A SYMPLECTIC INTEGRATION SCHEME THAT ALLOWS CLOSE ENCOUNTERS BETWEEN MASSIVE BODIES

J. E. CHAMBERS
Armagh Observatory, Armagh, North Ireland, UK

Mixed-variable symplectic integrators provide a fast, moderately accurate way to study the long-term evolution of a wide variety of N-body systems (Wisdom & Holman 1991). They are especially suited to planetary and satellite systems, in which a central body contains most of the mass. However, in their original form, they become inaccurate whenever two bodies approach one another closely. Here, I will show how to overcome this difficulty using a hybrid integrator that combines symplectic and conventional algorithms.

A symplectic integrator works by splitting the Hamiltonian, H, for an N-body system, into two or more parts $H = H_0 + H_1 + \cdots$, where $\epsilon_i = H_i/H_0 \ll 1$ for $i = 1, 2 \ldots$. An integration step consists of several substeps, each of which advances the system due to the effect of one part of the Hamiltonian only. The error incurred over the whole step is $\sim \epsilon\, \tau^{n+1}$, where τ is the timestep, n is the order of the integrator, and ϵ is the largest of ϵ_i.

A symplectic algorithm is efficient provided that the ϵ factors are small. In the planetary system, this is usually achieved by making H_0 the unperturbed Keplerian motion of the planets about the Sun, and H_1, H_2 etc. the perturbations between planets. For example, using mixed coordinates (heliocentric positions and barycentric velocities), H is split up as

$$
\begin{aligned}
H_0 &= \sum_{i=1}^{N} \left(\frac{p_i^2}{2m_i} - \frac{Gm_\odot m_i}{r_{i\odot}} \right) \\
H_1 &= -G \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{m_i m_j}{r_{ij}} \\
H_2 &= \frac{1}{2m_\odot} \left(\sum_{i=1}^{N} \mathrm{p}_i \right)^2 \qquad (1)
\end{aligned}
$$

where N is the number of planets, m denotes mass, and r, p are position and momentum respectively. Note that each part of the Hamiltonian can be integrated analytically in the absence of the others, and H_0 contains all of the large terms, provided that the planets are widely separated.

Now consider a close encounter between bodies a and b. During the encounter, the distance r_{ab} is small, and the corresponding term in H_1 is large. This means that ϵ_1 is no longer small and the integrator becomes inaccurate. An approximate solution to this difficulty is to transfer the offending term from H_1 to H_0 for the duration of the encounter. This ensures that ϵ_1 is always small. However, each time

J. Henrard and S. Ferraz-Mello (eds.), Impact of Modern Dynamics in Astronomy, 449–450.

H_0 and H_1 are changed in this way, the system undergoes a shift in energy, and the integrator's symplectic property is lost.

A better solution is to split each of the interaction terms between H_0 and H_1 as follows

$$
\begin{aligned}
H_0 &= \sum_{i=1}^{N}\left(\frac{p_i^2}{2m_i} - \frac{Gm_{\odot}m_i}{r_{i\odot}}\right) - G\sum_{i=1}^{N}\sum_{j=i+1}^{N}\frac{m_i m_j}{r_{ij}}[1 - K(r_{ij})] \\
H_1 &= -G\sum_{i=1}^{N}\sum_{j=i+1}^{N}\frac{m_i m_j}{r_{ij}}K(r_{ij})
\end{aligned}
\tag{2}
$$

where the function K is chosen so that $K \rightarrow 0$ when r_{ij} is small, and $K \rightarrow 1$ when r_{ij} is large. This ensures that ϵ_1 is always small, without requiring that terms move from one part of the Hamiltonian to another.

When all of the separations r_{ij} are large, H_0 can be advanced analytically as before (since $1 - K = 0$). If two bodies undergo a close encounter, the terms in H_0 due to these objects must be integrated numerically, but all the remaining terms can still be advanced analytically. By trial and error, I find that a good expression for K is

$$
K = \begin{cases} 0 & \text{for } y < 0 \\ y^2/(2y^2 - 2y + 1) & \text{for } 0 < y < 1 \\ 1 & \text{for } y > 1 \end{cases}
$$

where

$$
y = \left(\frac{r_{ij} - 0.1\, r_{crit}}{0.9\, r_{crit}}\right)
$$

and r_{crit} is the larger of 3 Hill radii and $0.5\tau v_{max}$, where v_{max} is the maximum likely orbital velocity of any of the objects.

References

Wisdom J., Holman M.: 1991, *Astron.J.*, **102**, 1528.