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ON WEIGHTED GEOMETRIC MEANS

BY
HORST ALZER

ABSTRACT. The aim of this paper is two-fold: First we prove the Rado-type inequality $P_{n-1}[G'_{n-1} - G_{n-1}] \leqq P_n[Gn'_n - G_n]$. Here G_n and G'_n denote the weighted geometric means of $x_1, \ldots, x_n$ and $1 - x_1, \ldots, 1 - x_n (x_i \in (0, 1/2], i = 1, \ldots, n)$, i.e.

$$G_n = \prod_{i=1}^{n} x_i^{p_i/P_n} \text{ and } G'_n = \prod_{i=1}^{n} (1 - x_i)^{p_i/P_n},$$

with $P_n = \sum_{i=1}^{n} p_i$ where the p_i are positive weights. Thereafter we investigate under which conditions the sequence

$$[(G'_n)^{P_n/P_{n+1}} - (G_n)^{P_n/P_{n+1}}]/[G'_n - G_n]$$

is convergent as $n \longrightarrow \infty$.

1. **Introduction.** In the development of the theory of inequalities the famous inequality between the geometric and arithmetic means of n positive real numbers $x_1, \ldots, x_n$:

$$(1.1) \qquad g_n = \prod_{i=1}^{n} x_i^{1/n} \leqq \frac{1}{n} \sum_{i=1}^{n} x_i = a_n$$

has played a central role. Because of its importance many mathematicians have published numerous proofs, sharpenings and extensions. We refer to the book "Inequalities" by E. F. Beckenbach and R. Bellman where one can find an interesting collection of twelve different proofs for what they call "probably the most important inequality, and certainly a keystone of the theory of inequalities" [5, p. 3]; see also [8] and [15].

Associated with (1. 1) there is the inequality of R. Radó [8]:

$$(1.2) \qquad (n-1)(a_{n-1} - g_{n-1}) \leqq n(a_n - g_n)$$

which was published for the first time in 1934, and there is its multiplicative analogue due to T. Popoviciu [18]:

$$(1.3) \qquad (a_{n-1}/g_{n-1})^{n-1} \leqq (a_n/g_n)^n$$

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proved in 1960. We remark that equality holds in (1.2) (resp. (1.3)) if and only if $x_n = g_{n-1}$ (resp. $x_n = a_{n-1}$). Many generalizations of (1.2) and (1.3) have been given, see, e.g., [6], [15], [21], [22], [25].

Inequality (1.1) can be written in the forms

$$a_n - g_n \geqq 0 \text{ and } a_n/g_n \geqq 1;$$

repeated applications of (1.2) and (1.3) lead to remarkable refinements of these inequalities:

$$(1.4) \qquad a_n - g_n \geqq \frac{1}{n} \max_{1 \leqq j,k \leqq n} \left[\sqrt{x_j} - \sqrt{x_k}\right]^2$$

and

$$a_n/g_n \geqq \max_{1 \leqq j,k \leqq n} \left[\frac{1}{2}(\sqrt{x_j/x_k} + \sqrt{x_k/x_j})\right]^{2/n}.$$

A very interesting counterpart of (1.1) is the celebrated Ky Fan inequality

$$(1.5) \qquad \left[\prod_{i=1}^{n} x_i/(1-x_i)\right]^{1/n} \leqq \sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n} (l - x_i)$$

$$(0 < x_i \leqq 1/2; i = 1, \ldots, n),$$

with equality holding only if $x_1 = \ldots = x_n$. Inequality (1.5) which "can be established by forward and backward induction" [5, p. 5] was published for the first time in 1961. Since then a lot of intensive work has been done to find new proofs and different extensions see [3], [7], [13], [19], [23–25].

C.-L. Wang who emphasized the resemblance between (1.1) and (1.5) called Fan's inequality "a replica of the A-G inequality" [23, p. 502].

If we denote by A_n and G_n (resp. A_n' and G_n') the weighted geometric and arithmetic means of $x_1, \ldots, x_n$ (resp. $1 - x_1, \ldots, 1 - x_n$), i.e.

$$A_n = \sum_{i=1}^{n} x_i p_i / P_n, \quad G_n = \prod_{i=1}^{n} x_i^{p_i/P_n},$$

and

$$A_n' = \sum_{i=1}^{n} (1 - x_i) p_i / P_n, \quad G_n' = \prod_{i=1}^{n} (1 - x_i)^{p_i/P_n},$$

with $P_n = \sum_{i=1}^{n} p_i$ and positive weights $p_1, \ldots, p_n$, then the following noteworthy Rado-type inequality holds for all $x_i \in (0, 1/2], i = 1, \ldots, n$:

$$(1.6) \qquad P_{n-1}[A_{n-1}(G_{n-1} + G_{n-1}') - G_{n-1}] \leqq P_n[A_n(G_n + G_n') - G_n].$$

This inequality is due to C.-L. Wang [23].

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From (1.6) we obtain inductively

$$G_n \leqq A_n(G_n + G'_n)$$

and because of $A_n + A'_n = 1$ the last inequality is equivalent to

$$G_n/G'_n \leqq A_n/A'_n, \tag{1.7}$$

which generalizes (1.5). Inequality (1.7) is called a "Ky Fan inequality of the complementary A-G type" [23, p. 503].

The aim of this paper is to prove new results concerning the weighted geometric means G_n and G'_n. A simple calculation yields that the Popoviciu-type inequality

$$(G'_{n-1}/G_{n-1})^{P_{n-1}} \leqq (G'_n/G_n)^{P_n} \tag{1.8}$$

is valid for all $x_i \in (0, 1/2], i = 1, \ldots, n$, with equality holding in (1.8) if and only if $x_n = 1/2$. It is natural to ask whether the corresponding (non-trivial) Rado-type inequality

$$P_{n-1}[G'_{n-1} - G_{n-1}] \leqq P_n[G'_n - G_n]$$

is also true. In Section 2 we will give an affirmative answer to this question. In the final section we shall investigate under which conditions the sequence

$$[(G'_n)^{P_n/P_{n+1}} - (G_n)^{P_n/P_{n+1}}]/[G'_n - G_n]$$

is convergent as $n \longrightarrow \infty$. In what follows we maintain the notations introduced in this section.

2. **A Rado-type inequality.** Let us begin by formulating a lemma that we shall need later.

LEMMA. If $L_r(x, y)$ (with distinct positive real numbers x and y and real parameters r) denotes the mean value family

$$L_r(x, y) = \left[\frac{x^r - y^r}{r(x - y)}\right]^{1/(r-1)}, \qquad r \neq 0, 1,$$

$$L_0(x, y) = \frac{x - y}{\ln(x) - \ln(y)},$$

$$L_1(x, y) = \frac{1}{e}(x^x/y^y)^{1/(x-y)},$$

then we have for all real r and s:

$$\textit{If } r < s \textit{ then } L_r(x, y) < L_s(x, y). \tag{2.1}$$

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An elegant *proof* of this lemma can be found in [20]. Further properties of $L_r(x,y)$ have been published in [1], [2], [4], [10–12], [20]. This remarkable family was introduced by K. B. Stolarsky [20] in 1975. Since it contains the logarithmic mean

$$L(x,y) = \frac{x-y}{\ln(x)-\ln(y)}$$

as a special case, $L_r(x,y)$ is sometimes called "Stolarsky's generalized logarithmic mean". The logarithmic mean which is frequently used in several physical, chemical and economical problems [9], [14], [16], [17] has been studied intensively by many authors in the last years, see, e.g., [1], [11] and the references therein.

Now we establish the Rado-type inequality for G_n and G'_n we have promised in the Introduction.

THEOREM 1. *For all real numbers $x_i \in (0,1/2], i = 1,\ldots,n$, we have*

$$P_{n-1}[G'_{n-1} - G_{n-1}] \leqq P_n[G'_n - G_n] \tag{2.2}$$

with equality holding in (2.2) if and only if $x_1 = \ldots = x_n = 1/2$.

PROOF. In order to establish (2.2) we consider the function f defined by

$$f : (0,1/2] \longrightarrow \mathbf{R}$$

$$f(x) = P_n[(G'_{n-1})^{P_{n-1}/P_n}(1-x)^{p_n/P_n} - (G_{n-1})^{P_{n-1}/P_n}x^{p_n/P_n}] - P_{n-1}[G'_{n-1} - G_{n-1}].$$

Differentiation of f yields

$$\frac{d}{dx}f(x) = -p_n[(G'_{n-1})^{P_{n-1}/P_n}(1-x)^{p_n/P_n-1} + (G_{n-1})^{P_{n-1}/P_n}x^{p_n/P_n-1}] < 0;$$

therefore we obtain

$$f(x) \geqq f(1/2) \tag{2.3}$$

with equality if and only if $x = 1/2$. If we set in (2.1) $0 < r < 1$ and $s = 2$ then we get for $y \geqq x > 0$:

$$r(y-x)((x+y)/2)^{r-1} \leqq y^r - x^r \tag{2.4}$$

where the sign of equality holds only if $x = y$.

Next, if we put $y = G'_{n-1}$, $x = G_{n-1}$ and $r = P_{n-1}/P_n$ in (2.4) then we have

$$\begin{aligned}(G'_{n-1})^{P_{n-1}/P_n} &- (G_{n-1})^{P_{n-1}/P_n} \\ &\geqq [G'_{n-1} - G_{n-1}]((G'_{n-1} + G_{n-1})/2)^{-p_n/P_n}P_{n-1}/P_n \\ &\geqq [G'_{n-1} - G_{n-1}](1/2)^{-p_n/P_n}P_{n-1}/P_n\end{aligned} \tag{2.5}$$

where the last inequality follows from

$$G'_{n-1} + G_{n-1} \leqq A'_{n-1} + A_{n-1} = 1.$$

Now we replace x by x_n in (2.3). Then (2.3) and (2.5) yield

$$f(x_n) \geqq f(1/2) = P_n(1/2)^{p_n/P_n}[(G'_{n-1})^{P_{n-1}/P_n} - (G_{n-1})^{P_{n-1}/P_n}] - P_{n-1}[G'_{n-1} - G_{n-1}] \geqq 0. \tag{2.6}$$

Hence we have established inequality (2.2).

To complete the proof of Theorem 1 we must consider the case

$$P_n[G'_n - G_n] = P_{n-1}[G'_{n-1} - G_{n-1}]. \tag{2.7}$$

From (2.6) we get

$$f(x_n) = f(1/2) = 0$$

which is valid (note the conditions for equality that are stated immediately after (2.3) and (2.4)) if and only if

$$x_n = 1/2 \qquad \text{and} \qquad G'_{n-1} = G_{n-1}. \tag{2.8}$$

A simple calculation yields that $G'_{n-1} = G_{n-1}$ holds only if $x_1 = \ldots = x_{n-1} = 1/2$; thus we conclude from (2.8) that (2.7) is valid if and only if $x_1 = \ldots = x_n = 1/2$. Theorem 1 is proved. □

REMARK. Repeated application of inequality (2.2) leads to the following counterpart of (1.4):

$$G'_n - G_n \geqq \frac{1}{P_n} \max_{1 \leqq j,k \leqq n} (p_j + p_k)[[(1-x_j)^{p_j}(1-x_k)^{p_k}]^{1/(p_j+p_k)} - [x_j^{p_j} x_k^{p_k}]^{1/(p_j+p_k)}].$$

3. **A limit theorem**. In the previous section we have introduced the logarithmic mean $L(x, y)$ of two distinct positive real numbers x and y and mentioned its importance for some practical problems. In the following limit theorem this mean value plays a central role, too.

THEOREM 2. *If not all of the numbers $x_i \in (0, 1/2], i = 1, \ldots, n$, are equal to $1/2$, if*

$$\lim_{n \to \infty} P_n/P_{n+1} = 1$$

and

(3.1) $L(G_n, G'_n) \geqq 1/e$ *for all sufficiently large integers n,*

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then we have

$$\lim_{n\to\infty}[(G_n')^{P_n/P_{n+1}} - (G_n)^{P_n/P_{n+1}}]/[G_n' - G_n] = 1. \tag{3.2}$$

PROOF. We shall establish that the double inequality

$$2^{p_{n+1}/P_{n+1}} P_n/P_{n+1} \leqq [(G_n')^{P_n/P_{n+1}} - (G_n)^{P_n/P_{n+1}}]/[G_n' - G_n] < 1 \tag{3.3}$$

is valid for all sufficiently large integers n.

Since $\lim_{n\to\infty} P_n/P_{n+1} = 1$, we obtain

$$\lim_{n\to\infty} p_{n+1}/P_{n+1} = \lim_{n\to\infty}[1 + [P_{n+1}/P_n - 1]^{-1}]^{-1} = 0$$

and as an immediate consequence of (3.3) we get the desired result (3.2).

It remains to prove (3.3). Since the numbers $x_1, \ldots, x_n$ are not all equal to 1/2 we have $G_n' - G_n > 0$. First we replace in inequality (2.5) $n-1$ by n and n by $n+1$; thereafter we divide both sides of (2.5) by $G_n' - G_n$, then we find that the left-hand inequality of (3.3) holds for all integers $n \geqq 1$.

In order to establish the right-hand inequality of (3.3) we set in (2.1) $r = 0$, $s = P_n/P_{n+1}$, $y = G_n'$ and $x = G_n$. Then we get for all sufficiently large integers n:

$$\begin{aligned}[(G_n')^{P_n/P_{n+1}} - (G_n)^{P_n/P_{n+1}}]/[G_n' - G_n] &< [L(G_n, G_n')]^{P_n/P_{n+1}-1} P_n/P_{n+1} \\ &\leqq (1/e)^{P_n/P_{n+1}-1} P_n/P_{n+1}\end{aligned}$$

where the last inequality follows immediately from hypothesis (3.1). An easy calculation yields that the inequality

$$a(1/e)^{a-1} < 1 \tag{3.4}$$

is fulfilled for all real numbers $a \in (0,1)$. Finally we replace a by P_n/P_{n+1} in (3.4) then we have

$$(1/e)^{P_n/P_{n+1}-1} P_n/P_{n+1} < 1,$$

which completes our proof of Theorem 2. □

REMARKS. 1. The inequality

$$L(G_n, G_n') \geqq 1/e$$

is fulfilled for arbitrary real numbers $x_i \in [1/e, 1/2]$, $i = 1, \ldots, n$.

2. We have proved that (3.1) is a sufficient condition for (3.2) to hold. Is it also necessary? The multiplicative analogue of Cauchy's limit theorem states: Let a_i, $i =$

$1, 2, \ldots$, be positive real numbers; if the sequence (a_n) tends to $a > 0$ as $n \longrightarrow \infty$, then

$$\lim_{n\longrightarrow\infty} \prod_{i=1}^{n} a_i^{1/n} = a.$$

As an immediate consequence of this theorem we obtain:

$$\text{Let } x_i \in (0, 1/2), i = 1, 2, \ldots; \text{if } \lim_{n\longrightarrow\infty} x_n = x \in (0, 1/2)$$

and

$$\lim_{n\longrightarrow\infty} P_n/P_{n+1} = 1,$$

then the ratio difference given in (3.2) tends to 1 as $n \longrightarrow \infty$. Since $L(x, 1-x) < 1/e$ for all sufficiently small positive x we conclude from $G_n \longrightarrow x$ and $G'_n \longrightarrow 1-x (n \longrightarrow \infty)$ that $L(G_n, G'_n) < 1/e$ for all sufficiently large integers n. This shows that the condition (3.1) is not necessary. It remains an open question to find sufficient and necessary conditions such that (3.2) is true.

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Horst Alzer
Morsbacher Str. 10
5220 Waldbröl
Federal Republic of Germany