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PARTITIONS WITH AN ARBITRARY NUMBER OF SPECIFIED DISTANCES

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Abstract

For positive integers $t_1, \ldots, t_k$, let $\tilde{p}(n, t_1, t_2, \ldots, t_k)$ (respectively $p(n, t_1, t_2, \ldots, t_k)$) be the number of partitions of n such that, if m is the smallest part, then each of $m + t_1, m + t_1 + t_2, \ldots, m + t_1 + t_2 + \cdots + t_{k-1}$ appears as a part and the largest part is at most (respectively equal to) $m + t_1 + t_2 + \cdots + t_k$. Andrews *et al.* ['Partitions with fixed differences between largest and smallest parts', *Proc. Amer. Math. Soc.* **143** (2015), 4283–4289] found an explicit formula for the generating function of $p(n, t_1, t_2, \ldots, t_k)$. We establish a q-series identity from which the formulae for the generating functions of $\tilde{p}(n, t_1, t_2, \ldots, t_k)$ and $p(n, t_1, t_2, \ldots, t_k)$ can be obtained.

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1. Introduction

A partition of a positive integer n is a weakly decreasing sequence of positive integers whose sum is n. Let $p(n, t)$ be the number of partitions of n with difference t between its largest and smallest parts. Andrews *et al.* [2] established the following formula for the generating function of $p(n, t)$ for $t > 1$:

$$\sum_{n=1}^{\infty} p(n, t)q^n = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q;q)_t} + \frac{q^t}{(1-q^{t-1})(q;q)_t}. \quad (1.1)$$

Here and in the rest of paper, we will adopt the usual q-series notation:

$$\begin{aligned}(a;q)_0 &= 1,\\ (a;q)_n &= \prod_{k=1}^{n}(1-aq^{k-1}), \quad n \in N,\\ (a;q)_\infty &= \prod_{k=1}^{\infty}(1-aq^{k-1}).\end{aligned}$$

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In [2], Andrews *et al.* also generalised (1.1) by considering partitions with specified distances. Define $p(n, t_1, t_2, \ldots, t_k)$ to be the number of partitions of n such that, if m is the smallest part, then each of $m + t_1, m + t_1 + t_2, \ldots, m + t_1 + t_2 + \cdots + t_{k-1}$ appears as a part and the largest part is $m + t_1 + t_2 + \cdots + t_k$, where $t_i \geq 1$ for $1 \leq i \leq k$. Let $P_{t_1,t_2,\ldots,t_k}(q)$ denote the generating function of $p(n, t_1, t_2, \ldots, t_k)$. Andrews *et al.* proved the following theorem by using Heine's transformation.

Theorem 1.1 (Andrews *et al.* [2]). *For $t = t_1 + t_2 + \cdots + t_k > k$,*

$$P_{t_1,t_2,\ldots,t_k}(q) = \frac{(-1)^k q^{T-\binom{k+1}{2}}\left(\sum_{j=0}^{k}\begin{bmatrix} t \\ j \end{bmatrix}(-1)^j q^{\binom{j+1}{2}} - (q;q)_t\right)}{\begin{bmatrix} t-1 \\ k \end{bmatrix}(1-q^t)(q;q)_t},$$

where $T := kt_1 + (k-1)t_2 + \cdots + 2t_{k-1} + t_k$ and

$$\begin{bmatrix} A \\ B \end{bmatrix} := \frac{(q;q)_A}{(q;q)_B(q;q)_{A-B}} \quad \text{for } 0 \leq B \leq A.$$

Later, Breuer and Kronholm [3] studied the number $\tilde{p}(n, t)$ of partitions of n with the difference between largest and smallest parts bounded by t and obtained

$$\sum_{n\geq 1} \tilde{p}(n,t)q^n = \frac{1}{1-q^t}\left(\frac{1}{(q;q)_t} - 1\right). \tag{1.2}$$

Chapman [4] gave another proof of (1.2) by using elementary q-series manipulation, involving no results deeper than the q-binomial theorem. Overpartitions with bounded differences between largest and smallest parts have also been examined (see [6, 7]). Chern [5] established an interesting identity which includes (1.1) and the results in [6, 7] as special cases.

Chapman [4] asked for an elementary proof of Theorem 1.1 and that is the goal of this paper. To this end, we consider the function $\tilde{p}(n, t_1, t_2, \ldots, t_k)$ counting the number of partitions of n such that, if m is the smallest part, then each of $m + t_1, m + t_1 + t_2, \ldots, m + t_1 + t_2 + \cdots + t_{k-1}$ appears as a part and the largest part is not greater than $m + t_1 + t_2 + \cdots + t_k$. We will establish the following formula for the generating function $\tilde{P}_{t_1,t_2,\ldots,t_k}(q)$ of $\tilde{p}(n, t_1, t_2, \ldots, t_k)$.

Theorem 1.2. *For $t = t_1 + t_2 + \cdots + t_k > k$,*

$$\tilde{P}_{t_1,t_2,\ldots,t_k}(q) = \frac{(-1)^{k+1} q^{T_{k-1}-\binom{k}{2}}\left(\sum_{j=0}^{k-1}\begin{bmatrix} t \\ j \end{bmatrix}(-1)^j q^{\binom{j+1}{2}} - (q;q)_t\right)}{\begin{bmatrix} t-1 \\ k-1 \end{bmatrix}(1-q^t)(q;q)_t},$$

where $T_k := kt_1 + (k-1)t_2 + \cdots + 2t_{k-1} + t_k$.

2. Proof of Theorem 1.2

We first establish the following identity, which is useful for our proofs.

Lemma 2.1. *We have*

$$\sum_{r=k}^{\infty}\frac{q^r}{1-q^r}\begin{bmatrix}t+r-k\\ t\end{bmatrix}=\frac{(-1)^{k+1}q^{-\binom{k}{2}}}{(q;q)_t(1-q^t)\begin{bmatrix}t-1\\ k-1\end{bmatrix}}\bigg(\sum_{j=0}^{k-1}(-1)^j q^{j(j+1)/2}\begin{bmatrix}t\\ j\end{bmatrix}-(q;q)_t\bigg). \tag{2.1}$$

Proof. We first observe that we can rewrite the left-hand side of (2.1) in the form

$$\begin{aligned}\sum_{r=k}^{\infty}\frac{q^r}{1-q^r}\begin{bmatrix}t+r-k\\ t\end{bmatrix}&=\sum_{r=k}^{\infty}\frac{q^r}{1-q^r}\frac{(q^{t+1};q)_{r-k}}{(q;q)_{r-k}}\\ &=\frac{1}{(q^{t-k+1};q)_k}\sum_{r=k}^{\infty}\frac{q^r(q^{t-k+1};q)_r}{(q;q)_r}(q^{r-k+1};q)_{k-1}.\end{aligned} \tag{2.2}$$

Define

$$S:=\sum_{r=k}^{\infty}\frac{q^r(q^{t-k+1};q)_r}{(q;q)_r}(q^{r-k+1};q)_{k-1}.$$

Since $(q^{r-k+1};q)_{k-1}=0$ for $1\le r<k$, we can take the summation in the definition of S from $r=1$ to ∞. Applying the identity

$$(-qz;q)_n=\sum_{j=0}^{n}q^{(n-j)(n-j+1)/2}z^{n-j}\begin{bmatrix}n\\ j\end{bmatrix} \tag{2.3}$$

from [1, Theorem 3.3],

$$\begin{aligned}S&=\sum_{r=1}^{\infty}\frac{q^r(q^{t-k+1};q)_r}{(q;q)_r}\sum_{j=0}^{k-1}q^{(k-1-j)(k-j)/2}(-q^{r-k})^{k-1-j}\begin{bmatrix}k-1\\ j\end{bmatrix}\\ &=(-1)^{k+1}q^{-\binom{k}{2}}\sum_{j=0}^{k-1}(-1)^j q^{j(j+1)/2}\begin{bmatrix}k-1\\ j\end{bmatrix}\sum_{r=1}^{\infty}\frac{(q^{t-k+1};q)_r}{(q;q)_r}(q^{k-j})^r.\end{aligned}$$

By the q-binomial theorem [1, Theorem 2.1],

$$\begin{aligned}S&=(-1)^{k+1}q^{-\binom{k}{2}}\sum_{j=0}^{k-1}(-1)^j q^{j(j+1)/2}\begin{bmatrix}k-1\\ j\end{bmatrix}\bigg(\frac{(q^{t-j+1};q)_\infty}{(q^{k-j};q)_\infty}-1\bigg)\\ &=(-1)^{k+1}q^{-\binom{k}{2}}\sum_{j=0}^{k-1}(-1)^j q^{j(j+1)/2}\begin{bmatrix}k-1\\ j\end{bmatrix}\bigg(\frac{(q;q)_{k-j-1}}{(q;q)_{t-j}}-1\bigg).\end{aligned}$$

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Applying (2.3) again,

$$\begin{aligned}S &= (-1)^{k+1}q^{-\binom{k}{2}}\Bigg(\sum_{j=0}^{k-1}(-1)^j q^{j(j+1)/2}\begin{bmatrix}k-1\\ j\end{bmatrix}\frac{(q;q)_{k-j-1}}{(q;q)_{t-j}} - (q;q)_{k-1}\Bigg)\\ &= (-1)^{k+1}q^{-\binom{k}{2}}(q;q)_{k-1}\Bigg(\sum_{j=0}^{k-1}(-1)^j q^{j(j+1)/2}\frac{1}{(q;q)_j(q;q)_{t-j}} - 1\Bigg).\end{aligned}$$

From (2.2),

$$\sum_{r=k}^{\infty}\frac{q^r}{1-q^r}\begin{bmatrix}t+r-k\\ t\end{bmatrix} = \frac{(-1)^{k+1}q^{-\binom{k}{2}}(q;q)_{k-1}}{(q^{t-k+1};q)_k}\Bigg(\sum_{j=0}^{k-1}(-1)^j q^{j(j+1)/2}\frac{1}{(q;q)_j(q;q)_{t-j}} - 1\Bigg),$$

which is easily seen to be equivalent to (2.1). □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $\tilde{\mathcal{P}}_{t_1,t_2,\ldots,t_k,m,r}$ be the set of partitions with the restriction that for each $\lambda \in \tilde{\mathcal{P}}_{t_1,t_2,\ldots,t_k,m,r}$, there are r parts in λ with smallest part m, largest part $\le m+t$ and $m+t_1, m+t_1+t_2, \ldots, m+t_1+\cdots+t_{k-1}$ appear as parts in λ. Then

$$\tilde{P}_{t_1,t_2,\ldots,t_k}(q) = \sum_{m=1}^{\infty}\sum_{r\ge k}\sum_{\lambda\in\tilde{\mathcal{P}}_{t_1,t_2,\ldots,t_k,m,r}} q^{|\lambda|},$$

where $|\lambda|$ denotes the sum of the parts in λ. For $\lambda \in \tilde{\mathcal{P}}_{t_1,t_2,\ldots,t_k,m,r}$, delete the k parts $m, m+t_1, \ldots, m+t_1+\cdots+t_{k-1}$ and reduce the remaining parts by m. This gives a partition μ with largest part $\le t$ and with at most $r-k$ parts. From [1, Theorem 3.1], the generating function of such partitions is given by the q-binomial coefficient $\begin{bmatrix}t+r-k\\ t\end{bmatrix}$. Hence,

$$\sum_{\lambda\in\tilde{\mathcal{P}}_{t_1,t_2,\ldots,t_k,m,r}} q^{|\lambda|} = q^{rm}q^{(k-1)t_1+(k-2)t_2+\cdots+t_{k-1}}\begin{bmatrix}t+r-k\\ t\end{bmatrix}$$

and

$$\begin{aligned}\tilde{P}_{t_1,t_2,\ldots,t_k}(q) &= \sum_{m=1}^{\infty}\sum_{r\ge k} q^{rm}q^{(k-1)t_1+(k-2)t_2+\cdots+t_{k-1}}\begin{bmatrix}t+r-k\\ t\end{bmatrix}\\ &= q^{T_{k-1}}\sum_{r=k}^{\infty}\frac{q^r}{1-q^r}\begin{bmatrix}t+r-k\\ t\end{bmatrix}.\end{aligned}$$

By Lemma 2.1, we get the desired result. □

To end this paper, we present another proof of Theorem 1.1.

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Proof of Theorem 1.1. Let $\mathcal{P}_{t_1,t_2,\dots,t_k,m,r}$ be the partitions in $\tilde{\mathcal{P}}_{t_1,t_2,\dots,t_k,m,r}$ with the additional condition that the largest part is $m+t$. Then it is not hard to see that

$$\sum_{\lambda\in\mathcal{P}_{t_1,t_2,\dots,t_k,m,r}} q^{|\lambda|} = q^{rm}q^{T_k}\begin{bmatrix} t+r-k-1 \\ t \end{bmatrix}.$$

Hence,

$$P_{t_1,t_2,\dots,t_k}(q) = \sum_{m=1}^{\infty}\sum_{r=k+1}^{\infty} q^{rm}q^{T_k}\begin{bmatrix} t+r-k-1 \\ t \end{bmatrix} = q^{T_k}\sum_{r=k+1}^{\infty}\frac{q^r}{1-q^r}\begin{bmatrix} t+r-k-1 \\ t \end{bmatrix}.$$

By Lemma 2.1,

$$P_{t_1,t_2,\dots,t_k}(q) = \frac{(-1)^k q^{T_k-\binom{k+1}{2}}}{(q;q)_t(1-q^t)\begin{bmatrix} t-1 \\ k \end{bmatrix}}\Bigg(\sum_{j=0}^{k}(-1)^j q^{j(j+1)/2}\begin{bmatrix} t \\ j \end{bmatrix} - (q;q)_t\Bigg).$$

This completes the proof. □

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