Can. J. Math., Vol. XXXIII, No. 6, 1981, pp. 1279–1290

RECURSIVE COLORINGS OF HIGHLY RECURSIVE GRAPHS

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0. Introduction. One of the attractions of finite combinatorics is its explicit constructions. This paper is part of a program to enlarge the domain of finite combinatorics to certain infinite structures while preserving the explicit constructions of the smaller domain. The larger domain to be considered consists of the recursive structures. While recursive structures may be infinite they are still amenable to explicit constructions. In this paper we shall concentrate on recursive colorings of highly recursive graphs.

A function $f: N^k \rightarrow N$, where N is the set of natural numbers, is *recursive* if and only if there exists an algorithm (i.e., a finite computer program) which upon input of a sequence of natural numbers $\bar{n}$, after a finite number of steps, outputs $f(\bar{n})$. A subset of N^k is *recursive* provided that its characteristic function is recursive. For a more thorough definition of recursive functions and recursive relations see [**10**]. A graph $G = (V, E)$ is recursive if and only if V and E are recursive sets. (We are treating the edge set E as a symmetric relation.) A recursive coloring of G is a recursive function f such that $f \upharpoonright V$ is a coloring of G. If $f \upharpoonright V$ is a k-coloring then we say that G is recursively k-colorable. Recursive edge colorings and recursively k-edge colorable are defined similarly. We shall use the adjectives "algorithmically", "effectively" and "recursively" interchangeably.

A k-colorable recursive graph may not be recursively k-colorable. In fact, Bean [**1**] has shown that there is a 2-colorable recursive graph which is not recursively k-colorable for any finite k. This negative result motivated him to make the following crucial definition. A graph $G = (V, E)$ is *highly recursive* if and only if it is recursive and there is a recursive function δ such that for every vertex $v \in V$, v is adjacent to exactly $\delta(v)$ other vertices. The significance of this definition is two-fold. Firstly, it assures that every vertex has only finitely many neighbors. Secondly, it allows us to algorithmically determine this finite subset of neighbors. If G is just recursive, then for any vertex $w \in V$ we can algorithmically determine whether w is a neighbor of v and we can systematically search the vertices of V for neighbors of v, but since there are infinitely many candidates for neighbors of v and we do not know

Received August 21, 1979 and in revised form April 27, 1981.

how many neighbors v has, we will never know when we have found them all. On the other hand, if G is highly recursive we need only search until we have found $\delta(v)$ neighbors of v. Schmerl [**11**] improved a result of Bean [**1**] by showing that every k-colorable highly recursive graph is recursively $2k - 1$-colorable and if $k \geqq 2$ then there is a k-colorable highly recursive graph that is not recursively $2k - 2$-colorable. This theorem suggests a classification of highly recursive graphs according to the ratio of the number of colors required to recursively color them to the number of colors required to color them.

In this paper we investigate the recursive colorability of various classes of highly recursive graphs. There are three main results. The first is the general technique for constructing recursive colorings that is developed throughout the paper. The finitary nature of the problem is emphasized by Lemma 1.1 which reduces all recursion theoretic considerations to a condition on finite subgraphs. The latter two results are the following applications of this method.

THEOREM 2.1. *Every perfect, k-colorable, highly recursive graph is recursively $k + 1$-colorable.*

THEOREM 4.1. *Every k-edge colorable, highly recursive graph is recursively $k + 1$-edge colorable.*

Theorem 2.1 is generalized to nearly perfect graphs is Section 3.

Notation. Let $G = (V, E)$ be a graph and X a set (of colors). A function $f: V \rightarrow X$ is an *X-coloring* of G provided that for each $x \in X$, $f^{-1}(x)$ is an independent set of vertices. Similarly $f: E \rightarrow X$ is an *X-edge coloring* of G provided that for each $x \in X$, $f^{-1}(x)$ is an independent set of edges. We shall identify the natural number n with the set $\{0, \ldots, n - 1\}$ and often refer to n-colorings. $\chi(G)$[resp. $\chi'(G)$] is the least n such that G is n-colorable [resp. n-edge colorable]. $\chi_r(G)$[resp. $\chi_r'(G)$] is the least n such that G is recursively n-colorable [resp. n-edge colorable]. The *degree* $\delta_G(v)$ of a vertex v is the number of vertices adjacent to v in G. If G is clear from the context we shall drop the subscript. The *degree* $\Delta(G)$ of a graph is $\max_{v \in V} \delta_G(v)$. A complete subgraph is a set of vertices that are pairwise adjacent. The maximum k such that G has a complete subgraph of cardinality k is denoted by $\omega(G)$. Let $X \subset V$.

$$N[X] = \{v \in V: (v, x) \in E \text{ for some } x \in X\} \cup X.$$

For $i \in \mathbf{N}$, $N^i[X]$ is defined inductively by

$$N^0[X] = X \text{ and } N^{i+1}[X] = N[N^i[x]].$$

If U is a set of vertices of G then $E(U) = E \cap U^2$. If H is a graph then $E(H)$ is the set of edges of H. If U is a subset of V we shall sometimes identify U with the subgraph $(U, E(U))$ of V induced by U. In particular we shall write $\omega(U)$ and $\chi(U)$ for $\omega(U, E(U))$ and $\chi(U, E(U))$.

1. Recursive colorings. Let $G = (V, E)$ be a highly recursive graph and U a finite subset of V. If we are to construct a recursive k-coloring of G we must irrevocably assign colors to the vertices of U before we have considered infinitely many of the vertices of G. Thus based on information only about some finite induced subgraph $(W, E(W))$ of G containing U, we must specify a k-coloring f_0 of U such that f_0 can be extended to a k-coloring of G; moreover we must be able to recognize how to make this extension. The following definition and lemma make these ideas more precise.

Definition 1.0. Let $G = (V, E)$ be a graph. A coloring of U (resp. $E(U)$) is said to be *d, α-prudent* if it can be extended to an α-coloring of $N^d[U]$ (resp. $E(N^d[U])$) that uses only $\chi(G)$ (resp. $\chi'(G)$) colors on $N^d[U] - U$ (resp. $E(N^d[U]) - E(U)$).

LEMMA 1.1. *Let $G = (V, E)$ be a highly recursive graph. Suppose there exist natural numbers d and α such that whenever $N^d[U] \subset W \subset V$ and f is a d, α-prudent coloring of U (resp. $E(U)$) then f can be extended to a d, α-prudent coloring of W (resp. $E(W)$). Then $\chi_r(G)$ (resp. $\chi_r'(G)$) $\leqq \alpha$.*

Proof. We prove the lemma only for vertex colorings because the proof for edge colorings is analogous. Let $L[U] = N^d[U]$. Since G is highly recursive, for any finite subset $U \subset V$ we can effectively compute $L[U]$. Thus, since any function on U into α has only finitely many extensions to $L[U]$ whose range is contained in α, we can effectively determine whether a function on U is d, α-prudent by checking each extension to $L[U]$. Similarly, if f is a d, α-prudent coloring of U, W is a finite subset such that $L[U] \subset W \subset V$, and g is a function on W, we can effectively determine whether g is a d, α-prudent extension of f. Finally, given that f has a d, α-prudent extension to W, we can effectively construct such an extension.

Let $\langle v_i: i \in \omega \rangle$ be an effective enumeration of V. Let $U_0 = \{v_0\}$, $\bar{f}_0$ be a $\chi(G)$-coloring of $L[U_0]$, and $f_0 = \bar{f}_0 \upharpoonright U_0$. Thus f_0 is a d, α-prudent coloring of U_0. Now arguing inductively, suppose that we have a uniform algorithm for determining U_i and f_i such that f_i is a d, α-prudent coloring of U_i, U_i is finite, and $\{v_0, \ldots v_i\} \subset U_i$. Set $U_{i+1} = L[U_i] \cup \{v_{i+1}\}$. U_{i+1} is finite and can be calculated algorithmically. By the hypothesis of the lemma there is a d, α-prudent coloring of U_{i+1} which extends f_i. Thus by our opening remarks we can effectively construct such a d, α-prudent coloring f_{i+1} of U_{i+1}. Let $f = \bigcup_{i \in N} f_i$. f is a recursive function on V since to determine $f(v_i)$ we need only carry out the algorithm for constructing f_i and observe the value of $f_i(v_i)$. Clearly f is an α-coloring of V.

Lemma 1.1 reduces the recursion theoretic problem of producing a recursive α-coloring of V to a finite graph theoretic problem. In the remainder of the paper whenever we show that a highly recursive graph is α-colorable we will use Lemma 1.1.

2. Perfect graphs. We begin our development of a method for recursively coloring highly recursive graphs by applying Lemma 1.1 to perfect graphs.

Definition 2.0. A graph G is *perfect* if and only if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

Proof of Theorem 2.1. Let $d = 2\chi(G)$, $L[U] = N^d[U]$, and $\alpha = \chi(G) + 1$. By Lemma 1.1 it suffices to show that if U and W are finite subsets of V such that $L[U] \subset W$ and f is an α-coloring of $L[U]$ such that $f \upharpoonright (L[U] - U)$ is a $\chi(G)$-coloring, then there exists an α-coloring g of $L[W]$ that extends $f \upharpoonright U$ and is a $\{1, \ldots, \alpha - 1\}$-coloring of $L(W) - W$. We shall construct g one color at a time. To describe this construction we need considerably more notation.

Suppose that we have constructed $g^{-1}(i)$ for $i < j$. Let $C_j = U_{i<j}g^{-1}(i)$ be the set of vertices already colored. Let

$$I_j = N^{d-2j}[U] - C_j \text{ and } O_j = L[W] - (N[I_j] \cup C_j).$$

This is illustrated in Figure 1.

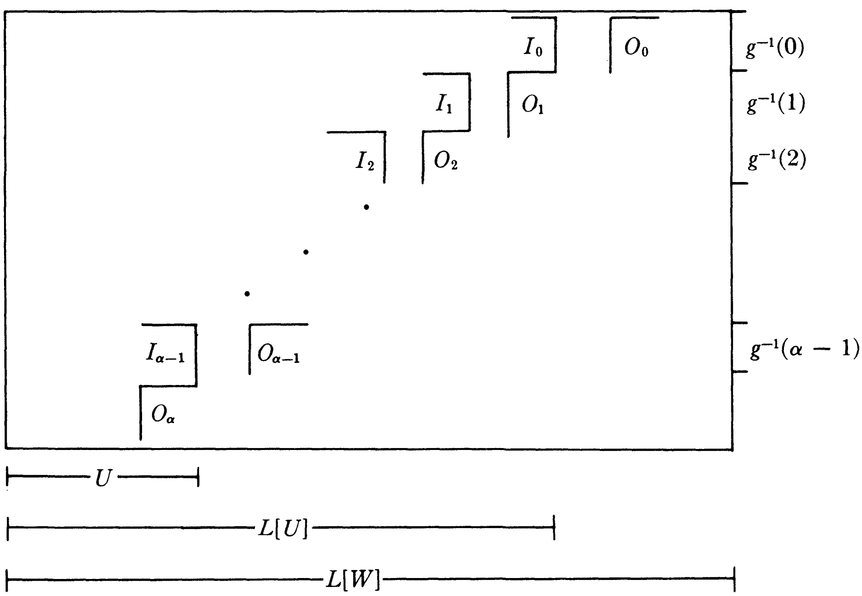

FIGURE 1.

At stage j of the construction we shall specify those vertices $g^{-1}(j)$ to be colored j. The "inner portion" I_j of $L[W]$ will be colored according to f. The "outer portion" O_j of $L[W]$ will be colored according to a new coloring t_j chosen at stage j. These colorings will not conflict because no vertex in I_j is adjacent to any vertex in O_j.

Set $g^{-1}(0) = f^{-1}(0) \cap I_0$. Arguing inductively suppose that we have constructed $g^{-1}(\mathrm{i})$ for $i < j \leqq \alpha$ such that:

(0) $g^{-1}(i)$ is independent;

(1) $g^{-1}(i) \cap I_i = f^{-1}(i) \cap I_i$; and

(2) $\omega(O_j) \leqq \alpha - j$.

Since G is perfect $\chi(O_j) = \omega(O_j)$. Let t_j be a $\{j, \ldots, \alpha - 1\}$-coloring of O_j. Set

$$g^{-1}(j) = (f^{-1}(j) \cap I_j) \cup t_j^{-1}(j).$$

Clearly $g^{-1}(j)$ is independent and

$$g^{-1}(j) \cap I_j = f^{-1}(j) \cap I_j.$$

To see that the inductive hypotheses is preserved we still must check that $\omega(O_{j+1}) \leqq \alpha - j - 1$. Since the diameter of a complete subgraph is one, any complete subgraph contained in O_{j+1} is contained in O_j or I_{j-1}. Thus it suffices to show that if K is a complete subgraph of cardinality $\alpha - j$ that is contained in O_j or I_{j-1} then K is not contained in O_{j+1}. Firstly suppose K is contained in O_j. Since t_j is a $\{j, \ldots, \alpha - 1\}$-coloring of O_j, for some vertex $v \in K$, $g(v) = t_j(v) = j$. So $v \in C_{j+1}$ and thus $v \notin O_{j+1}$. Secondly suppose K is contained in I_{j-1}. Since $f \restriction (L[U] - U)$ is a $\chi(G)$-coloring and $g^{-1}(i) \cap I_i = f^{-1}(i) \cap I_i$ for $i < j - i$, $f \restriction (I_{j-1} - U)$ is a $\{j - 1, \ldots, \alpha - 2\}$-coloring. Thus for some vertex $v \in K, f(v) = j - 1$. Since

$$g^{-1}(j-1) \cap I_{j-1} = f^{-1}(j-1) \cap I_{j-1},$$

$g(v) = j - 1$. Hence $v \notin O_{j+1}$.

Finally we check that g works. By (0) g is a coloring of its domain. Since $U - C_i \subset I_i$ for $i < \alpha$ (1) applied to $j = \alpha$ shows that $f \restriction U = g \restriction U$. By (2) applied to $j = \alpha$, $g \restriction (O_\alpha \cup C_\alpha)$ is an α-coloring of $O_\alpha \cup C_\alpha$. Since $U \cup O_\alpha \cup C_\alpha = L[W]$ g is an α-coloring $L[W]$. Finally, since $g^{-1}(0) \subset I_0 = L[U] \subset W$, g is a $\{1, \ldots, \alpha - 1\}$-coloring of $L(W) - W$.

COROLLARY 2.2. *If G is a recursive k-regular bipartite graph then* $\chi'(G) \leqq k + 1$.

Proof. Since G is k-regular and recursive the line graph of G, $l(G)$, is highly recursive. Using Hall's Theorem [**6**] it is easy to see that the line graph of a k-regular bipartite graph is perfect and has clique size k. The recursive $k + 1$-vertex coloring of $l(G)$ provided by Theorem 2.1 induces a recursive $k + 1$-edge coloring of G.

The author originally proved Corollary 2.2 to answer a question of Schmerl. Then Schmerl pointed out that essentially the same argument could be used to prove Theorem 2.2. Manaster and Rosenstein [**9**] have constructed recursive k-regular bipartite graphs that cannot be recursively k-edge colored. Thus the bounds of Theorem 2.1 and Corollary 2.2 are best possible.

COROLLARY 2.3. *If G is the comparability graph of a partial ordering P of width w and G is highly recursive, then $\chi_r(G) \leqq w + 1$. Dually, P can be covered by $w + 1$ recursive chains.*

Corollary 2.3 should be compared with Dilworth's Theorem [**3**] and the results in [**7**]. It is easy to construct partial orderings of width w whose comparability graphs are highly recursive but are not recursively w-colorable.

3. Nearly perfect graphs. In this section we apply the technique of the previous section to a larger class of graphs. Notice that in the proof of Theorem 2.1 we used a succession of different colorings t_j to color $L[W] - W$, i.e.,

$$g^{-1}(j) \cap (L[W] - W) = t_j^{-1}(j) \cap L([W] - W).$$

This was possible because G was perfect. Now we are going to weaken the hypothesis that G is perfect. This will require that we use a fixed coloring on $L[W] - W$.

Definition 3.0. A graph G is *p-nearly perfect* if and only if $\chi(H) \leqq \omega(H) + p$ for every induced subgraph H of G.

Notice that a perfect graph is 0-nearly perfect. By Vizing's Theorem [**13**] the line graph of a graph is 1-nearly perfect, and more generally, the line graph of a multigraph of multiplicity p is p-nearly perfect. Schmerl and the author [**8**] have shown that if G is a graph that does not induce $K_{1,3}$ or $K_5 - e$ (i.e., the result of removing an edge from the complete graph on five vertices) then G is 2-nearly perfect. The next theorem generalizes Theorem 2.1.

THEOREM 3.1. *If G is a p-nearly perfect highly recursive graph then $\chi_r(G) \leqq \chi(G) + p(\beta + 1) + 1$, where β is the least integer k such that*

$$\sum_{i=0}^{k} (pi + 1) \geqq \omega(G).$$

Proof. Let $d = 4\beta + 2$, $L[U] = N^d[U]$, and $\alpha = \chi(G) + p(\beta + 1) + 1$. By Lemma 1.1, it suffices to show that if U and W are finite subsets of V

https://doi.org/10.4153/CJM-1981-097-8 Published online by Cambridge University Press

such that $L[U] \subset W$ and f is an α-coloring of $L[U]$ such that $f \restriction (L[U] - U)$ is a $\chi(G)$-coloring, then there exists an α-coloring g that extends $f \restriction U$ and is a $\chi(G)$-coloring of $L[W] - W$. The construction of g will be similar to the construction in the proof of Theorem 2.1, but will involve the extra complication that we must use a fixed $\chi(G)$-coloring on $L[W] - W$. Fix a $\chi(G)$-coloring h of $L[W] - N^{2\beta+1}[U]$.

For any $j \leqq \beta$, let

$$\bar{j} = \chi(G) - \omega(G) + \sum_{i=0}^{j} (pi + 1).$$

In particular $\overline{-1} = \chi(G) - \omega(G)$. Let

$$\overline{\beta + 1} = \chi(G) + p(\beta + 1) + 1.$$

Suppose that we have constructed $g^{-1}(i)$ for $i < \overline{j-1}$. Let $C_j = \bigcup_{i<j-1} g^{-1}(i)$ be the set of vertices already colored. Let

$$I_j = N^{2(\beta-j)}[U] - C_j,$$
$$O_j = L[W] - (N^{2(\beta+j+1)}[U] \cup C_j), \text{ and}$$
$$B_j = L[W] - (N[I_j] \cup N[O_j] \cup C_j).$$

This is illustrated in Figure 2. At stage j of the construction we shall specify those vertices $g^{-1}(\overline{j-1}), g^{-1}(\overline{j-1} + 1), \ldots g^{-1}(\bar{j} - 1)$ that will be colored $\overline{j-1}, j - 1 + 1, \ldots \bar{j} - 1$. The "innner portion" I_j of

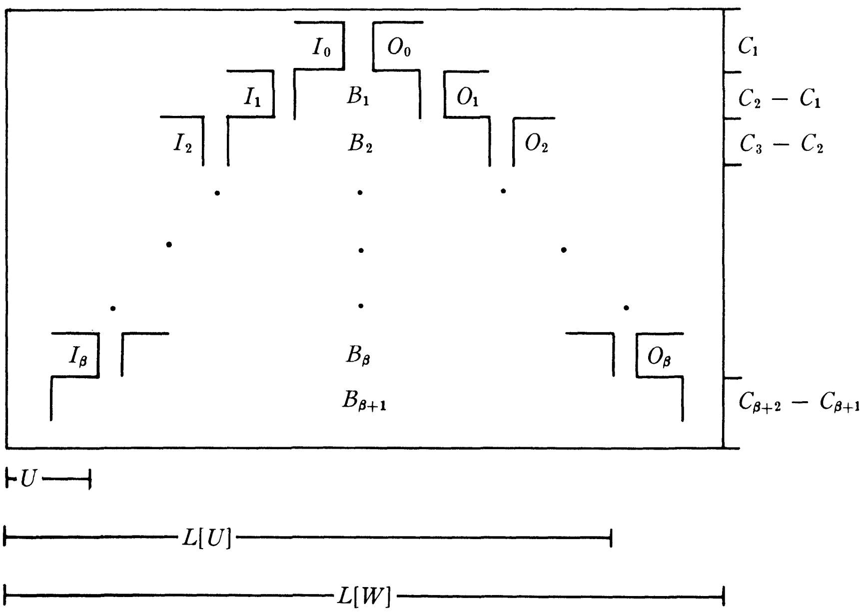

FIGURE 2.

$L[W]$ will be colored according to f and the "outer portion" O_j of $L[W]$ will be colored according to h. B_j will be colored according to a new coloring t_j chosen at stage j. These colorings will not conflict because their domains are sufficiently separated.

Set

$$g^{-1}(i) = (f^{-1}(i) \cup I_0) \cup (h^{-1}(i) \cap I_0) \text{ for } 0 \leqq i < \bar{0}.$$

Arguing inductively suppose that at stage $j \leqq \beta + 2$ we have constructed $g^{-1}(i)$ for $i <$ for $i < j - 1$, such that

(0) $g^{-1}(i)$ is independent;

(1) $g^{-1}(i) \cap I_k = \bar{f}^{-1}(i) \cap I_k$ and $g^{-1}(i) \cap O_k = h^{-1}(i) \cap O_k$, where $i < \bar{k}$; and

(2) $\omega(B_j) \leqq \chi(G) - \overline{j-2}$ or $\omega(B_j) = 0$.

Since G is p-nearly perfect $\chi(B_j) \leqq \omega(B_j) + p$. Let t_j be a $\{\overline{j-1}, \ldots \chi(G) + pj\}$-coloring of B_j. Set

$$g^{-1}(i) = (f^{-1}(i) \cap I_j) \cup (t_j^{-1}(i) \cap B_j) \cup (h^{-1}(i) \cap O_j)$$

for all i such that $\overline{j-1} \leqq i < \bar{j}$. Clearly g^{-1} is independent and both

$$g^{-1}(i) \cap I_i = \bar{f}^{-1}(i) \cap I_i \text{ and}$$

$$g^{-1}(i) \cap O_k = h^{-1}(i) \cap O_k \text{ for all } i < \bar{j}.$$

To see that the inductive hypothesis is preserved we still must check that

$$\omega(B_{j+1}) \leqq \chi(G) - \overline{j-1}.$$

Since the diameter of a complete subgraph is one, any complete subgraph contained in B_{j+1} is contained in I_{j-1}, O_{j-1}, or B_j. Thus it suffices to show that $\omega(I_{j-1} - C_{j+1})$, $\omega(O_{j-1} - C_{j+1})$, and $\omega(B_j - C_{j+1})$ are each less than or equal to $\chi(G) - \overline{j-1}$. Firstly, consider $B_j - C_{j+1}$. $t_j \upharpoonright (B_j - C_{j+1})$ is a $\{\bar{j}, \ldots, \chi(G) + pj\}$-coloring of $B_j - C_{j+1}$. Thus

$$\omega(B_j - C_{j+1}) \leqq \chi(G) + pj + 1 - \bar{j} = \chi(G) - \overline{j-1}.$$

Secondly, consider $I_{j-1} - C_j$. Since

$$g^{-1}(i) \cap I_{j-1} = \bar{f}^{-1}(i) \cap I_{j-1} \text{ for } i < \overline{j-1},$$

$f \upharpoonright (I_{j-1} \cap C_j)$ is a $\{\overline{j-1}, \ldots, \chi(G) - 1\}$-coloring of $I_{j-1} \cap C_j \supset I_{j-1} \cap C_{j+1}$. Thus

$$\omega(I_{j-1} - C_{j+1}) \leqq \chi(G) - \overline{j-1}.$$

An analogous argument shows that

$$\omega(O_{j-1} - C_{j+1}) \leqq \chi(G) - \overline{j-1}.$$

Finally we check that g works. By (0) g is a coloring in its domain.

Since $U - C_i \subset I_i$ and $L[W] - (L[U] \cup C_i)$ for $i \leqq \beta$, (1) applied to $j = \beta + 1$ shows that

$$f \restriction U = g \restriction U \text{ and } h \restriction L[W] - W = g \restriction L[W] - W.$$

In particular $U \cup (L[W] - L[U])$ is contained in the domain of g. Also $L[W] - (U \cup (L[W] - W))$ is contained in $B_{\beta+2} \cup C_{\beta+2}$. Thus by (2) applied to $j = \beta + 2$, the domain of g is all of $L[W]$.

COROLLARY 3.2. *Let M be a highly recursive multigraph of multiplicity p. Then*

$$\chi_r'(G) \leqq \chi'(G) + p(\beta + 1) + 1$$

where β is the least integer k such that

$$\sum_{i=0}^{k} (pi + 1) \geqq \Delta(G).$$

4. Edge colorings. Notice that in the proof of Theorem 3.1, every time we used a new coloring t_j we had to "waste" p colors. This did not happen in the proof of Theorem 2.1 since G was perfect. In the next application the graph is not perfect but, even so, the following weak form of the Vizing Adjacency Lemma will allow us to avoid wasting colors when new t_j are used.

LEMMA 4.0. *Let $G = (V, E)$ be a graph and $S \subset V$. If $\chi'(G - S) = \Delta(G)$ and no vertex in S is adjacent to any vertex in V of degree $\Delta(G)$, then $\chi'(G) = \Delta(G)$.*

Proof. See [**13**] or [**5**].

Proof of Theorem 4.1. Let $d = 3\chi'(G)$, $L[U] = N^d[U]$, and $\alpha = \chi'(G) + 1$. By Lemma 1.1 it suffices to show that if U and W are finite subsets of V such that $L[U] \subset W$ and f is an α-coloring of $E(L[U])$ such that $f \restriction (E(L[U]) - E(U))$ is a $\chi'(G)$-coloring, then there exists an α-coloring g of $E(L[W])$ that extends $f \restriction E(U)$ and is a $\{1, \ldots, \alpha - 1\}$-coloring of $E(L[W]) - E(W)$. As in the proof of Theorem 2.1 we shall construct g one color at a time. Similar notation is needed to describe the construction.

Suppose that we have constructed $g^{-1}(i)$ for all $i < j$. Let $C_j = \bigcup_{i<j} g^{-1}(i)$ be the set of edges already colored. Let $I_j = N^{3(\alpha-1-j)}[U]$ and I_j be the partial subgraph $(I_j, E(I_j) - C_j)$. Similarly let $O_j = L[W] - I_j$ and $\mathscr{O}_j$ be the partial subgraph $(O_j, E(O_j) - C_j)$. Finally let $F_j^s = N^s[I_j] - I_j$. F_j^s is the set of vertices of O_j that are at a distance of at most s from some vertex in I_j. At stage j of the construction we shall specify those edges $g^{-1}(j)$ to be colored j. The "inner portion" $\mathscr{I}_j$ will be colored according to f. The "outer portion" $\mathscr{O}_j$ will be colored

according to a new coloring t_j chosen at stage j. These colorings will not conflict because no edge in $\mathscr{I}_j$ is incident to any edge in $\mathscr{O}_j$.

Set $g^{-1}(0) = \bar{f}^{-1}(0) \cap E(\mathscr{I}_0)$, which of course is the same as $\bar{f}^{-1}(0)$. Arguing inductively, suppose that we have constructed $g^{-1}(i)$ for $i < j \leqq \alpha$ such that:

(0) $g^{-1}(i)$ is independent;
(1) $f^{-1}(i) \cap E(I_i) \subset C_j$ and $g^{-1}(i) \cap E(U) = f^{-1}(i) \cap E(U)$
(2) $\chi'(O_j) \leqq \alpha - j$.

Let t_j be a $\{j, \ldots, \alpha - 1\}$-coloring of $E(O_j)$. Consider the independent set

$$J = (f^{-1}(j) \cap E(\mathscr{I}_j)) \cup t_j^{-1}(j).$$

Let $\hat{J}$ be a maximum independent set of edges such that

$$J \subset \hat{J} \subset E(L[W]) - (C_j \cup E(U)).$$

Finally let $g^{-1}(j) = \hat{J}$. Clearly $g^{-1}(j)$ is independent. Every edge of $f^{-1}(j) \cap E(I_j)$ is in C_{j+1} and

$$g^{-1}(j) \cap E(U) = f^{-1}(j) \cap E(U).$$

To see that the inductive hypotheses is preserved we still must check that $\chi'(\mathscr{O}_{j+1}) \leqq \alpha - j - 1$. We first show that $\Delta(\mathscr{O}_{j+1}) \leqq \alpha - j - 1$ and that any two vertices in $F_{j+1}{}^5$ of degree $\alpha - j - 1$ in O_{j+1} are independent. If $v \in O_{j+1} - F_{j+1}{}^5$ then $\delta_{\mathscr{O}_{j+1}}(v) \leqq \alpha - j - 1$ since $t_j \restriction (E(O_j) - C_{j+1})$ is a $\{j + 1, \ldots, \alpha - 1\}$-coloring, $t_j^{-1}(j) \subset g^{-1}(j) \cap E(O_j)$, and $N[\{v\}] \subset O_j$. So suppose $v \in F_{j+1}{}^5$. Then $N[\{v\}] \subset I_{j-1}$. Using (1), $f \restriction (E(\mathscr{I}_{j-1}) - E[U])$ is a $\{j - 1, \ldots, \alpha - 2\}$-coloring. Thus $\delta_{\mathscr{I}_{j-1}} \leqq \alpha - j$. Furthermore if $\delta_{\mathscr{I}_{j-1}}(v) = \alpha - j$, then some edge incident with v is colored $j - 1$ by f and hence is in C_j. Thus, depending on whether $v \in I_j$ or $v \in O_j$,

$$\delta_{\mathscr{I}_j}(v) \leqq \alpha - j - 1 \text{ or } \delta_{\mathscr{O}_j}(v) \leqq \alpha - j - 1.$$

In either case, $\delta_{\mathscr{O}_{j+1}}(v) \leqq \alpha - j - 1$ and if $\delta_{\mathscr{O}_{j+1}}(v) = \alpha - j - 1$ then no edge incident with v was colored j. Thus by the maximality of $\hat{J}$ and the generality of this argument, no other vertex in $F_s{}^5$ of degree $\alpha - j - 1$ in $\mathscr{O}_{j+1}$ can be adjacent to v.

Now let M be the set of vertices in $F_{j+1}{}^4$ of degree $\alpha - j - 1$ in O_{j+1}. No vertex in $F_{j+1}{}^3 - M$ is adjacent to any vertex of $O_j - M$ whose degree in O_{j+1} is $\alpha - j - 1$. Also $t_j \restriction (E(\mathscr{O}_j) - C_{j+1})$ is a $\{j + 1, \ldots, \alpha - 1\}$-coloring. Thus by Lemma 4.0 there exists a $\{j + 1, \ldots, \alpha - 1\}$-coloring of

$$(E(O_j - M) - C_{j+1}) \cup (E(F_{j+1}^4 - M) - C_{j+1}) = E(\mathscr{O}_{j+1} - M).$$

Since no vertex in M is adjacent to any vertex in O_{j+1} of degree $\alpha - j - 1$

in $\mathscr{O}_{j+1}$, further use of Lemma 4.0 provides a $\{j+1, \ldots, \alpha - 1\}$-coloring of $E(\mathscr{O}_{j+1})$. Thus $\chi'(\mathscr{O}_{j+1}) \leqq \alpha - j - 1$.

Finally, we check that g is an α-coloring of $E(L[W])$ which extends $f \upharpoonright E(U)$ and is a $\{1, \ldots, \alpha - 1\}$-coloring of $E(L[W]) - E[W]$. By (0) g is a coloring of its domain. By (1) $g \upharpoonright E(U) = f \upharpoonright E(U)$. $E(L[W]) = E(U) \cup E(\mathscr{O}_\alpha)$. Thus by (2) applied to $j = \alpha$, g is an α-coloring of $E(L[W])$. Since $I_0 = L[U] \subset W$,

$$g^{-1}(0) = f^{-1}(0) \cap E(I_0) \subset E(W).$$

Thus $g \upharpoonright (E(L[W]) - E(W))$ is a $\{1, \ldots, \alpha - 1\}$-coloring.

One should notice that Corollary 2.2 is also a corollary of Theorem 4.1.

By Vizing's Theorem [**13**] $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$. Manaster's and Rosenstein's example [**9**] shows that Theorem 4.1 cannot be improved when $\chi'(G) = \Delta(G)$. If $\chi'(G) = \Delta(G) + 1$ then by Theorem 4.1

$$\chi_r'(G) = \Delta(G) + 1 \text{ or } \chi_r'(G) = \Delta(G) + 2.$$

If $\Delta(G) = 1$ or 2 then clearly $\chi_r'(G) \leqq \Delta(G) + 1$. Suppose $\Delta(G) = 3$. Schmerl [**12**] has shown that Brooks' Theorem [**2**] is effective, i.e., if H is a highly recursive graph, $\Delta(H) \geqq 3$, and H has no complete subgraph of cardinality $\Delta(H) + 1$, then $\chi_r(H) \leqq \Delta(H)$. If G is a highly recursive graph with line graph H then H is highly recursive and $2(\Delta(G) - 1) = \Delta(H)$. Thus, for $\Delta(G) = 3$,

$$\chi_r'(G) = \chi_r(H) \leqq \Delta(H) = 4.$$

For $\Delta(G) > 3$ it is an open question whether $\chi_r'(G) \leqq \chi(G) + 1$. Another open problem is to improve the bound provided by Corollary 3.2 on $\chi_r'(M)$ where M is a multigraph.

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