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COMBINATORIAL SOLUTION OF CERTAIN SYSTEMS OF LINEAR EQUATIONS INVOLVING (0, 1) MATRICES

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Abstract

If m and n are natural numbers satisfying $1 \leqq m < n$ let $\left\langle {n \atop m} \right\rangle$ denote the least integer k such that the statement:

'*Every* (0, 1) *matrix with n columns, with constant row-sum m, and with at least k distinct rows, has rank n*'

is true. Then $\left\langle {n \atop m} \right\rangle = \binom{n-1}{m} + 1$ for $m \geqq 2$, $n \geqq m^2 + 2$. Further, $\left\langle {n \atop m} \right\rangle = \left\langle {n \atop n-m} \right\rangle$ for $1 \leqq m < n$.

1. Introduction

Let m and n be natural numbers satisfying $1 \leqq m < n$. With every non-empty family $\mathcal{F}$ of distinct m-element subsets of $\{1, 2, 3, \cdots, n\}$ can be associated a homgeneous system of equations in the real unknowns $x_1, x_2, x_3, \cdots, x_n$:

$$\sum_{i \in A} x_i = 0 \qquad (A \in \mathcal{F}).$$

The coefficient matrix of this system is a (0, 1) matrix with n columns, $|\mathcal{F}|$ distinct rows and constant row-sum m. Which such matrices have rank n? Equivalently: Which such systems have $x_1 = x_2 = x_3 = \cdots = x_n = 0$ as their only solution? Let $\left\langle {n \atop m} \right\rangle$ denote the least integer k such that the statement:

'For every non-empty family $\mathcal{F}$ of m-element subsets of $\{1, 2, 3, \cdots, n\}$ with at least k distinct elements the system $\Sigma_{i \in A} x_i = 0$ $(A \in \mathcal{F})$ has $x_1 = x_2 = x_3 = \cdots = x_n = 0$ as its only solution' is true. Notice that $k = \binom{n}{m}$ has this property and that $\left\langle{n \atop 1}\right\rangle = n$ for $n \geqq 2$.

For each $m \geqq 1$, $\left\langle{n \atop m}\right\rangle$ is evaluated for all but a finite number of n. In particular, it is shown that $\left\langle{n \atop m}\right\rangle = \binom{n-1}{m} + 1$ for $m \geqq 2$, $n \geqq m^2 + 2$. The proof uses certain combinatorial inequalities. Also, it is shown that $\left\langle{n \atop m}\right\rangle = \left\langle{n \atop n-m}\right\rangle$ for $1 \leqq m < n$. Applications, concerning uniqueness of solution, are made to more general real systems of linear equations.

2. Main theorem

Let $1 \leqq m < n$. The family $\mathscr{G}$, consisting of those m-element subsets of $\{1, 2, 3, \cdots, n\}$ which do not contain n, has $\binom{n-1}{m}$ elements and the system

$$\sum_{i \in A} x_i = 0 \qquad (A \in \mathscr{G})$$

does not have $x_1 = x_2 = x_3 = \cdots = x_n = 0$ as its only solution since x_n can be arbitrary. Thus $\left\langle{n \atop m}\right\rangle \geqq \binom{n-1}{m} + 1$.

THEOREM 2.1. $\left\langle{n \atop m}\right\rangle = \binom{n-1}{m} + 1$ *for* $m \geqq 2$ *and* $n \geqq m^2 + 2$.

REMARK. The cancellation law of addition is used in the proof. The proof is divided into five steps and involves several combinatorial inequalities. Establishing the validity of these inequalities, although necessary for the proof, adds nothing to its understanding. The more difficult inequalities are numbered I_1 to I_4 and are dealt with in the fifth and final step.

PROOF. It suffices to show that if $\mathscr{S}$ is a family of m-element subsets of $\{1, 2, 3, \cdots, n\}$ satisfying $|\mathscr{S}| = \binom{n-1}{m} + 1$ and $x_1, x_2, x_3, \cdots, x_n$ are real numbers satisfying $\Sigma_{i \in A} x_i = \Sigma_{j \in B} x_j$ $(A, B \in \mathscr{S})$ then $x_1 = x_2 = x_3 = \cdots = x_n$.

Step 1. It is shown that there is a subset $\mathscr{X}$ of $\{1, 2, 3, \cdots, n\}$ with $|\mathscr{X}| = 2m - 3$ such that $x_i = x_j$ whenever $i, j \in \mathscr{X}^c$ (where $\mathscr{X}^c$ denotes the complement of $\mathscr{X}$):

For every $(m-1)$-element subset S of $\{1,2,3,\cdots,n\}$ let q_S be the number of elements of $\mathcal{S}$ which contain S. Then

$$\sum_{\text{all } S} q_S = m\left\{\binom{n-1}{m}+1\right\}.$$

If $q_S \geqq n-2m+3$ we may take $\mathcal{X}$ to be $\mathcal{Y}^c$ where $\mathcal{Y}$ is any subset of $\{i : S \cup \{i\} \in \mathcal{S}\}$ of order $n-2m+3$. Suppose then, that $q_s \leqq n-2m+2$ for every S. There exists S_1 satisfying

$$q_{S_1} \geqq \frac{m\left\{\binom{n-1}{m}+1\right\}}{\binom{n}{m-1}}.$$

Since

$$\frac{m\left\{\binom{n-1}{m}+1\right\}}{\binom{n}{m-1}} > \frac{m\binom{n-1}{m}}{\binom{n}{m-1}} = \frac{(n-m+1)(n-m)}{n} \geqq n-2m+1,$$

it follows that $q_{S_1} = n-2m+2$. Suppose that k distinct $(m-1)$-element subsets S_i have been found satisfying $q_{S_i} = n-2m+2$ and that $1 \leqq k \leqq m\binom{n-1}{m-2}+m-1$. Since

$$(I_1) \qquad m\binom{n-1}{m-2}+m-1 < \binom{n}{m-1} \qquad (m \geqq 2,\ n \geqq m^2+2),$$

there is an $(m-1)$-element subset S_{k+1} distinct from S_i $(1 \leqq i \leqq k)$ such that

$$q_{S_{k+1}} \geqq \frac{m\left\{\binom{n-1}{m}+1\right\}-k(n-2m+2)}{\binom{n}{m-1}-k}.$$

It is easy to verify that

$$\frac{m\left\{\binom{n-1}{m}+1\right\}-k(n-2m+2)}{\binom{n}{m-1}-k} > n-2m+1,$$

so $q_{S_{k+1}} = n-2m+2$. It follows that there are $m\binom{n-1}{m-2}+m$ distinct $(m-1)$-element subsets S of $\{1,2,3,\cdots,n\}$ satisfying $q_S = n-2m+2$. Let

$\sigma = m\binom{n-1}{m-2} + m$. By using the cancellation law of addition it follows that there are σ (possibly not distinct) $(2m-2)$-element subsets $B_p (1 \leqq p \leqq \sigma)$ of $\{1, 2, 3, \cdots, n\}$ such that

(a) $i, j \in B_p^c$ (some p) implies $x_i = x_j$;

(b) From $\bigcup_{1 \leqq p \leqq \sigma} B_p$ can be chosen σ distinct $(m-1)$-element subsets.

If $\kappa = |\bigcup_{1 \leqq p \leqq \sigma} B_p|$ then, by (b), $\binom{\kappa}{m-1} \geqq \sigma$. Since

$$(I_2) \qquad \binom{2m-2}{m-1} < m\binom{n-1}{m-2} + m = \sigma \qquad (m \geqq 2,\ n \geqq m^2 + 2),$$

it follows that $\kappa > 2m-2$. Hence $B_{p_1} \neq B_{p_2}$ for some p_1 distinct from p_2. Now $|B_{p_1} \cap B_{p_2}| \leqq 2m-3$ and $(B_{p_1} \cup B_{p_2})^c$ is non-empty (since $n \geqq 4m-3$). If $i_0 \in (B_{p_1} \cup B_{p_2})^c$ and $j \in (B_{p_1} \cap B_{p_2})^c$ we have $x_j = x_{i_0}$ by (a). The subset $\mathscr{X}$ can be taken to be any $(2m-3)$-element subset of $\{1, 2, 3, \cdots, n\}$ containing $B_{p_1} \cap B_{p_2}$.

Step 2. It is shown that there is a member A of $\mathscr{S}$ which is disjoint from $\mathscr{X}$:

This follows from the inequality:

$$(I_3) \qquad \binom{n-1}{m} \geqq \binom{n}{m} - \binom{n-2m+3}{m} \qquad (m \geqq 2,\ n \geqq m^2 + 2).$$

Step 3. It is shown that, given any t distinct elements of $\{1, 2, 3, \cdots, n\}$ with $1 \leqq t \leqq 2m-3$ there is an element of $\mathscr{S}$ which contains one and only one of these t elements:

In all, there are $t\binom{n-t}{m-1}$ m-element subsets of $\{1, 2, 3, \cdots, n\}$ which contain precisely one of the t given elements. The desired result follows from the inequality:

$$(I_4) \quad \binom{n-1}{m} \geqq \binom{n}{m} - t\binom{n-t}{m-1} \qquad (m \geqq 2,\ n \geqq m^2 + 2,\ 1 \leqq t \leqq 2m-3).$$

Step 4. Assume that the combinatorial inequalities I_1–I_4 are valid. By the result of Step 1, $x_i = x_j$ whenever $i, j \in \mathscr{X}^c$ and $|\mathscr{X}| = 2m-3$. Let the subset A be as in Step 2. By the result of Step 3, there is an element B of $\mathscr{S}$ which contains precisely one element, r_1 say, of $\mathscr{X}$. Since $\Sigma_{i \in B} x_i = \Sigma_{j \in A} x_j$, it follows by cancellation that $x_{r_1} = x_j$ $(j \in \mathscr{X}^c)$. Thus $x_i = x_j$ whenever $i, j \in \mathscr{X}^c \cup \{r_1\}$. Similarly, by applying the result of Step 3 to the set $\mathscr{X} \backslash \{r_1\}$ we

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deduce that $x_{r_2} = x_j$ $(j \in \mathscr{X}^c \cup \{r_1\})$ for some r_2 satisfying $r_2 \in \mathscr{X} \backslash \{r_1\}$. Thus $x_i = x_j$ whenever $i, j \in \mathscr{X}^c \cup \{r_1, r_2\}$. Continuing in this way, we finally obtain the result that $x_1 = x_2 = x_3 = \cdots = x_n$.

Step 5. It remains to establish the validity of inequalities I_1–I_4:

$$(I_1): \qquad m\binom{n-1}{m-2} + m - 1 < \binom{n}{m-1} \qquad (m \geqq 2,\ n \geqq m^2+2).$$

This inequality is easily shown to be equivalent to

$$\binom{n-1}{m-2}\left[\frac{n}{m(m-1)} - 1\right] \geqq 1.$$

The latter is clearly valid for $m = 2$, $n \geqq 6$. If $m \geqq 3$, $n \geqq m^2 + 2$ we have $\dfrac{n}{m(m-1)} - 1 \geqq 0$ and $\binom{n-1}{m-2} \geqq n-1$. Hence,

$$\binom{n-1}{m-2}\left[\frac{n}{m(m-1)} - 1\right] \geqq (n-1)\left[\frac{n}{m(m-1)} - 1\right] \geqq 1.$$

$$(I_2): \qquad \binom{2m-2}{m-1} < m\binom{n-1}{m-2} + m \qquad (m \geqq 2,\ n \geqq m^2+2).$$

Clearly, it suffices to show that

$$\binom{2m-2}{m-1} \leqq \binom{m^2+1}{m-2} \qquad (m \geqq 3).$$

This is equivalent to showing that

$$\frac{2^{m-1}}{m-1}\prod_{r=1}^{m-2}(2m-2r-1) \leqq \prod_{r=1}^{m-2}(m^2+2-r) \qquad (m \geqq 3).$$

Since $2^{m-1}/(m-1) \leqq 2^{m-2}$ and $m^2 \geqq 4m - 7$ for $m \geqq 3$ it follows that

$$\frac{2^{m-1}}{m-1}\prod_{r=1}^{m-2}(2m-2r-1) \leqq \prod_{r=1}^{m-2}(4m-4r-2) \leqq \prod_{r=1}^{m-2}(m^2+2-r).$$

$$(I_3): \qquad \binom{n-1}{m} \geqq \binom{n}{m} - \binom{n-2m+3}{m} \qquad (m \geqq 2,\ n \geqq m^2+2).$$

This inequality is easily shown to be equivalent to

$$\prod_{r=1}^{m-1}\left(\frac{n-3m+4+r}{n-m+r}\right)\cdot\frac{n-3m+4}{m} \geqq 1 \qquad (m \geqq 2,\ n \geqq m^2+2).$$

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Notice that, if $m \geqq 2$ and $n_1 \geqq n_2 \geqq m^2+2$ we have

$$\frac{n_1 - 3m + 4 + r}{n_1 - m + r} \geqq \frac{n_2 - 3m + 4 + r}{n_2 - m + r} \qquad (1 \leqq r \leqq m-1).$$

Therefore, it suffices to show that

$$\prod_{r=1}^{m-1} \left(\frac{m^2 - 3m + 6 + r}{m^2 - m + 2 + r}\right) \cdot \frac{m^2 - 3m + 6}{m} \geqq 1 \qquad (m \geqq 2).$$

If

$$a_m = \prod_{r=1}^{m-1} \left(\frac{m^2 - 3m + 6 + r}{m^2 - m + 2 + r}\right) \quad \text{for} \quad m \geqq 2$$

then

$$a_m \geqq \left(\frac{m^2 - 3m + 7}{m^2 - m + 3}\right)^{m-1} = \left(1 - \frac{2b_m}{m-1}\right)^{m-1},$$

where

$$b_m = \frac{(m-1)(m-2)}{m^2 - m + 3} \qquad (m \geqq 2).$$

Since $b_m \leqq 1$ we have $a_m \geqq \left(1 - \frac{2}{m-1}\right)^{m-1}$ $(m \geqq 2)$. The sequence $c_m = \left(1 - \frac{2}{m-1}\right)^{m-1}$ is monotone non-decreasing for $m \geqq 2$. Hence $a_m \geqq c_4 = \frac{1}{27}$ for $m \geqq 4$. Thus,

$$a_m \cdot \frac{m^2 - 3m + 6}{m} \geqq \frac{m^2 - 3m + 6}{27m} \geqq 1 \quad \text{for} \quad m \geqq 30.$$

That $a_m \cdot (m^2 - 3m + 6)/m \geqq 1$ is true for $2 \leqq m \leqq 29$ is easily checked directly.

$$(I_4)\colon \binom{n-1}{m} \geqq \binom{n}{m} - t\binom{n-t}{m-1} \qquad (m \geqq 2,\ n \geqq m^2+2,\ 1 \leqq t \leqq 2m-3).$$

We may suppose $m \geqq 3$, $n \geqq m^2+2$, $2 \leqq t \leqq 2m-3$. It is easily shown that this inequality is equivalent to

$$\prod_{r=0}^{t-2} \left(\frac{n - t + 1 + r}{n - m - t + 2 + r}\right) \leqq t \qquad (m \geqq 3,\ n \geqq m^2+2,\ 2 \leqq t \leqq 2m-3).$$

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If $n_1 \geqq n_2 \geqq m^2+2$ and $m \leqq 3$, $2 \leqq t \leqq 2m-3$ we have

$$\frac{n_1 - t + 1 + r}{n_1 - m - t + 2 + r} \leqq \frac{n_2 - t + 1 + r}{n_2 - m - t + 2 + r}.$$

Therefore, it suffices to show that

$$\prod_{r=0}^{t-2}\left(\frac{m^2+3-t+r}{m^2-m+4-t+r}\right) \leqq t \qquad (m \geqq 3,\ 2 \leqq t \leqq 2m-3).$$

Since

$$\prod_{r=0}^{t-2}\left(\frac{m^2+3-t+r}{m^2-m+4-t+r}\right) \leqq \left(\frac{m^2+3-t}{m^2-m+4-t}\right)^{t-1}$$

it suffices to show that

$$\frac{m^2+3-t}{m^2-m+4-t} \leqq t^{\frac{1}{t-1}}$$

or equivalently, that

$$t + \frac{m-1}{t^{\frac{1}{t-1}}-1} \leqq m^2 - m + 4 \qquad (m \geqq 3,\ 2 \leqq t \leqq 2m-3).$$

Since the function $t \mapsto t^{\frac{1}{t-1}}$ is monotone decreasing it suffices to show that

$$2m - 3 + \frac{m-1}{(2m-3)^{\frac{1}{2m-4}}-1} \leqq m^2 - m + 4 \qquad (m \geqq 3)$$

or, that

$$\left(1 + \frac{m-1}{m^2-3m+7}\right)^{2m-4} \leqq 2m-3 \qquad (m \geqq 3).$$

Clearly, this is true for $m = 3$ or 4. If $m \geqq 5$ and $x = 2m-4$ we have to show that

$$1 + x \geqq \left(1 + \frac{2x+4}{x^2+2x+20}\right)^x.$$

Now

$$\frac{2x+4}{x^2+2x+20} \leqq \frac{2}{x} \quad \text{for} \quad x \geqq 6.$$

Also, $1 + x \geqq (1 + (2/x))^x$ for $x \geqq 6$. The result follows.

This completes the proof of the theorem.

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3. Extension and application

The number $\left\langle {n \atop m} \right\rangle$ has been evaluated for $m = 1$, $n \geqq 2$ and for $m \geqq 2$, $n \geqq m^2 + 2$. In fact, $\left\langle {n \atop m} \right\rangle = \binom{n-1}{m} + 1$ in these cases. The domain of evaluation of $\left\langle {n \atop m} \right\rangle$ can be extended by the following observation:

If $1 \leqq m < n$ we have $\left\langle {n \atop m} \right\rangle = \left\langle {n \atop n-m} \right\rangle$.

For, suppose $\mathscr{S}$ is a family of m-element subsets of $\{1, 2, 3, \cdots, n\}$ with $|\mathscr{S}| = \left\langle {n \atop n-m} \right\rangle$ and take real numbers $x_1, x_2, x_3, \cdots, x_n$ satisfying $\Sigma_{i \in A} x_i = 0$ $(A \in \mathscr{S})$. If $s = \Sigma_{1 \leqq i \leqq n} x_i$ and $y_i = x_i - s/(n-m)$ we have $\Sigma_{i \in A^c} y_i = 0$ $(A \in \mathscr{S})$. Hence, by definition of $\left\langle {n \atop n-m} \right\rangle$, $y_i = 0$ $(1 \leqq i \leqq n)$. Thus $x_i = 0$ $(1 \leqq i \leqq n)$. This shows that $\left\langle {n \atop m} \right\rangle \leqq \left\langle {n \atop n-m} \right\rangle$. The reverse inequality follows by symmetry.

Thus, for example, $\left\langle {6 \atop 4} \right\rangle = \left\langle {6 \atop 2} \right\rangle = \left\langle {5 \atop 2} \right\rangle + 1 = 11$ and $\left\langle {n \atop n-1} \right\rangle = \left\langle {n \atop 1} \right\rangle = n$ for $n \geqq 2$.

The numbers $\left\langle {n \atop m} \right\rangle$ provide sufficient conditions for the uniqueness of solution of certain homgeneous real linear systems of equations. We now show how these conditions can be applied to a more general class of real linear systems to establish uniqueness of solution.

Let n, m and k be natural numbers satisfying $1 \leqq m < n$ and $k \geqq \left\langle {n \atop m} \right\rangle$. Let E be a real linear system of equations in unknowns $x_1, x_2, x_3, \cdots, x_n$ such that

(I) The system reads "$\Sigma_{j=1}^{n} a_{ij} x_j = b$ $(i = 1, 2, 3, \cdots, k)$"; (That is, b is the same for all equations of the system).

(II) The matrix (a_{ij}) has precisely m non-zero entries in each row;

(III) The non-zero coefficients of x_j are equal;

(IV) No two equations are identical.

THEOREM 3.1. *Each system E (as above) has a unique solution, $x_j = b/a_j m$ $(j = 1, 2, 3, \cdots, n)$ where a_j is any non-zero coefficient of x_j.*

PROOF. Since $k \geqq \left\langle {n \atop m} \right\rangle \geqq \binom{n-1}{m} + 1$, every x_j has a non-zero coeffi-

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cient a_j. If we put $y_j = a_j x_j - b/m$ the system becomes $\Sigma_{j\in A} y_j = 0$ $(A \in \mathcal{S})$, where $\mathcal{S}$ is a family of m-element subsets of $\{1,2,3,\cdots,n\}$ with $|\mathcal{S}| = k \geqq \left\langle {n \atop m} \right\rangle$. It follows from the definition of $\left\langle {n \atop m} \right\rangle$ that $y_j = 0$ $(j = 1,2,3,\cdots,n)$. Hence $x_j = b/a_j m$ $(j = 1,2,3,\cdots,n)$ and the theorem is proved.

EXAMPLE. The system

$$\begin{bmatrix} 0 & 0 & 4 & 3 & 2 & 1 \\ 0 & 5 & 0 & 3 & 2 & 1 \\ 0 & 5 & 4 & 0 & 2 & 1 \\ 0 & 5 & 4 & 3 & 2 & 0 \\ 6 & 0 & 0 & 3 & 2 & 1 \\ 6 & 0 & 4 & 3 & 0 & 1 \\ 6 & 0 & 4 & 3 & 2 & 0 \\ 6 & 5 & 0 & 3 & 0 & 1 \\ 6 & 5 & 0 & 3 & 2 & 0 \\ 6 & 5 & 4 & 0 & 0 & 1 \\ 6 & 5 & 4 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 3 \end{bmatrix}$$

has unique solution $x_1 = \frac{1}{8}$, $x_2 = \frac{3}{20}$, $x_3 = \frac{3}{16}$, $x_4 = \frac{1}{4}$, $x_5 = \frac{3}{8}$, $x_6 = \frac{3}{4}$.

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