological vector spaces, let K be a nonempty convex subset of X, and let $T:K\to 2^Y$ be a set-valued mapping. Let $\bar{x}$ be a solution of* (VI). *Assume that $T(\bar{x})$ is a compact, convex set. Then there exists $\overline{f}\in T(\bar{x})$ such that*

$$\langle y-\bar{x}, \overline{f}\rangle \geq 0 \quad \forall y\in K.$$

PROOF. Define $P:T(\bar{x})\times K\to\mathbb{R}$ by $P(f,y)=\langle y-\bar{x}, f\rangle$, and put $D=T(\bar{x})$. It is clear that the mapping P is concave and upper semi-continuous in the first variable, and convex in the second variable. Now since $\bar{x}$ is a solution of (VI) and $T(\bar{x})$ is compact we have

$$\max_{f\in T(\bar{x})}P(f,y)=\max_{f\in T(\bar{x})}\langle y-\bar{x}, f\rangle=\sup_{f\in T(\bar{x})}\langle y-\bar{x}, f\rangle \geq 0.$$

Consequently P satisfies all of the assumptions of Lemma 2.11 and so there exists $\overline{f}\in T(\bar{x})$ such that

$$P(\overline{f}, y)=\langle y-\bar{x}, \overline{f}\rangle \geq 0 \quad \forall y\in K.$$ □

https://doi.org/10.1017/S144618110900008X Published online by Cambridge University Press

Acknowledgements

The author would like to thank a referee for valuable suggestions and remarks. The author was in part supported by the School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395–5746, Tehran, Iran (research code 87490015).

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