Compositio Mathematica **114:** 3–35, 1998.

Linearity properties of Shimura varieties, II

BEN MOONEN
Westfälische Wilhelms-Universität Münster, Mathematisches Institut, Einsteinstraße 62, 48149 Münster, Germany

Abstract. Let $\mathsf{A}{=}\mathsf{A}_{g,1,n}$ denote the moduli scheme over $\mathbb{Z}[1/n]$ of p.p. g-dimensional abelian varieties with a level n structure; its generic fibre can be described as a Shimura variety. We study its 'Shimura subvarieties'. If $x\in\mathsf{A}$ is an ordinary moduli point in characteristic p, then we formulate a local 'linearity property' in terms of the Serre–Tate group structure on the formal deformation space ($=$ formal completion of A at x). We prove that an irreducible algebraic subvariety of A is a 'Shimura subvariety' if, locally at an ordinary point x, it is 'formally linear'. We show that there is a close connection to a differential-geometrical linearity property in characteristic 0.

We apply our results to the study of Oort's conjecture on subvarieties $Z\hookrightarrow\mathsf{A}$ with a dense collection of CM-points. We give a reformulation of this conjecture, and we prove it in a special case.

Mathematics Subject Classifications (1991): 14G, 14D, 11G.

Key words: Shimura varieties, Serre–Tate theory, Oort's conjecture.

Introduction

In this second part of our work on 'linearity properties', which for a large part is independent of Part I ($=$ [19]), we continue our investigation of subvarieties of Hodge type in a given Shimura variety. We study their properties, and in particular the question of how such subvarieties of Hodge type can be characterized. Here we are not looking for a *description* of all subvarieties (which can be given in terms of the Deligne formalism of Shimura varieties) but rather for direct *characterizations* of when an algebraic subvariety $Z\hookrightarrow \mathrm{Sh}_K(G,\mathfrak{X})$ is of Hodge type. The conjectures of Coleman and Oort (see the introduction of Part I; for Oort's conjecture see also below and Section 5) can be seen as motivating problems.

In this paper, we restrict our attention to subvarieties $Z\hookrightarrow\mathsf{A}_{g,1,n}$ of the moduli space of g-dimensional abelian varieties ($+$ polarization and a level structure). Similar to Th. 4.3 in Part I, we prove that an algebraic subvariety Z is of Hodge type if and only if it satisfies a certain 'linearity property'. In this case, however, we work with a linearity property (called 'formal linearity') of more arithmetic flavour than the 'total geodesicness' considered in Part I. The set-up is as follows.

Let $Z\hookrightarrow\mathsf{A}_{g,1,n}\otimes F$ be an absolutely irreducible algebraic subvariety of $\mathsf{A}_{g,1,n}$ over a number field F. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_F$ not dividing n, write $\mathcal{A}_g=\mathsf{A}_{g,1,n}\otimes\widehat{\mathcal{O}}_{\mathfrak{p}}$, and define $\mathcal{Z}\hookrightarrow\mathcal{A}_g$ as the Zariski closure of Z inside $\mathcal{A}_g$. Suppose $x\in\mathcal{Z}\otimes\kappa(\mathfrak{p})$ is a closed ordinary moduli point. Taking formal completions at the point x, we obtain formal schemes $\mathfrak{Z}_x\hookrightarrow\mathfrak{A}_x$ over $\mathrm{Spf}(\Lambda)$, where $\Lambda=$

$W(\kappa(x)) \otimes_{W(\kappa(\mathfrak{p}))} \widehat{\mathcal{O}}_{\mathfrak{p}}$. By Serre–Tate theory, $\mathfrak{A}_x$ has a natural structure of a formal torus. We say that $\mathcal{Z}$ is formally linear at x if $\mathfrak{Z}_x \hookrightarrow \mathfrak{A}_x$ is a formal subtorus.

Our study of this notion of formal linearity was motivated by results of Rutger Noot ([21], see also [22]). He proved that if we start with a subvariety Z of Hodge type, then the model $\mathcal{Z}$ as above is formally linear at all closed ordinary points x (possibly excluding finitely many primes $\mathfrak{p}$). We give a precise formulation of Noot's results in Section 4.

One of our objects in this paper is to prove that, conversely, a weakened version of formal linearity implies that the subvariety Z is of Hodge type. More precisely:

THEOREM. *Let $Z \hookrightarrow \mathsf{A}_{g,1,n} \otimes F$ be an irreducible algebraic subvariety of the moduli space $\mathsf{A}_{g,1,n}$, defined over a number field F. Suppose there is a prime $\mathfrak{p}$ of $\mathcal{O}_F$ such that the model $\mathcal{Z}$ of Z (as in Section 3.3) has formally quasi-linear components at some closed ordinary point $x \in (\mathcal{Z} \otimes \kappa(\mathfrak{p}))^{\circ}$. Then Z is of Hodge type, i.e., every irreducible component of $Z \otimes_F \mathbb{C}$ is a subvariety of Hodge type.*

Notice that this statement is very similar to Corollary 5.5 in Part I, to which, in fact, we reduce the proof. The definitions and preliminaries that are needed to establish the main results, are discussed in Sections 1–3. The above theorem is proved in Section 4.

Next we try to apply our characterization to Oort's conjecture. Recall that this conjecture says that an irreducible algebraic subvariety $Z \hookrightarrow \mathsf{A}_{g,1,n} \otimes \mathbb{C}$ containing a Zariski dense collection of CM-points should be a subvariety of Hodge type. What we would like to prove therefore, is that the existence of such a Zariski dense collection of CM-points implies that, for some prime $\mathfrak{p}$, we obtain a model $\mathcal{Z}$ which is formally linear at some ordinary point x. Unfortunately, we can only prove this under an additional assumption. Although this does not settle Oort's conjecture in general, we think that both the methods used and the resulting variant of the conjecture (see 5.3) are interesting in their own right.

We conclude with some applications. In Section 5 we apply the main results discussed above to prove Oort's conjecture in a particular situation. For a precise statement, see 5.7. In the last section we study the Zariski closure of the moduli point of X^{can}, where X is an ordinary abelian variety in characteristic p (not necessarily defined over a finite field). First we show that this Zariski closure, call it Z, is a subvariety of Hodge type. Knowing this, one wonders how $\dim(Z)$ compares to the dimension of the Zariski closure $\{x\}^{\mathrm{Zar}} \subseteq \mathsf{A}_{g,1,n} \otimes \mathbb{F}_p$ of the moduli point of X. Clearly, if x is a closed point, then both dimensions are zero. In general, $\dim(Z) \geqslant \dim(\{x\}^{\mathrm{Zar}})$. We show (joint work with A.J. de Jong and F. Oort) that there exist ordinary moduli points x with $\dim(\{x\}^{\mathrm{Zar}}) = 1$ and $\dim(Z) = g(g+1)/2$.

1. Definitions and preliminaries

1.1. Recall that a Shimura datum $(G, \mathfrak{X})$ is a pair consisting of an algebraic group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class $\mathfrak{X} \subseteq \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ satisfying the axioms [8, (2.1.1.1-3)]. We write $\mathrm{Sh}_K(G, \mathfrak{X})$ for the canonical model, defined over the reflex field $E(G, \mathfrak{X})$, of the Shimura variety associated to a Shimura datum $(G, \mathfrak{X})$ and a compact open subgroup $K \subset G(\mathbb{A}_f)$; by definition we thus have

$$\mathrm{Sh}_K(G, \mathfrak{X})(\mathbb{C}) = G(\mathbb{Q})\backslash \mathfrak{X} \times G(\mathbb{A}_f)/K.$$

In this context, we adopt the notational convention that symbols $\mathfrak{X}$, $\mathfrak{Y}$ etc. represent the conjugacy classes which are part of a Shimura datum, and that symbols X, Y etc. represent connected components (which are hermitian symmetric domains).

Let $f\colon (G_1, \mathfrak{X}_1) \to (G_2, \mathfrak{X}_2)$ be a morphism of Shimura data (i.e., a homomorphism $f\colon G_1 \to G_2$ of algebraic groups over $\mathbb{Q}$ inducing a map from $\mathfrak{X}_1$ to $\mathfrak{X}_2$) and let $K_1 \subseteq G_1(\mathbb{A}_f)$, $K_2 \subseteq G_2(\mathbb{A}_f)$ be compact open subgroups with $f(K_1) \subseteq K_2$. We obtain a morphism

$$f_{(K_1,K_2)}\colon \mathrm{Sh}_{K_1}(G_1, \mathfrak{X}_1) \to \mathrm{Sh}_{K_2}(G_2, \mathfrak{X}_2)$$

of associated Shimura varieties. If $S \hookrightarrow \mathrm{Sh}_{K_2}(G_2, \mathfrak{X}_2)$ is an irreducible component of the image of $f_{(K_1,K_2)}$ then we call S a subvariety of Shimura type.

In general, the class of subvarieties of Shimura type is not stable under Hecke correspondences. We say that an algebraic subvariety $S \hookrightarrow \mathrm{Sh}_K(G, \mathfrak{X})$ is a subvariety of Hodge type if there is a subvariety $S' \hookrightarrow \mathrm{Sh}_K(G, \mathfrak{X})$ of Shimura type and a Hecke correspondence $\mathcal{T}_\eta$ (with $\eta \in G(\mathbb{A}_f)$) such that S is an irreducible component of the image $\mathcal{T}_\eta(S')$. For further discussion of this notion we refer to [19]; here we only recall the following fact (loc. cit., Section 2).

PROPOSITION 1.2. *Let $S \subseteq \mathrm{Sh}_K(G, \mathfrak{X})$ be a subvariety of Hodge type. Then there exists a compact open subgroup K' of $G(\mathbb{A}_f)$ contained in K, a representation $\xi\colon G \to \mathrm{GL}(V)$, and an algebraic subgroup $M \subseteq \mathrm{GL}(V)$, such that* (i) *$\xi$ induces a polarizable VHS $\mathcal{V}(\xi)$ over $\mathrm{Sh}_{K'}(G, \mathfrak{X})$, and* (ii) *$S$ is the image under the natural map $\mathrm{Sh}_{K'}(G, \mathfrak{X}) \to \mathrm{Sh}_K(G, \mathfrak{X})$ of an irreducible subvariety $S' \subseteq \mathrm{Sh}_{K'}(G, \mathfrak{X})$ such that S' is a maximal irreducible subvariety with generic Mumford–Tate group M.*

1.3. Let F be a field with $E = E(G, \mathfrak{X}) \subseteq F \subseteq \mathbb{C}$. We extend the previous definitions by defining an irreducible subvariety $S \hookrightarrow \mathrm{Sh}_K(G, \mathfrak{X}) \otimes F$ to be of Hodge type (resp. of Shimura type) if all irreducible components of $S \otimes_F \mathbb{C}$ are of Hodge (resp. Shimura) type.

For such a subvariety S to be of Hodge type, it then suffices to check that *one* of the irreducible components of $S_{\mathbb{C}} = S \otimes_F \mathbb{C}$ is of Hodge type. (Similarly for subvarieties of Shimura type.) This is because the class of subvarieties of

Hodge type is stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on the (non-connected) variety $\mathrm{Sh}_K(G,\mathfrak{X}) \otimes \mathbb{C}$ – an easy consequence of the results on conjugation of Shimura varieties, discussed in [17, Sect. II.4].

1.4. Let CSp_{2g} be the group scheme of symplectic similitudes of the space $\mathbb{Z}^{2g}$ with its standard symplectic form ψ, and let $\mathfrak{h}_g^{\pm}$ denote the Siegel double space, considered as a $\mathrm{CSp}_{2g}(\mathbb{R})$-conjugacy class of homomorphisms $h\colon \mathbb{S} \to \mathrm{CSp}_{2g,\mathbb{R}}$. Set $K_n = \{g \in \mathrm{CSp}_{2g}(\widehat{\mathbb{Z}}) \mid g \equiv 1 \pmod n\}$. As is well-known (see [7, Sect. 4]), we can identify the Shimura variety $\mathrm{Sh}_{K_n}(\mathrm{CSp}_{2g}, \mathfrak{h}_g^{\pm})$ with $\mathsf{A}_{g,1,n} \otimes \mathbb{Q}$. Here we write $\mathsf{A}_{g,1,n}$ for the (coarse) moduli scheme over $\mathbb{Z}[1/n]$ of principally polarized abelian varieties with a Jacobi level n structure (by which we mean a level n structure which, for *some* choice of an nth root of unity, is symplectic in the sense of [12, p. 121]). If $n \geqslant 3$ then $\mathsf{A}_{g,1,n}$ is a fine moduli scheme.

1.5 FORMAL SCHEMES. The theory of formal schemes is set up in [13, I. Sect. 10 and III, Sects 3–5]. Unfortunately, not everything we need is treated there. Lacking a good reference, let us briefly discuss some definitions. Convention: *all formal schemes we use are noetherian and adic.*

Let $\mathfrak{X}$ be a formal scheme. We write $\mathfrak{X}_{\mathrm{red}}$ for the associated reduced scheme ([13, I, Sect. 10.5]), which has the same underlying topological space as $\mathfrak{X}$. We call $\mathfrak{X}$ connected if the underlying topological space is connected. We call $\mathfrak{X}$ formally reduced, if for all points $x \in \mathfrak{X}$ the local ring $\mathcal{O}_x$ is reduced. Let $\mathfrak{Y}_1$ and $\mathfrak{Y}_2$ be closed formal subschemes of $\mathfrak{X}$, defined by coherent ideal sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively. We define the closed formal subscheme $\mathfrak{Y}_1 \cup \mathfrak{Y}_2 \subseteq \mathfrak{X}$ by the sheaf of ideals $\mathcal{I}_1 \cap \mathcal{I}_2$, which again is coherent.

Suppose $\mathfrak{X}$ is formally reduced. We say $\mathfrak{X}$ is formally irreducible if for all closed formal subschemes $\mathfrak{Y}_1, \mathfrak{Y}_2$ with $\mathfrak{X} = \mathfrak{Y}_1 \cup \mathfrak{Y}_2$ we have $\mathfrak{Y}_1 = \mathfrak{X}$ or $\mathfrak{Y}_2 = \mathfrak{X}$. An excellent formal scheme $\mathfrak{X}$ has a well-defined decomposition into (formal) irreducible components.

Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a finite morphism of formal (noetherian and adic) schemes (as defined in [13, III, Sect. 4.8]). The $\mathcal{O}_{\mathfrak{Y}}$-algebra $f_*\mathcal{O}_{\mathfrak{X}}$ is coherent (loc. cit., Proposition 4.8.6), hence $\mathcal{K} = \mathcal{K}er(\mathcal{O}_{\mathfrak{Y}} \to f_*\mathcal{O}_{\mathfrak{X}})$ is also coherent (op. cit., 0_{I}, Corollary 5.3.4). We define the image of f (in the sense of formal schemes) as the closed formal subscheme $f(\mathfrak{X}) \subseteq \mathfrak{Y}$ defined by the ideal $\mathcal{K}$. It is the smallest closed formal subscheme of $\mathfrak{Y}$ through which f factors.

The following two lemma's will be used in Section 3. We leave the proofs to the reader (or see [18, Chap. III, Sect. 3]).

LEMMA 1.6. *Let $\widehat{\mathcal{O}}$ be a complete discrete valuation ring, and write $\mathfrak{S} = \mathrm{Spf}(\widehat{\mathcal{O}})$. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be formally reduced, noetherian, adic formal schemes which are flat and of finite type over $\mathfrak{S}$. Suppose $\mathfrak{X}_{\mathrm{red}}$ and $\mathfrak{Y}_{\mathrm{red}}$ are equidimensional of the same dimension. Let $p\colon \mathfrak{X} \to \mathfrak{Y}$ be a finite $\mathfrak{S}$-morphism. Then for every irreducible* *component $\mathfrak{C} \subseteq \mathfrak{X}$, the image $p(\mathfrak{C}) \subseteq \mathfrak{Y}$ (in the sense of formal schemes) is an irreducible component of $\mathfrak{Y}$.*

LEMMA 1.7. *Let $\widehat{\mathcal{O}}$ be a complete discrete valuation ring, and write $S = \mathrm{Spec}(\widehat{\mathcal{O}})$, $\mathfrak{S} = \mathrm{Spf}(\widehat{\mathcal{O}})$. Let $f: X \to S$ be an S-scheme, flat and of finite type over S. Write $\widehat{f}: \mathfrak{X} \to \mathfrak{S}$ for the formal completion of X along its closed fibre. Let $\mathcal{R}$ be a complete local domain which is finite and flat over $\widehat{\mathcal{O}}$, and let $t: \mathrm{Spec}(\mathcal{R}) \to X$ be an S-morphism. Write $\hat{t}: \mathrm{Spf}(\mathcal{R}) \to \mathfrak{X}$ for the induced $\mathfrak{S}$-morphism. Assume t maps the generic point of $\mathrm{Spec}(\mathcal{R})$ into the regular locus of X. Then there is a unique irreducible component $\mathfrak{C} \subseteq \mathfrak{X}$ such that $\hat{t}$ factors through $\mathfrak{C}$.*

2. Local moduli of abelian varieties

2.1. Fix an integer $n \geqslant 3$ and a prime number p with $p \nmid n$. We also fix a perfect field k of characteristic p. Write $W = W(k)$ for its ring of (infinite) Witt vectors, and write $\mathcal{A}_g$ for $\mathsf{A}_{g,1,n} \otimes W$. Let $(\mathcal{A}_g \otimes k)^\circ$ be the ordinary locus in characteristic p. This is a locally closed subscheme of $\mathcal{A}_g$, hence we can take the formal completion along it to obtain a formal scheme $\widehat{\mathcal{A}}_g = \mathcal{A}_{g/(\mathcal{A}_g \otimes k)^\circ}$ over $\mathrm{Spf}(W)$.

Let $U \subset \mathcal{A}_g$ be an open subscheme such that the ordinary locus $U \cap (\mathcal{A}_g \otimes k)^\circ$ is a closed subscheme of U, defined by an ideal sheaf $\mathcal{J}$. For $m \geqslant 0$, let Y_m be the subscheme of U defined by $\mathcal{J}^m$ and let $(X_m, \lambda_m, \theta_m)$ be the universal object over Y_m. Then X_m is an ordinary abelian scheme over Y_m and the multiplicative part $X_m[p]_\mu$ of its p-torsion is a finite, locally free subgroup scheme of X_m of rank p^g, which moreover is maximal totally isotropic for the Weil pairing e_{λ_m}. It follows that the abelian scheme $X'_m = X_m / X_m[p]_\mu$ has a principal polarization λ'_m such that $\pi^* \lambda'_m = p \cdot \lambda_m$, where $\pi: X_m \to X'_m$ is the canonical map. Also, since $p \nmid n$, the level n structure θ_m naturally induces a level n structure θ'_m on X'_m.

The new triplet $(X'_m, \lambda'_m, \theta'_m)$ corresponds to a morphism $\Phi_m: Y_m \to \mathcal{A}_g$, which factors through Y_m. These morphisms Φ_m form a projective system. Taking the inverse limit we obtain an endomorphism $\Phi_U: \widehat{U} \to \widehat{U}$ on the formal completion. Finally, we can glue these Φ_U to obtain a morphism $\Phi_{\mathrm{can}}: \widehat{\mathcal{A}}_g \to \widehat{\mathcal{A}}_g$ over $\mathrm{Spf}(W)$. (Alternatively, we can take for U the complement of the non-ordinary locus in characteristic p, in which case $\widehat{U} = \widehat{\mathcal{A}}_g$.) It lifts the endomorphism of $(\mathcal{A}_g \otimes k)^\circ$ which is obtained by pulling back the Frobenius endomorphism of $(\mathsf{A}_{g,1,n} \otimes \mathbb{F}_p)^\circ$ via $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{F}_p)$.

2.2. Let $(X_0, \lambda_0, \theta_0)$ be a principally polarized abelian variety of dimension g with a Jacobi level n structure over $\mathrm{Spec}(k)$. It corresponds to some closed point x of $\mathcal{A}_g$. Let $\mathfrak{A}_x \to \mathrm{Spf}(W)$ be the formal completion of $\mathcal{A}_g$ at x. If x is an ordinary point (i.e., if X_0 is ordinary) then, by Serre–Tate theory, $\mathfrak{A}_x$ has a canonical structure of a formal torus over $\mathfrak{S} = \mathrm{Spf}(W)$. (See Section 3.1 for a brief discussion on formal tori.) We will review some results that we need in the next sections. For proofs we refer to [10], [15] and [16].

2.3. We will work with categories $\mathcal{C}_W$ and $\widehat{\mathcal{C}}_W$ as in [27]. The objects of $\mathcal{C}_W$ are the artinian local W-algebras R such that the structure homomorphism $W \to R$ is local and induces an isomorphism $k \xrightarrow{\sim} R/\mathfrak{m}_R$. Then $\widehat{\mathcal{C}}_W$ is defined as the category of complete noetherian local W-algebras $\mathcal{R}$ such that $\mathcal{R}/\mathfrak{m}^i_{\mathcal{R}}$ is in $\mathcal{C}_W$ for all i. We have a formal deformation functor $\mathcal{D}efo_{X_0}\colon \mathcal{C}_W \to \mathsf{Sets}$, given by

$$\mathcal{D}efo_{X_0}(R) = \left\{ \begin{array}{l} \text{isomorphism classes of pairs } (X, \varphi), \text{ where } X \text{ is an abelian} \\ \text{scheme over } \mathrm{Spec}(R) \text{ and } \varphi \text{ is an isomorphism } \varphi\colon X \otimes k \xrightarrow{\sim} X_0 \end{array} \right\}.$$

In a similar way one defines deformation functors $\mathcal{D}efo_{(X_0,\lambda_0)}$ and $\mathcal{D}efo_{(X_0,\lambda_0,\theta_0)}$ of the pair (X_0, λ_0), and the triplet $(X_0, \lambda_0, \theta_0)$, respectively. Since we consider level n structures with $p \nmid n$, the natural morphism $\mathcal{D}efo_{(X_0,\lambda_0,\theta_0)} \to \mathcal{D}efo_{(X_0,\lambda_0)}$ is an isomorphism. The functor $\mathcal{D}efo_{(X_0,\lambda_0,\theta_0)}$ is represented by $\mathfrak{A}_x$.

Let $\overline{k}$ be an algebraic closure of k, and write $\overline{W} = W(\overline{k})$ for its ring of Witt vectors. Write $T_pX_0 = T_pX_0(\overline{k})$ for the 'physical' Tate module of X_0. Over $\overline{k}$ we have an isomorphism of functors $\mathcal{D}efo_{X_0 \otimes \overline{k}} \xrightarrow{\sim} \mathrm{Hom}(T_pX_0^{\otimes 2}, \widehat{\mathbb{G}}_{\mathrm{m}})$. The deformation space $\mathfrak{A} := \mathfrak{A}_x \otimes \overline{W}$ is isomorphic to the formal subtorus $\mathrm{Hom}(\mathrm{Sym}^2(T_pX_0), \widehat{\mathbb{G}}_{\mathrm{m}})$.

2.4. Let $\sigma\colon W \to W$ be the Frobenius automorphism of W. Assume that k is finite. The Frobenius morphism $\mathrm{Frob}\colon X_0 \to X_0^{(p)}$ induces an isomorphism $X_0/X_0[p]_\mu \xrightarrow{\sim} X_0^{(p)}$. Using this we see that the formal completion of $\mathcal{A}_g$ at the point $\Phi_{\mathrm{can}}(x)$ is isomorphic to $\mathfrak{A}_x^{(\sigma)} = \mathfrak{A}_x \times_{\mathfrak{S},\sigma} \mathfrak{S}$. The morphism Φ_{can} introduced in 2.1 therefore induces a morphism $\mathfrak{A}_x \to \mathfrak{A}_x^{(\sigma)}$, which we again call Φ_{can} (cf. [10, p. 135]). It is not difficult to see that this is a group homomorphism. If $N = {}^p\log(\#k)$ (i.e., $k \cong \mathbb{F}_{p^N}$) then Φ_{can}^N is an endomorphism of $\mathfrak{A}_x$. It is the endomorphism 'raising to the p^Nth power' in the group $\mathfrak{A}_x$.

The next lemma, which we quote from the appendix to [10] by Katz, shows that the group structure is uniquely determined by the fact that it is compatible with Φ_{can}.

LEMMA 2.5 (Katz, [10, A.1]). *Let k be a perfect field of characteristic $p > 0$, let $W = W(k)$ its ring of Witt vectors, and let $\sigma\colon W \xrightarrow{\sim} W$ be the automorphism induced by the Frobenius automorphism of k. Let $\mathcal{M}$ be a formally smooth affine formal scheme of finite type over W, i.e., $\mathcal{M} \cong \mathrm{Spf}(W[\![t_1, \ldots, t_n]\!])$. Suppose we are given a morphism $\Phi\colon \mathcal{M} \to \mathcal{M}^{(\sigma)}$ of formal schemes over W whose reduction modulo p is the Frobenius morphism* $\mathrm{Frob}\colon \mathcal{M} \otimes_W k \to (\mathcal{M} \otimes_W k)^{(p)}$.

(i) *Given $(\mathcal{M}, \Phi)$, there exists at most one structure of commutative formal group over W on $\mathcal{M}$ for which the given map $\Phi\colon \mathcal{M} \to \mathcal{M}^{(\sigma)}$ is a group homomorphism.*

https://doi.org/10.1023/A:1000411631772 Published online by Cambridge University Press

(ii) *If this structure exists, it makes* $\mathcal{M}$ *into a formal torus and the given* Φ *is the unique homomorphism lifting Frobenius.*

(iii) *If* $(\mathcal{M}_1, \Phi_1)$ *and* $(\mathcal{M}_2, \Phi_2)$ *both admit group structures as in* (i)*, then a morphism* $f\colon \mathcal{M}_1 \to \mathcal{M}_2$ *of formal schemes over* W *is a group homomorphism if and only if* $\Phi_2 \circ f = f^{(\sigma)} \circ \Phi_1$.

2.6. Choose a $\mathbb{Z}_p$-basis $\alpha_1, \dots, \alpha_g$ for T_pX_0. Writing $A := \overline{W}[\![q_{ij}-1]\!]/(q_{ij}-q_{ji})$ we obtain an isomorphism $\mathfrak{A} = \mathrm{Hom}(\mathrm{Sym}^2(T_pX_0), \widehat{\mathbb{G}}_{\mathrm{m}}) \cong \mathrm{Spf}(A)$, where q_{ij} represents the character sending $\varphi \in \mathrm{Hom}_R(\mathrm{Sym}^2(T_pX_0), \widehat{\mathbb{G}}_{\mathrm{m}})$ to $\varphi(\alpha_i \otimes \alpha_j) \in \widehat{\mathbb{G}}_{\mathrm{m}}(R) = 1 + \mathfrak{m}_R$.

Let $\mathfrak{X} \to \mathfrak{A}$ be the universal (polarized) formal abelian scheme. The A-module $H = \mathrm{H}^1_{\mathrm{DR}}(\mathfrak{X}/\mathfrak{A})$ with the Gauß-Manin connection ∇ and its Hodge filtration

$$\mathcal{F}^0 = H \supset \mathcal{F}^1 = \mathrm{H}^0(X, \Omega^1_{\mathfrak{X}/\mathfrak{A}})$$

has the structure of an ordinary Hodge F-crystal of level 1 (see [10] and [15]). To the chosen basis $\alpha_1, \dots, \alpha_g$ one associates elements $a_1, \dots, a_g$ and $b_1, \dots, b_g$ of H such that $\mathcal{F}^1 = A \cdot b_1 \oplus \cdots \oplus A \cdot b_g$ and H is the direct sum of $U = A \cdot a_1 \oplus \cdots A \cdot a_g$ and $\mathcal{F}^1$. For the connection ∇ we have

$$\nabla(a_i) = 0, \qquad \nabla(b_j) = \sum_i a_i \otimes \eta_{ij}$$

for certain forms $\eta_{ij} \in \Omega^1_{\mathfrak{A}/W}$ (continuous differential forms). Furthermore,

$$F(\Phi_{\mathrm{can}})\Phi^*_{\mathrm{can}}(a_i^{(\sigma)}) = a_i, \qquad F(\Phi_{\mathrm{can}})\Phi^*_{\mathrm{can}}(b_i^{(\sigma)}) = pb_i,$$
$$\Phi^*_{\mathrm{can}}(\eta_{ij}^{(\sigma)}) = p\eta_{ij} \quad \text{and} \quad \mathrm{d}\eta_{ij} = 0.$$

In particular, U is a unit sub-F-crystal of H.

Let K be the fraction field of $W(\overline{k})$ and write $\tau_{ij} = \log(q_{ij}) \in K[\![q_{ij}-1]\!]$. Let $B = K[\![\tau_{ij}]\!]/(\tau_{ij} - \tau_{ji})$, then we obtain a homomorphism $A \to B$ by sending q_{ij} to $\exp(\tau_{ij})$. We have the identities $\Phi^*_{\mathrm{can}}(q_{ij}^{(\sigma)}) = q_{ij}^p$, $\Phi^*_{\mathrm{can}}(\tau_{ij}^{(\sigma)}) = p\tau_{ij}$ and $\eta_{ij} = \mathrm{d}\tau_{ij}$. If $\mathbf{0}\colon A \to W(\overline{k})$ is the Teichmüller lift of the augmentation map $A \twoheadrightarrow \overline{k}$ with respect to Φ_{can} then $q_{ij}(\mathbf{0}) = 1$ and $\tau_{ij}(\mathbf{0}) = 0$.

Write $c_j = b_j - \sum_i \tau_{ij} a_i$. The elements $a_1, \dots, a_g, c_1, \dots, c_g$ form a horizontal B-basis for $H \widehat{\otimes}_A B$, and the Hodge flag $\mathcal{F}^1 \widehat{\otimes}_A B$ is spanned by the elements $c_j + \sum_i \tau_{ij} a_i$.

2.7. As above, assume that k is finite, and let $N = {}^p\log(\#k)$. We still assume that X_0 is an ordinary abelian variety. Write $\pi = \pi_{X_0}\colon X_0 \to X_0$ for the Frobenius endomorphism of X_0 (so '$\pi = \mathrm{Frob}^N$'). Let $\mathcal{R}$ be an object of $\widehat{\mathcal{C}}_W$ and let $s\colon \mathrm{Spf}(\mathcal{R}) \to \mathfrak{A}_x$ be an $\mathcal{R}$-valued point of $\mathfrak{A}_x$ over $\mathrm{Spf}(W)$. Let X_s denote the corresponding abelian scheme over $\mathrm{Spec}(\mathcal{R})$.

We say that an abelian scheme X over $\mathrm{Spec}(\mathcal{R})$ is of CM-type if $\mathrm{End}^0(X \otimes \overline{\mathcal{R}})$ contains a commutative semi-simple $\mathbb{Q}$-subalgebra of rank $2g$ over $\mathbb{Q}$. Here $\overline{\mathcal{R}} = \mathcal{R}\widehat{\otimes}\overline{W}$. If $\mathcal{R}$ is a normal domain then X is of CM-type if and only if its generic fibre is of CM-type.

LEMMA 2.8. (i) *The following conditions are equivalent.*

(a) s *is the identity element of* $\mathfrak{A}_x(\mathcal{R})$.
(b) $\mathrm{End}_{\mathcal{R}}(X_s) \xrightarrow{\sim} \mathrm{End}_k(X_0)$.
(c) π^m *lifts to an endomorphism of* X_s *for some* $m \geqslant 1$.

In case $\mathcal{R} = W(k)$, *these conditions are also equivalent to*

(d) *the Frobenius* $\mathrm{Frob}\colon X_0 \to X_0^{(p)}$ *lifts to a homomorphism* $\widetilde{F}\colon X \to X^{(\sigma)}$.

(ii) The following conditions are equivalent.

(a) s *is a torsion element of* $\mathfrak{A}_x(\mathcal{R})$.
(b) $\mathrm{End}_{\mathcal{R}}(X_s) \otimes \mathbb{Z}[1/p] \xrightarrow{\sim} \mathrm{End}_k(X_0) \otimes \mathbb{Z}[1/p]$.
(c) X_s *is isogenous to the lifting* X_1 *(where* $1 \in \mathfrak{A}_x(\mathcal{R})$ *is the identity element).*
(d) X_s *is of CM-type.*

Up to some details (which we leave to the reader), a proof is obtained by combining [16, Appendix] and [6, Sect. 3].

DEFINITION 2.9. Suppose $\mathcal{R} \in \widehat{\mathcal{C}}_W$ is a flat W-algebra. The lifting of X_0 over $\mathrm{Spec}(\mathcal{R})$ corresponding to the identity element of $\mathfrak{A}_x(\mathcal{R})$ is called the canonical lifting of X_0. We denote it by X_0^{can}. The liftings of X_0 over $\mathrm{Spec}(\mathcal{R})$ corresponding to the torsion elements of $\mathfrak{A}_x(\mathcal{R})$ are called quasi-canonical liftings; by (ii) of the lemma these are precisely the CM-liftings of X_0.

The property in 2.8(i) that Frob lifts can be formulated in terms of Φ_{can}. Let us give the statement in the form we need it. The proof of the following lemma is left to the reader.

LEMMA 2.10. *Consider the formal scheme* $\widehat{\mathcal{A}}_g$ *over* $\mathrm{Spf}(W)$ *as in Section* 2.1, *with* $k \cong \mathbb{F}_{p^N}$. *Let* $k \to k'$ *be a finite field extension, and let* $s\colon \mathrm{Spf}(W(k')) \to \widehat{\mathcal{A}}_g$ *be a* $W(k')$*-valued point, giving rise to a triplet* (X, λ, θ) *over* $\mathrm{Spec}(W(k'))$. *Let* $\mathrm{Frob} = \mathrm{Frob}_{X_{k'}/k'}\colon X_{k'} \to X_{k'}^{(p)}$ *be the Frobenius morphism. If* a *is a multiple of* N *then* Frob^a *lifts to a morphism* $\widetilde{F}\colon X \to X^{(\sigma^a)}$ *over* $\mathrm{Spec}(W)$ *if and only if* $\Phi_{\mathrm{can}}^a \circ s = s \circ \sigma^a$.

2.11. So far we only discussed the ‘unramified’ case, studying formal completions of the scheme $\mathsf{A}_{g,1,n}$ over a ring of Witt vectors. By base change we can extend most of the above results to a slightly more general situation, which is what we need for the next sections. Since most of this is obvious, the following remarks are mainly intended to fix notations, which agree with the ones used before.

Let F be a number field with ring of integers $\mathcal{O}_F$, and let $\mathfrak{p}$ be a prime of $\mathcal{O}_F$ lying over p. We write $\widehat{\mathcal{O}}_\mathfrak{p}$ for the completion of the local ring $\mathcal{O}_\mathfrak{p}$. Write $\mathcal{A}_g = \mathsf{A}_{g,1,n} \otimes \widehat{\mathcal{O}}_\mathfrak{p}$, and let $\widehat{\mathcal{A}}_g$ be the formal completion of $\mathcal{A}_g$ along the ordinary locus in characteristic p. We obtain a morphism $\Phi_{\mathrm{can}}\colon \widehat{\mathcal{A}}_g \to \widehat{\mathcal{A}}_g$ over $\mathrm{Spf}(\widehat{\mathcal{O}}_\mathfrak{p})$ by pulling back the Φ_{can} defined in 2.1 via $\mathrm{Spf}(\widehat{\mathcal{O}}_\mathfrak{p}) \to \mathrm{Spf}\big(W(\kappa(\mathfrak{p}))\big)$.

Let x be a closed point of the ordinary locus $\big(\mathcal{A}_g \otimes \kappa(\mathfrak{p})\big)^\circ$. Consider the ring $\Lambda = W(\kappa(x)) \otimes_{W(\kappa(\mathfrak{p}))} \widehat{\mathcal{O}}_\mathfrak{p}$, which is a complete local ring with residue field $\kappa(x)$, and write $\mathfrak{S} = \mathrm{Spf}(\Lambda)$. We let $\mathfrak{A}_x \to \mathfrak{S}$ be the formal completion of $\mathcal{A}_g$ at x (which has a natural morphism to $\mathfrak{S}$). It is obtained via base change $\mathfrak{S} \to \mathrm{Spf}\big(W(\kappa(x))\big)$ from a formal deformation space as studied above, and therefore has the structure of a formal torus over $\mathfrak{S}$. Via this base change and the results of 2.6 we also get a description of the de Rham cohomology $\mathrm{H}^1_{\mathrm{DR}}(\mathfrak{X}/\mathfrak{A})$ in this more general setting.

2.12. Recall that a p-isogeny between principally polarized abelian schemes (X,λ) and (X',λ') of relative dimension g over a base scheme S is an isogeny $f\colon X \to X'$ such that $f^*\lambda' = p^e\cdot\lambda$ for some $e \in \mathbb{Z}_{\geqslant 1}$. If this holds then f has degree p^{eg}. If X and X' are equipped with level n structures θ and θ' ($p \nmid n$) then we further require that $f^*\theta' = \theta$ (meaning that θ '=' θ' via the isomorphism $T^pX \xrightarrow{\sim} T^pX'$ on the 'prime-to-p Tate modules' induced by f).

Let A_g be the moduli stack of principally polarized abelian schemes, as in [12, Chap. I, 4.3]. The p-isogenies form a stack p-Isog, with two natural morphisms $\mathrm{pr}_1, \mathrm{pr}_2\colon p\text{-}\mathsf{Isog} \to \mathsf{A}_g$ obtained by associating to an isogeny $f\colon (X,\lambda) \to (X',\lambda')$ its source (X,λ), and its target (X',λ'), respectively. Bounding the degree of the isogeny gives a substack of p-Isog which is representable by a relative scheme over $\mathsf{A}_g \times \mathsf{A}_g$. We write $\mathsf{Isog}(p^{eg})$ for the stack of p-isogenies of degree p^{eg} (it is empty if the degree is not a power of p^g).

As a variant, we can take level structures into account. Choose an integer $n \geqslant 3$ with $p \nmid n$ and, as before, write $\mathsf{A}_{g,1,n}$ for the moduli scheme over $\mathrm{Spec}(\mathbb{Z}[1/n])$ of principally polarized g-dimensional abelian varieties with a Jacobi level n structure. It is a fine moduli scheme ($n \geqslant 3$). By considering isogenies which respect level structures we obtain a scheme $\mathsf{Isog}(p^{eg})$ over $\mathsf{A}_{g,1,n} \times \mathsf{A}_{g,1,n}$; to keep notations simple we here omit the subscript '$g,1,n$'.

2.13. We use the notations $F, \mathfrak{p}$ and $\widehat{\mathcal{O}}_\mathfrak{p}$ of 2.11, and we write $\mathcal{A}_g = \mathsf{A}_{g,1,n} \otimes \widehat{\mathcal{O}}_\mathfrak{p}$, $\mathcal{I}sog(p^{eg}) = \mathsf{Isog}(p^{eg}) \otimes \widehat{\mathcal{O}}_\mathfrak{p}$. Write $\mathcal{A}_g^\circ \subset \mathcal{A}_g$ for the open subscheme obtained by deleting the non-ordinary locus in characteristic p. The isogenies lying over $\mathcal{A}_g^\circ \times \mathcal{A}_g^\circ$ form an open subscheme $\mathcal{I}sog(p^{eg})^\circ$ of $\mathcal{I}sog(p^{eg})$. The restricted projection morphisms $\mathrm{pr}_i\colon \mathcal{I}sog(p^{eg})^\circ \to \mathcal{A}_g^\circ$ are finite and flat.

The ordinary locus $\mathcal{I}sog(p^{eg})^\circ \otimes \kappa(\mathfrak{p})$ in characteristic p is a locally closed subscheme of $\mathcal{I}sog(p^{eg})$. We can take the formal completion along it to obtain a formal scheme $\mathcal{I}sog(p^{eg})^\wedge$ over $\mathrm{Spf}(\widehat{\mathcal{O}}_\mathfrak{p})$, with projection maps $\mathrm{pr}_i\colon \mathcal{I}sog(p^{eg})^\wedge \to \widehat{\mathcal{A}}_g$.

https://doi.org/10.1023/A:1000411631772 Published online by Cambridge University Press

PROPOSITION 2.14. *There is an open and closed formal subscheme* $\widehat{\mathcal{I}} \subseteq \mathcal{I}sog(p^{eg})^\wedge$ *such that the restriction* $\widehat{\mathrm{pr}}_1: \widehat{\mathcal{I}} \to \widehat{\mathcal{A}}_g$ *is an isomorphism, and such that the composition*

$$\widehat{\mathcal{A}}_g \xrightarrow{\widehat{\mathrm{pr}}_1^{-1}} \widehat{\mathcal{I}} \xrightarrow{\widehat{\mathrm{pr}}_2} \widehat{\mathcal{A}}_g$$

is equal to the morphism Φ_{can}^e, *where* Φ_{can} *is defined as in* 2.1 *and* 2.11. *The reduced underlying scheme* $\widehat{\mathcal{I}}_{\mathrm{red}}$ *is a disjoint union of irreducible components.*

Proof. This is essentially [12, Prop. VII. 4.1]; in the notation of loc. cit., our $\widehat{\mathcal{I}}$ is the formal completion along the subscheme of $\mathsf{Isog}(p^{eg})$ classifying isogenies of type LmL with $m = \mathrm{diag}(p^e \cdot \mathrm{Id}_g, 1 \cdot \mathrm{Id}_g)$. In other words, $\widehat{\mathcal{I}}$ is the formal completion along the pull-back (via $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathrm{Spec}(\mathbb{F}_p)$) of the graph of the Frobenius morphism. Let us nevertheless sketch a proof.

Except for the last statement, it suffices to prove the proposition over $\mathbb{Z}_p$, since all ingredients over $\widehat{\mathcal{O}}_\mathfrak{p}$ are obtained via pull-back over $\mathrm{Spf}(\widehat{\mathcal{O}}_\mathfrak{p}) \to \mathrm{Spf}(\mathbb{Z}_p)$. We only do the case $e = 1$; the general argument only differs in that the notations are more complicated.

Write $(X_m, \lambda_m, \theta_m)$ for the universal object over $\mathcal{A}_g^\circ \otimes (\mathbb{Z}/p^m)$. In Section 2.1, we have defined Φ_{can} on $\widehat{\mathcal{A}}_g$ as the limit of morphisms Φ_m such that $\Phi_m^*(X_m, \lambda_m, \theta_m) = (X_m', \lambda_m', \theta_m')$, obtained by dividing out $X_m[p]_\mu$. (Here we apply the discussion of Section 2.1 to $U = \mathcal{A}_g^\circ$, in which case $\mathcal{J} = p \cdot \mathcal{O}_U$.)

We obtain a section s_m of $\mathrm{pr}_1: \mathcal{I}sog(p^g)^\circ \otimes (\mathbb{Z}/p^m) \to \mathcal{A}_g^\circ \otimes (\mathbb{Z}/p^m)$ by associating to $(X_m, \lambda_m, \theta_m)$ the natural isogeny $\pi_m: X_m \to X_m' = X_m/X_m[p]_\mu$ (compatible with polarizations and level structures). Clearly, $\mathrm{pr}_2 \circ s_m = \Phi_m$ on Y_m.

Define $I_m \subseteq \mathcal{I}sog(p^g)^\circ \otimes (\mathbb{Z}/p^m)$ as the (scheme-theoretic) image of s_m. The section s_m maps into the open subscheme of $\mathcal{I}sog(p^g)^\circ \otimes (\mathbb{Z}/p^m)$ of isogenies with a kernel of multiplicative type. Over this locus, the first projection is finite étale (by rigidity of group schemes of multiplicative type). It follows that I_m is an open and closed subscheme of $\mathcal{I}sog(p^g)^\circ \otimes (\mathbb{Z}/p^m)$, with $s_m: \mathcal{A}_g^\circ \otimes (\mathbb{Z}/p^m) \xrightarrow{\sim} I_m$. Moreover, $I_m = I_{m+k} \otimes_{(\mathbb{Z}/p^{m+k})} (\mathbb{Z}/p^m)$ for every $k \geqslant 0$.

Define $\widehat{\mathcal{I}} \subseteq \mathcal{I}sog(p^{eg})^\wedge$ as the formal subscheme with $\widehat{\mathcal{I}} \otimes (\mathbb{Z}/p^m) = I_m$ for every $m \geqslant 0$. It follows from the preceding remarks that $\widehat{\mathcal{I}}$ is an open and closed formal subscheme of $\mathcal{I}sog(p^{eg})^\wedge$. The section $s: \widehat{\mathcal{A}}_g \to \widehat{\mathcal{I}}$ obtained by taking the limit over all s_m is an isomorphism, and $\widehat{\mathrm{pr}}_2 \circ s = \Phi_{\mathrm{can}}$. This proves the proposition, except for the statement that $\widehat{\mathcal{I}}_{\mathrm{red}}$ is a disjoint union of irreducible components. To see this, remark that the topological space underlying $\widehat{\mathcal{I}}_{\mathrm{red}}$ is homeomorphic to that of $(\mathcal{A}_g \otimes \kappa(\mathfrak{p}))^\circ = \mathcal{A}_g^\circ \otimes \kappa(\mathfrak{p})$. Since this is the disjoint union of irreducible components, the same holds for $\widehat{\mathcal{I}}_{\mathrm{red}}$. □

3. Formal linearity

3.1. Let k be a perfect field of characteristic $p > 0$, let $\overline{k}$ an algebraic closure of k, and write $W = W(k), \overline{W} = W(\overline{k})$. Let Λ be a complete local noetherian ring with residue field k. A formal group $\mathcal{M}$ over $\mathrm{Spf}(\Lambda)$ (defined as in [11, Exposé VII$_\mathrm{B}$]) is called a formal torus if $\mathcal{M}\widehat{\otimes}\overline{\Lambda}$ is isomorphic to $\widehat{\mathbb{G}}_\mathrm{m}^d$ for some $d \geqslant 0$, where $\overline{\Lambda} = \Lambda\widehat{\otimes}_W\overline{W}$ (which again is a complete local noetherian ring). Such a formal torus is completely determined by its fibre $\mathcal{M}\widehat{\otimes}_\Lambda k$. In particular, every formal torus is defined over W, and there is an (anti-)equivalence of categories

$$\left\{\begin{array}{l}\text{formal tori}\\ \text{over } \mathrm{Spf}(\Lambda)\end{array}\right\}^\circ$$

$$\xrightarrow{\text{eq.}} \left\{\begin{array}{l}\text{free } \mathbb{Z}_p\text{-modules of finite rank with a}\\ \text{continuous action of } \mathrm{Gal}_\mathrm{cont}(\overline{\Lambda}/\Lambda) \cong \mathrm{Gal}(\overline{k}/k)\end{array}\right\}$$

by associating to $\mathcal{M}$ its character group $X^*(\mathcal{M}) = \mathrm{Hom}(\mathcal{M}\widehat{\otimes}\overline{\Lambda}, \widehat{\mathbb{G}}_\mathrm{m})$.

For a Galois submodule $Y \subseteq X^*(\mathcal{M})$ we write $\mathcal{N}(Y)$ for the common kernel of the characters $\chi \in Y$. If Y is primitive (meaning that the quotient group is torsion free) this is a formal subtorus of $\mathcal{M}$ with character group $X^*(\mathcal{M})/Y$. For general Y it has the form $\mathcal{N}(Y) = \mathfrak{T}\cdot\mathcal{N}$, where $\mathcal{N}$ is a formal subtorus of $\mathcal{M}$ and $\mathfrak{T}$ is a torsion subgroup.

LEMMA 3.2. *Let k be a finite field, and let $W, \mathcal{M}$ and Φ be as in Lemma* 2.5. *Suppose $\mathcal{M}$ has the structure of a formal torus such that $\Phi\colon \mathcal{M} \to \mathcal{M}^{(\sigma)}$ is a group homomorphism. Let Λ be a complete discrete valuation ring with residue field k, and let $\mathcal{N} \subseteq \mathcal{M}_\Lambda$ be an irreducible closed formal subscheme of $\mathcal{M}_\Lambda = \mathcal{M} \times_{\mathrm{Spf}(W)} \mathrm{Spf}(\Lambda)$ which is flat over Λ. Take an integer $m \geqslant 1$ such that the automorphism σ^m of W lifts to an automorphism τ of Λ.*

(i) *The following properties are equivalent.*

(a) *$\mathcal{N}$ is a formal subtorus of $\mathcal{M}_\Lambda$.*
(b) *There is a primitive Galois submodule $Y \subseteq X^*(\mathcal{M})$ such that $\mathcal{N} = \mathcal{N}(Y)_\Lambda$.*
(c) *$\Phi^m|_\mathcal{N}\colon \mathcal{N} \to \mathcal{M}_\Lambda^{(\tau)}$, obtained by restricting*

$$\Phi^m \otimes \mathrm{Id}\colon \mathcal{M}_\Lambda \to \mathcal{M}^{(\sigma^m)} \times_{\mathrm{Spf}(W)} \mathrm{Spf}(\Lambda) \cong \mathcal{M}_\Lambda^{(\tau)}$$

to $\mathcal{N}$, factors through $\mathcal{N}^{(\tau)} \hookrightarrow \mathcal{M}_\Lambda^{(\tau)} = (\mathcal{M}^{(\sigma^m)})_\Lambda$.

(ii) *The following properties are equivalent.*

(a) *$\mathcal{N} = \mathfrak{T}\cdot\mathcal{N}'$ is the translate of a formal subtorus $\mathcal{N}' \subseteq \mathcal{M}_\Lambda$ over an irreducible closed formal subscheme $\mathfrak{T}$, flat over Λ, contained in the p^n-torsion subgroup $\mathcal{M}_\Lambda[p^n]$ for some $n \geqslant 0$.*
(b) *There is a (possibly non-primitive) Galois submodule $Y \subseteq X^*(\mathcal{M})$ such that $\mathcal{N}$ is an irreducible component of $\mathcal{N}(Y)_\Lambda$.*

(c) *There are integers* $k, l \geqslant 1$ *such that the morphism* $\Phi^{(k+l)m}|_{\mathcal{N}}\colon \mathcal{N} \to \mathcal{M}_{\Lambda}^{(\tau^{k+l})}$ *factors through* $(\Phi^{km}|_{\mathcal{N}})^{(\tau^l)}\colon \mathcal{N}^{(\tau^l)} \to \mathcal{M}_{\Lambda}^{(\tau^{k+l})}$.

Proof. Statement (i) is proven in [5]. In (ii), the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear. Assume that (c) holds, and let $\mathcal{N}'$ be the image of $\Phi^{lm}|_{\mathcal{N}}$. Then $\mathcal{N}'$ is mapped into $(\mathcal{N}')^{(\tau^k)}$ under $\Phi^{km}\colon \mathcal{M}_{\Lambda}^{(\tau^l)} \to \mathcal{M}_{\Lambda}^{(\tau^{k+l})}$. By (i), it follows that $\mathcal{N}'$ is a formal subtorus of $\mathcal{M}^{(\tau^l)}$, compatible with Φ^m. From the description of Φ^m over $\overline{\Lambda}$ it readily follows that

$$(\phi^{lm})^{-1}(\mathcal{N}'_{\overline{\Lambda}}) \subseteq \mathcal{M}_{\overline{\Lambda}}[p^{lm}] \cdot \mathcal{N}'_{\overline{\Lambda}}, \quad \text{hence}$$

$$\mathcal{N} \subseteq (\phi^{lm})^{-1}(\mathcal{N}') \subseteq \mathcal{M}_{\Lambda}[p^{lm}] \cdot \mathcal{N}'.$$

Because we assumed $\mathcal{N}$ to be irreducible we conclude that (a) holds. □

3.3. At this point, let us set up the situation that we will study in the next sections. Fix $n \geqslant 3$, and write $\mathsf{A}_g = \mathsf{A}_{g,1,n}$. We consider a closed, absolutely irreducible algebraic subvariety Z of $\mathsf{A}_g \otimes F$, where F is a number field.

Next we introduce models in mixed characteristic. So, let $\mathfrak{p}$ be a finite prime of F with residue field κ of characteristic $p > 0$, with $p \nmid n$. Write $\mathcal{A}_g = \mathsf{A}_{g,1,n} \otimes \widehat{\mathcal{O}}_{\mathfrak{p}}$, and define $\mathcal{Z} \hookrightarrow \mathcal{A}_g$ as the Zariski closure of Z inside $\mathcal{A}_g$. We write $\widehat{\mathcal{Z}} \hookrightarrow \widehat{\mathcal{A}}_g$ for the formal completion along the ordinary locus in characteristic p, and for a closed ordinary point $x \in \big(\mathcal{Z} \otimes \kappa(\mathfrak{p})\big)^{\circ}$, let $\mathfrak{Z}_x \hookrightarrow \mathfrak{A}_x$ over $\mathfrak{S} = \mathrm{Spf}(\Lambda)$ (with $\Lambda = W(\kappa(x)) \otimes_{W(\kappa(\mathfrak{p}))} \widehat{\mathcal{O}}_{\mathfrak{p}}$) be the formal completion at x.

DEFINITION 3.4. We say that $\mathcal{Z}$ is formally linear at the closed point $x \in \big(\mathcal{Z} \otimes \kappa(\mathfrak{p})\big)^{\circ}$ if $\mathfrak{Z}_x$ is a formal subtorus of $\mathfrak{A}_x$. If all irreducible components of $\mathfrak{Z}_x$ have the properties described in (ii) of Lemma 3.2 (w.r.t. $\mathfrak{A}_x$ and Φ_{can}) then we say that $\mathcal{Z}$ is formally quasi-linear at x.

If $\mathfrak{Z}_x$ has at least one irreducible component which is a formal subtorus of $\mathfrak{A}_x$ (respectively the translate of a formal subtorus over a torsion point) then we say that $\mathcal{Z}$ has formally linear (respectively formally quasi-linear) components at x.

DEFINITION 3.5. Let X be an abelian variety of CM-type, defined over a number field K. If $\mathfrak{p}$ is a finite prime of K then we say that X is canonical at $\mathfrak{p}$ if there exists an abelian scheme $\mathcal{X}_{\mathfrak{p}}$ over $\mathrm{Spec}(\mathcal{O}_{K,\mathfrak{p}})$ with generic fibre X and ordinary special fibre $\mathcal{X}_{\mathfrak{p}} \otimes \kappa(\mathfrak{p})$, such that $\mathcal{X}_{\mathfrak{p}}$ is the canonical lifting of $\mathcal{X}_{\mathfrak{p}} \otimes \kappa(\mathfrak{p})$. We say that a CM-point $t \in \mathsf{A}_{g,1,n}(K)$ is canonical at $\mathfrak{p}$ if the corresponding abelian variety has this property.

3.6. Suppose that $\mathcal{Z}$ has formally quasi-linear components at the closed ordinary point x. Let $\mathcal{R}$ be a complete local algebra which is finite and flat over Λ, and let $\hat{t} \in \mathfrak{Z}_x(\mathcal{R})$ be a torsion point. The formal abelian scheme over $\mathrm{Spf}(\mathcal{R})$ corresponding to $\hat{t}$ is algebraizable, so we get an abelian scheme $\mathcal{X} = \mathcal{X}_t$ over $\mathrm{Spec}(\mathcal{R})$, and $\hat{t}$ extends to a section $t \in \mathcal{Z}(\mathcal{R})$. It follows from Lemma 2.8 that $\mathcal{X}$ is of CM-type.

Let $\widehat{\mathcal{T}}$ be the collection of all points $\hat{t} \in \mathfrak{Z}_x(\mathcal{R})$, where $\mathcal{R}$ ranges over all complete local domains, finite and flat over Λ, and $\hat{t}$ is a torsion point of $\mathfrak{Z}_x(\mathcal{R})$. Let $\mathcal{T}$ be the collection of corresponding points $t \in \mathcal{Z}(\mathcal{R})$. We claim that $\mathcal{T}$ is Zariski dense in $\mathcal{Z}$. To see this, write $\mathcal{Z}' \subseteq \mathcal{Z}$ for the Zariski closure of $\mathcal{T}$. By assumption, there is an irreducible component $\mathfrak{C} \subseteq \mathfrak{Z}_x$ which is the translate of a formal subtorus of $\mathfrak{A}_x$ over a torsion point. From the definition of the set $\mathcal{T}$ we see that $\mathfrak{C}$ is contained in the formal completion of $\mathcal{Z}'$ at x. The claim follows by a dimension argument: $\mathcal{Z}'$ and $\mathcal{Z}$ are flat over $\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathfrak{p}})$ of relative dimensions $d' \leqslant d$. Then the closed fibres $\mathcal{Z}' \otimes \kappa(\mathfrak{p})$ and $\mathcal{Z} \otimes \kappa(\mathfrak{p})$ are equidimensional of dimension d' and d respectively, and $\mathfrak{C} \subseteq (\mathcal{Z}')_{/\{x\}}$ implies that $d' = d$. Since $Z \otimes Q(\widehat{\mathcal{O}}_{\mathfrak{p}})$ is irreducible, the generic fibre of $\mathcal{Z}'$ is equal to $Z \otimes Q(\widehat{\mathcal{O}}_{\mathfrak{p}})$, and by definition of $\mathcal{Z}$ this implies that $\mathcal{Z}' = \mathcal{Z}$.

Let $Q = Q(\mathcal{R})$ denote the quotient field of a complete domain $\mathcal{R}$ as above, then we have a collection T of CM-points $t \in Z(Q)$, corresponding to the characteristic zero fibres X_t of the abelian schemes $\mathcal{X}_t$. From the fact that $\mathcal{T}$ is Zariski dense in $\mathcal{Z}$ it follows that T is dense in Z. Notice that the abelian varieties X_t are all p-isogenous, i.e., given two torsion points $\hat{t} \in \mathfrak{Z}_x(\mathcal{R})$ and $\hat{t}' \in \mathfrak{Z}_x(\mathcal{R}')$ in the collection $\widehat{\mathcal{T}}$, then over a common field extension of $Q(\mathcal{R})$ and $Q(\mathcal{R}')$ the abelian varieties X_t and $X_{t'}$ are isogenous via an isogeny whose degree is a power of p. This is because X_t and $X_{t'}$ are CM-liftings of the same ordinary abelian variety in characteristic p.

Choose one of the points $t \in T$, and consider the corresponding abelian variety X_t. As X_t is of CM-type, it is defined over some number field $K_t \supseteq F$, which we take large enough so that all endomorphisms of $X_t \otimes \overline{K}_t$ are defined over K_t. The endomorphism ring $\mathrm{End}(X_t)$ is an order in $\mathrm{End}^0(X_t)$. It has a well-determined index in a maximal order of $\mathrm{End}^0(X_t)$, which we call the conductor of $\mathrm{End}(X_t)$, and which we denote by $\mathfrak{f}(X_t)$.

Now choose a prime number $\ell \neq p$, with the following properties:

($*$) ℓ does not divide the conductor $\mathfrak{f}(X_t)$, i.e., $\mathrm{End}(X_t)$ is maximal at ℓ,

($**$) the prime ℓ splits completely in the endomorphism algebra $\mathrm{End}^0(X_t)$, i.e., $\mathrm{End}^0(X_t) \otimes \mathbb{Q}_\ell$ is a product of algebras $\mathrm{M}_m(\mathbb{Q}_\ell)$.

Possibly after first replacing K_t by a finite extension, X_t has good reduction $X_{t,\mathfrak{l}}$ at all primes $\mathfrak{l}$ above ℓ. The fact that ℓ splits completely in $\mathrm{End}^0(X_t)$ implies that the reduction is ordinary (using [32, Lem. 5]). By Lemma 2.8, X_t is isogenous to the canonical lifting of $X_{t,\mathfrak{l}}$, so $\mathrm{End}^0(X_t) = \mathrm{End}^0(X_{t,\mathfrak{l}})$. The conductors of the endomorphism rings can only differ by an ℓ-power, see [24, Lem. 2.1], and it then follows from the first condition on ℓ that $\mathrm{End}(X_t) \cong \mathrm{End}(X_{t,\mathfrak{l}})$. We conclude that X_t is canonical at all primes of K_t above ℓ.

As remarked, all abelian varieties X_t with $t \in T$ are p-isogenous. Therefore, the above conditions on ℓ do not depend on the chosen t, and our conclusion is valid for all X_t simultaneously. This shows that if $\mathcal{Z}$ has formally quasi-linear components at an ordinary point x in characteristic p, then there is a different characteristic ℓ and a Zariski dense collection T of CM-points $t \in Z(K_t)$ such that each X_t is canonical at all primes $\mathfrak{l}$ of K_t above ℓ.

We will now show that, conversely, this last property implies that $\mathcal{Z}$ has formally linear components at some of its ordinary points in prime characteristic.

PROPOSITION 3.7. *Let Z, $\mathfrak{p}$, $\mathcal{Z}$ etc. be as in* 3.3. *Suppose there is a collection T of CM-points $t \in Z(K_t)$ (K_t a number field containing F) which is Zariski dense in Z (over F). Also suppose that each X_t is canonical at some prime $\mathfrak{q}$ of K_t above $\mathfrak{p}$. Then there is a non-empty union U of irreducible components of $(\mathcal{Z} \otimes \kappa(\mathfrak{p}))^\circ$ such that Z has formally linear components at all closed points $x \in U$.*

Proof. For each t, choose a prime $\mathfrak{q}$ of K_t above $\mathfrak{p}$ such that X_t is canonical at $\mathfrak{q}$. Write $\mathcal{R}_\mathfrak{q} = W(\kappa(\mathfrak{q})) \otimes_{W(\kappa(\mathfrak{p}))} \widehat{\mathcal{O}}_\mathfrak{p}$, then X_t gives rise to an $\mathcal{R}_\mathfrak{q}$-valued point $t_\mathfrak{q}\colon \mathrm{Spec}(\mathcal{R}_\mathfrak{q}) \to \mathcal{Z}$, corresponding to the abelian scheme $\mathcal{X}_{t,\mathfrak{q}}$ over $\mathcal{R}_\mathfrak{q}$.

Let $N = {}^p\log(\#\kappa(\mathfrak{p}))$. The automorphism σ^N of $W(\kappa(\mathfrak{q}))$ lifts to an automorphism τ of $\mathcal{R}_\mathfrak{q}$. Since $\mathcal{X}_{t,\mathfrak{q}}$ is the canonical lifting of $\mathcal{X}_{t,\mathfrak{q}} \otimes \kappa(\mathfrak{q})$, the morphism

$$\mathrm{Frob}^N\colon \mathcal{X}_{t,\mathfrak{q}} \otimes \kappa(\mathfrak{q}) \to \big(\mathcal{X}_{t,\mathfrak{q}} \otimes \kappa(\mathfrak{q})\big)^{(p^N)}$$

lifts to a morphism $F_t\colon \mathcal{X}_{t,\mathfrak{q}} \to \mathcal{X}_{t,\mathfrak{q}}^{(\tau)}$ over $\mathrm{Spec}(\mathcal{R}_\mathfrak{q})$. We consider this as a point $F_t \in \mathcal{I}sog^\circ(\mathcal{R}_\mathfrak{q})$, where $\mathcal{I}sog = \mathcal{I}sog(p^{Ng})$. Define $\mathcal{Y} \subseteq \mathcal{I}sog^\circ$ as the Zariski closure over $\mathrm{Spec}(\widehat{\mathcal{O}}_\mathfrak{p})$ of these points.

For the projection maps $\mathrm{pr}_i\colon \mathcal{I}sog^\circ \to \mathcal{A}_g^\circ$ we have $\mathrm{pr}_1(F_t) = t_\mathfrak{q}$ and $\mathrm{pr}_2(F_t) = t_\mathfrak{q} \circ \tau$, and since the points $t_\mathfrak{q}$ (hence also the points $t_\mathfrak{q} \circ \tau$) are Zariski dense in $\mathcal{Z}^\circ = \mathcal{Z} \cap \mathcal{A}_g^\circ$, it follows that pr_i ($i \in \{1,2\}$) restricts to a finite surjective morphism $\mathrm{pr}_i\colon \mathcal{Y} \to \mathcal{Z}^\circ$. Possibly after replacing the collection of CM-points T by a subcollection T' (which is still dense in Z), and replacing $\mathcal{Y}$ by the Zariski closure of the points F_t with $t \in T'$, we may assume that, moreover, every irreducible component of $\mathcal{Y}$ maps surjectively to an irreducible component of $\mathcal{Z}^\circ$.

Write $k = \kappa(\mathfrak{p})$. By construction, $\mathcal{Y}$ and $\mathcal{Z}^\circ$ are flat over $\mathrm{Spec}(\widehat{\mathcal{O}}_\mathfrak{p})$ and, as remarked above, the projections pr_i are finite. The closed fibres $\mathcal{Y}_k$ and $\mathcal{Z}_k^\circ$ are therefore equidimensional of the same dimension, and every irreducible component of $\mathcal{Y}_k$ maps surjectively to an irreducible component of $\mathcal{Z}_k^\circ$.

We have seen before that there is a disjoint union of irreducible components $\mathcal{I}_k \subset \mathcal{I}sog_k^\circ$ classifying the Nth power of Frobenius. Then $\mathcal{Y}_k = \mathcal{Y}'_k \sqcup \mathcal{Y}''_k$, where $\mathcal{Y}'_k$ and $\mathcal{Y}''_k$ are unions of connected components of $\mathcal{Y}_k$, chosen such that $\mathcal{Y}'_k \subseteq \mathcal{I}_k$ and $\mathcal{Y}''_k \cap \mathcal{I}_k = \emptyset$. Now $\mathrm{pr}_{1|\mathcal{I}}\colon \mathcal{I}_k \to \mathcal{A}_g^\circ \otimes k$ is an isomorphism and (by our choice of N) the composition $\mathrm{pr}_2 \circ (\mathrm{pr}_{1|\mathcal{I}})^{-1}\colon \mathcal{A}_g^\circ \otimes k \to \mathcal{A}_g^\circ \otimes k$ is the identity on the underlying topological space. The image of $\mathcal{Y}'_k$ under both projections to $\mathcal{Z}_k^\circ$ is therefore the same; call it $\mathcal{Z}_k^{\circ\prime} \subseteq \mathcal{Z}_k^{\circ}$. It is a union of irreducible components of $\mathcal{Z}_k^{\circ}$, which is non-empty because the special fibre of every F_t factors through $\mathcal{Y}_k'$.

Next we look at formal completions. Write $\widehat{\mathcal{Y}}$, $\widehat{\mathcal{Y}}'$ and $\widehat{\mathcal{Z}}$ for the formal completions of $\mathcal{Y}, \mathcal{Y}, \mathcal{Z}$ along $\mathcal{Y}_k, \mathcal{Y}_k'$ and $\mathcal{Z}_k^{\circ}$ respectively. Notice that these formal schemes are formally reduced, noetherian and adic, flat and of finite type over $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathfrak{p}})$, since $\mathcal{Y}$ and $\mathcal{Z}^{\circ}$ are excellent schemes (being of finite type over a complete local ring) with the corresponding properties.

Since $\mathcal{Y}_k'$ and $\mathcal{Y}_k''$ are disjoint, $\widehat{\mathcal{Y}}'$ is an open and closed formal subscheme of $\widehat{\mathcal{Y}}$. Let $\widehat{\mathcal{Z}} = \cup_{\alpha\in A}\widehat{\mathcal{Z}}_{\alpha}$ be the decomposition of $\widehat{\mathcal{Z}}$ into irreducible components. The projections $\widehat{\mathrm{pr}}_i\colon \widehat{\mathcal{Y}}' \to \widehat{\mathcal{Z}}$ are finite (using [13, III, Cor. 4.8.4]), so by Lemma 1.6 we have

$$\widehat{\mathrm{pr}}_1\colon \widehat{\mathcal{Y}}' \xrightarrow{\sim} \bigcup_{\alpha\in A_1} \widehat{\mathcal{Z}}_{\alpha}, \qquad \widehat{\mathrm{pr}}_2(\widehat{\mathcal{Y}}') = \bigcup_{\alpha\in A_2} \widehat{\mathcal{Z}}_{\alpha}$$

for some $A_1 \subseteq A$, $A_2 \subseteq A$. Proposition 2.14 shows that the composition

$$\widehat{\mathrm{pr}}_2 \circ (\widehat{\mathrm{pr}}_1)^{-1}\colon \bigcup_{\alpha\in A_1} \widehat{\mathcal{Z}}_{\alpha} \to \bigcup_{\alpha\in A_2} \widehat{\mathcal{Z}}_{\alpha}$$

is the restriction of Φ_{can}^N.

At this point we apply Lemma 1.7. We take $t \in T$ (the subcollection with which we replaced the original T) corresponding to a point in the regular locus of Z. This is certainly possible, since the collection T is Zariski dense in Z. We conclude that there is a unique component $\widehat{\mathcal{Z}}_{\alpha(t)} \subseteq \widehat{\mathcal{Z}}$ with $\alpha(t) \in A_1$ such that $\hat{t}_{\mathfrak{q}}\colon \mathrm{Spf}(\mathcal{R}_{\mathfrak{q}}) \to \widehat{\mathcal{Z}}$ factors through $\widehat{\mathcal{Z}}_{\alpha(t)}$. Let $\widehat{\mathcal{Y}}_{\alpha}(t)$ be the unique irreducible component with $\widehat{\mathrm{pr}}_1\colon \widehat{\mathcal{Y}}_{\alpha(t)} \xrightarrow{\sim} \widehat{\mathcal{Z}}_{\alpha(t)}$. Since $t_{\mathfrak{q}} = \mathrm{pr}_1 \circ F_t$, the section $\hat{F}_t\colon \mathrm{Spf}(\mathcal{R}_{\mathfrak{q}}) \to \widehat{\mathcal{Y}}'$ factors through $\widehat{\mathcal{Y}}_{\alpha(t)}$. The image of $\widehat{\mathcal{Y}}_{\alpha(t)}$ under $\widehat{\mathrm{pr}}_2$ is some irreducible component $\widehat{\mathcal{Z}}_{\alpha'(t)}$ through which $\widehat{\mathrm{pr}}_2 \circ \hat{F}_t$ factors. But $\widehat{\mathrm{pr}}_2 \circ \hat{F}_t = \hat{\tau} \circ \hat{t}_{\mathfrak{q}}\colon \mathrm{Spf}(\mathcal{R}_{\mathfrak{q}}) \to \widehat{\mathcal{Z}}$ and, τ being an automorphism of $\mathcal{R}$, we see that $\hat{t}_{\mathfrak{q}}$ factors through $\widehat{\mathcal{Z}}_{\alpha'(t)}$. By assumption we have $\widehat{\mathcal{Z}}_{\alpha(t)} = \widehat{\mathcal{Z}}_{\alpha'(t)}$. This shows that for every $t \in T$ corresponding to a point in the regular locus of Z, the formal irreducible component $\widehat{\mathcal{Z}}_{\alpha(t)}$ is mapped into itself under Φ_{can}^N.

Let x be a closed point on the component $\widehat{\mathcal{Z}}_{\alpha(t)}$. The formal completion of $\widehat{\mathcal{Z}}_{\alpha(t)}$ at x is the union of a number of irreducible components, say $\mathfrak{C}_1, \dots, \mathfrak{C}_r$ of $\mathfrak{Z}_x$. If m is a suitable multiple of both N and ${}^p\log(\#\kappa(x))$ then Φ_{can}^m induces a finite morphism $\Phi_{\mathrm{can}}^m\colon \mathfrak{A}_x \to \mathfrak{A}_x$ of formal schemes, and it follows from the above that this maps $\cup_j \mathfrak{C}_j \subseteq \mathfrak{A}_x$ onto itself. Then Φ_{can}^m acts by permutations on the set $\{\mathfrak{C}_1, \dots, \mathfrak{C}_r\}$, so after replacing m by a suitable multiple it preserves all $\mathfrak{C}_j$. By Lemma 3.2 these irreducible components $\mathfrak{C}_j \subseteq \mathfrak{Z}_x$ are therefore formal subtori of $\mathfrak{A}_x$. $\square$

COROLLARY 3.8. *Let $Z, \mathfrak{p}, \mathcal{Z} = \mathcal{Z}_{\mathfrak{p}}$ be as in* 3.3 *and suppose $\mathcal{Z}_{\mathfrak{p}}$ has formally quasi-linear components at some closed ordinary point x. Then there exist infinitely many primes $\mathfrak{l}$ of $\mathcal{O}_F$ such that the model $\mathcal{Z}_{\mathfrak{l}}$ of Z over* Spec$(\widehat{\mathcal{O}}_{\mathfrak{l}})$ *is formally linear at all closed points y in a non-empty open subset of $(\mathcal{Z}_{\mathfrak{l}} \otimes \kappa(\mathfrak{l}))^{\circ}$.*

Proof. We start as in 3.6. We have seen that for primes ℓ satisfying conditions $(*)$ and $(**)$, there is a Zariski dense collection T of CM-points $t \in Z(K_t)$ such that each X_t is canonical at all primes of K_t above ℓ.

We consider primes $\mathfrak{l}$ of $\mathcal{O}_F$ such that the residue characteristic ℓ satisfies these conditions $(*)$ and $(**)$ of 3.6, and such that no irreducible component of $\mathcal{Z}_{\mathfrak{l}} \otimes \kappa(\mathfrak{l})$ is contained in the singular locus of $\mathcal{Z}_{\mathfrak{l}}$. This last condition excludes only finitely many primes $\mathfrak{l}$. The model $\mathcal{Z}_{\mathfrak{l}}$ being an excellent scheme, it follows that for generic $y \in \mathcal{Z}_{\mathfrak{l}} \otimes \kappa(\mathfrak{l})$, the completed local ring $\mathcal{O}_y^{\wedge}$ of $\mathcal{Z}_{\mathfrak{l}}$ at y is regular. The corollary now results from the previous proposition and the remark that $(\mathcal{Z}_{\mathfrak{l}})_{/\{y\}}$ is irreducible if $\mathcal{O}_y^{\wedge}$ is regular. □

Remark 3.9. A posteriori we will get a much stronger conclusion, see Theorem 4.5.

4. Formal linearity and subvarieties of Hodge type

4.1. Our main motivation for introducing the concept of formal linearity is its relation to Shimura varieties, or rather to subvarieties of $\mathsf{A}_{g,1,n}$ of Hodge type. The first main result in this direction was established in the Ph.D. thesis of Noot ([21], see also [22]).

THEOREM 4.2 (Noot). (i) *Let F be a number field, and let $S \hookrightarrow \mathsf{A}_{g,1,n} \otimes F$ be a subvariety of Shimura type. Let $\mathfrak{p}$ be a prime of F above p, and write $\mathcal{S}_{\mathfrak{p}}$ for the Zariski closure of S inside $\mathsf{A}_{g,1,n} \otimes \widehat{\mathcal{O}}_{\mathfrak{p}}$. Let x be a closed point in the ordinary locus $(\mathcal{S}_{\mathfrak{p}} \otimes \kappa(\mathfrak{p}))^{\circ}$. Then $\mathcal{S}_{\mathfrak{p}}$ is formally quasi-linear at x. For $\mathfrak{p}$ outside a finite set Σ of primes of $\mathcal{O}_F$, the formal completion $\mathfrak{S}_x$ of $\mathcal{S}_{\mathfrak{p}}$ at x is a union of formal subtori of $\mathfrak{A}_x$. If S is non-singular then we can choose the (finite) set Σ such that $\mathcal{S}_{\mathfrak{p}}$ for $\mathfrak{p} \notin \Sigma$ is formally linear at all closed ordinary points $x \in (\mathcal{S}_{\mathfrak{p}} \otimes \kappa(\mathfrak{p}))^{\circ}$.*

(ii) *Let $S \hookrightarrow \mathsf{A}_{g,1,n} \otimes F$ be a subvariety of Hodge type, and let $\mathcal{S}_{\mathfrak{p}}$ and x be as in* (i). *For $\mathfrak{p}$ outside a finite set of primes of $\mathcal{O}_F$, the formal completion $\mathfrak{S}_x$ is a union of formal subtori of $\mathfrak{A}_x$.*

For a proof of this result we refer to [21, Prop. 2.2.3] and [22, Th. 3.7]; see also [18, Chap. III, Sect. 4].

The main theorem of this section is a converse to Noot's theorem. Taken together, the two results provide a characterization of subvarieties of $\mathsf{A}_{g,1,n}$ of Hodge type in terms of formal linearity. Notice that Theorem 4.5 below is very similar to Corollary 5.5 in Part I. Since we will actually reduce the proof of (4.5) to this corollary, let us briefly recall its statement, as well as the method used to prove it.

https://doi.org/10.1023/A:1000411631772 Published online by Cambridge University Press

4.3. Let $(G, \mathfrak{X})$ be an arbitrary Shimura datum, and let $K \subseteq G(\mathbb{A}_f)$ be a torsion-free compact open subgroup. In [19, Sect. 5] we have introduced a 'Serre–Tate group structure' on the formal completion of $\mathrm{Sh}_K(G, \mathfrak{X})_{\mathbb{C}}$ at an arbitrary closed point x. The definition works as follows.

The point x lies in the image Sh^0 of a uniformization map $u: X \to \mathrm{Sh}_K(G, \mathfrak{X})$, which, by our assumption on K, is a topological covering. Choose $\tilde{x} \in X$ with $u(\tilde{x}) = x$. We have a Borel embedding $X \hookrightarrow \check{X} = G^{\mathrm{ad}}(\mathbb{C})/P_{\tilde{x}}(\mathbb{C})$, where $P_{\tilde{x}} \subset G^{\mathrm{ad}}_{\mathbb{C}}$ is the parabolic subgroup stabilizing the point $\tilde{x}$. Using the Hodge decomposition of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathrm{Ad} \circ h_{\tilde{x}}$, we obtain a parabolic subgroup $P^-_{\tilde{x}} \subset G^{\mathrm{ad}}_{\mathbb{C}}$ opposite to $P_{\tilde{x}}$. Write $U^-_{\tilde{x}}$ for the unipotent radical of $P^-_{\tilde{x}}$, which is isomorphic to $\widehat{\mathbb{G}}^d_{\mathrm{a}}$ for $d = \dim(X)$. The natural map $U^-_{\tilde{x}}(\mathbb{C}) \to \check{X}$ gives an isomorphism of $U^-_{\tilde{x}}(\mathbb{C})$ onto its image $\mathcal{U} \subset \check{X}$ which is the complement of a divisor $D \subset \check{X}$.

On formal completions we obtain an isomorphism

$$\mathfrak{U}_{\tilde{x}} := U^-_{\tilde{x}\,/\{1\}} \xrightarrow{\sim} \mathcal{U}_{/\{\tilde{x}\}} = \check{X}_{/\{\tilde{x}\}} \xrightarrow[\sim]{u} \mathrm{Sh}^0_{/\{x\}} =: \mathfrak{S}\mathfrak{h}_x$$

and in this way $\mathfrak{S}\mathfrak{h}_x$ inherits the structure of a formal vector group. This is our Serre–Tate group structure over $\mathbb{C}$. In this context we define the notion of formal linearity analogous to the definition in 3.4.

In Part I we proved the following statement.

THEOREM (=[19, Cor. 5.5]). *An irreducible algebraic subvariety $Z \hookrightarrow \mathrm{Sh}_K\,(G, \mathfrak{X})_{\mathbb{C}}$ is of Hodge type if and only if* (i) *Z is formally linear at some closed point, and* (ii) *Z contains a CM-point.*

4.4. Let us sketch a proof of the 'if' statement in the theorem. (For details see [19, Sect. 5].) So, assume that Z is formally linear at the point x and that, moreover, Z contains a CM-point. Possibly after passing to a higher level of $\mathrm{Sh}(G, \mathfrak{X})$, we may assume that K is torsion-free.

We compare Z with the smallest subvariety of Hodge type $S \hookrightarrow \mathrm{Sh}_K(G, \mathfrak{X})$ containing it. There is a closed immersion of Shimura data $(M, \mathfrak{Y}) \hookrightarrow (G, \mathfrak{X})$, a connected component $Y \subseteq \mathfrak{Y}$, and a class $\eta K \in G(\mathbb{A}_f)/K$, such that S is the image of $Y \times \eta K$ in $\mathrm{Sh}_K(G, \mathfrak{X})$. If $\xi: G \twoheadrightarrow G^{\mathrm{ad}} \hookrightarrow \mathrm{GL}(V)$ is a representation inducing a VHS $\mathcal{V}(\xi)$ over $\mathrm{Sh}_K(G, \mathfrak{X})$, then $\xi(M)$ is the generic Mumford–Tate group of $\mathcal{V}(\xi)_{|Z}$. Our goal is to prove that $Z = S$.

Write $u_S: Y \to S$ for the uniformization map, and let $\widetilde{Z} \subseteq Y$ be a connected component of $u_S^{-1}(Z)$. For simplicity, let us assume that $\widetilde{Z}$ is analytically irreducible. In the general case the argument is essentially the same, but notationally more involved. The situation is summarized by the following diagram:

https://doi.org/10.1023/A:1000411631772 Published online by Cambridge University Press

$$\begin{tikzcd}
\tilde{x} \arrow[d] \arrow[r, phantom, "\in"] & \widetilde{Z} \arrow[r, hook] \arrow[d] & Y \arrow[r, hook] \arrow[d, "u_S"] & X \arrow[d] \arrow[r, phantom, "\subseteq"] & \mathfrak{X} \arrow[d] & \alpha \arrow[d] \\
x \arrow[r, phantom, "\in"] & Z \arrow[r, hook] & S \arrow[r, hook] & \mathrm{Sh}^0 \arrow[r, phantom, "\subseteq"] & \mathrm{Sh}_K(G, \mathfrak{X}) & {[\alpha, \eta K]}
\end{tikzcd}$$

Choose an isomorphism $U^-_{\tilde{x}} \cong \mathbb{G}_{\mathrm{a}}^d$, and let $t_1, \ldots, t_d$ denote the corresponding coordinates on $U^-_{\tilde{x}}$. The t_i, viewed as functions on $\mathcal{U} \subset \check{X}$, extend to global sections $t_i \in \Gamma(\check{X}, \mathcal{L})$, where $\mathcal{L} = \mathcal{O}_{\check{X}}(k \cdot D)$ for a suitable $k \geqslant 0$. This bundle $\mathcal{L}$ is a $G^{\mathrm{ad}}_{\mathbb{C}}$-bundle over $\check{X}$. (Strictly speaking, this is not completely true: $\mathcal{L}$ is in general only a projective $G^{\mathrm{ad}}_{\mathbb{C}}$-bundle. In this sketch of the argument we shall ignore this.) Define $I = \{s \in \Gamma(\check{X}, \mathcal{L}) \mid s_{|\widetilde{Z}} = 0\}$, and write $V(I)$ for the zero locus of I. From the assumption that Z is formally linear at x one deduces that $\widetilde{Z}$ is an irreducible component of $V(I) \cap X$.

At this point we invoke a monodromy argument. Corresponding to the local system underlying $\mathcal{V}(\xi)_{|Z}$, we have a monodromy homomorphism $\rho_Z \colon \pi_1(Z, x) \to \xi(M)(\mathbb{Q}) \subseteq G^{\mathrm{ad}}(\mathbb{Q})$, and since $\mathcal{L}$ is a G^{ad}-bundle over $\check{X}$ this induces an action of $\pi_1(Z, x)$ on $\Gamma(\check{X}, \mathcal{L})$. An easy lemma (see [19, Lem. 3.5]) shows that the subspace I is stable under this action. It follows that I is also stable under the action of the Zariski closure of the image of ρ_Z in $\xi(M)_{\mathbb{C}} \subset G^{\mathrm{ad}}_{\mathbb{C}}$. However, since Z contains a CM-point, it follows from a theorem of Y. André ([2, Prop. 2]) that this Zariski closure is the full derived group $\xi(M)^{\mathrm{der}}_{\mathbb{C}}$. (In this sketch of the argument it may seem that one needs a *regular* CM-point for this, but this condition can be avoided.) Consequently, $V(I) \cap X$ is stable under the action of $M^{\mathrm{der}}(\mathbb{R})^+$, and since this group acts transitively on Y, it follows that $Z = S$. □

THEOREM 4.5. *Let $Z \hookrightarrow \mathsf{A}_{g,1,n} \otimes F$ be an irreducible algebraic subvariety of the moduli space $\mathsf{A}_{g,1,n}$, defined over a number field F. Suppose there is a prime $\mathfrak{p}$ of $\mathcal{O}_F$ such that the model $\mathcal{Z}$ of Z (as in Section* 3.3*) has formally quasi-linear components at some closed ordinary point $x \in (\mathcal{Z} \otimes \kappa(\mathfrak{p}))^\circ$. Then Z is of Hodge type, i.e., every irreducible component of $Z \otimes_F \mathbb{C}$ is a subvariety of Hodge type.*

Proof. We divide the proof in a number of steps.

Step 1. First we reduce to the case that Z is absolutely irreducible and $\mathcal{Z}$ is formally linear at x. Choose a finite extension F' of F such that every irreducible component $Z' \hookrightarrow \mathsf{A}_{g,1,n} \otimes F'$ of $Z \otimes_F F'$ is absolutely irreducible. There exists a component Z' and a prime $\mathfrak{p}'$ of $\mathcal{O}_{F'}$ above $\mathfrak{p}$, such that the model $\mathcal{Z}'$ of Z' over $\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathfrak{p}'})$ has formally linear components at some point x' in the preimage of x. By the remarks in 1.3, it suffices to prove the theorem for Z'. However, since Z' is absolutely irreducible, we can apply Corollary 3.8, and it follows that we may even assume that $\mathcal{Z}'$ is formally linear at the point x'.

https://doi.org/10.1023/A:1000411631772 Published online by Cambridge University Press

From now on we may therefore assume that Z is absolutely irreducible and that $\mathcal{Z}$ is formally linear at x.

Step 2. Let us make 4.3 more explicit in the case $(G, \mathfrak{X}) = (\mathrm{CSp}_{2g,\mathbb{Q}}, \mathfrak{H}_g^{\pm})$. The compact dual $\mathfrak{H}_g^{\vee}$ of $\mathfrak{H}_g$ is the flag variety of g-dimensional subspaces of $\mathbb{C}^{2g}$ which are totally isotropic for the (standard) symplectic form ψ. Suppose $a_1, \ldots, a_g, c_1, \ldots, c_g$ is a symplectic basis for $\mathbb{C}^{2g}$, and write $\mathcal{F} = \mathrm{Span}\{c_1, \ldots, c_g\}$. Then $\mathfrak{H}_g^{\vee} \cong \mathrm{Sp}_{2g,\mathbb{C}}/P(\mathcal{F})$, where $P(\mathcal{F})$ is the stabilizer of $\mathcal{F}$, which is a parabolic subgroup of $\mathrm{Sp}_{2g,\mathbb{C}}$. Let $P(\mathcal{F})^{-} \subset \mathrm{Sp}_{2g,\mathbb{C}}$ be the stabilizer of $\mathcal{F}^{-} = \mathrm{Span}\{a_1, \ldots, a_g\}$, which is a parabolic subgroup opposite to $P(\mathcal{F})$. The image $\mathcal{U}$ of $P(\mathcal{F})^{-}$ in $\mathfrak{H}_g^{\vee}$ is the complement of a divisor D.

In terms of flags, $\mathcal{U}$ is the open part of $\mathfrak{H}_g^{\vee}$ corresponding to flags of the form

$$\mathcal{F}_T = \mathrm{Span}\left(\left\{c_j + \sum_{i=1}^{n} t_{ij} \cdot a_i\right\}_{j=1,\ldots,n}\right),$$

where $T = (t_{ij})$ is a symmetric $g \times g$ matrix. The coefficients t_{ij} of the matrix T are well-determined regular functions on $\mathcal{U}$.

Suppose that $\mathcal{F}$ corresponds to a point $\tilde{x} \in \mathfrak{H}_g$. We thus have a polarized Hodge structure $h_{\tilde{x}}\colon \mathbb{S} \to \mathrm{CSp}_{2g,\mathbb{R}}$ on the space $\mathbb{Q}^{2g}$ for which $\mathcal{F} = \mathrm{Fil}^1_{\tilde{x}}(\mathbb{C}^{2g}) = (\mathbb{C}^{2g})^{1,0}$. Assume, moreover, that $\mathcal{F}^{-} = (\mathbb{C}^{2g})^{0,1}$. In this case we see that the isomorphism $\mathfrak{H}^{\vee}_{g\,/\{\tilde{x}\}1} \cong \widehat{\mathbb{G}}_{\mathrm{a}}^{d}$ giving rise to the Serre–Tate group structure as in 4.3, is given by the coordinates t_{ij}, in the sense that it is obtained from the isomorphism $\mathcal{U} \cong \mathrm{Spec}(\mathbb{C}[t_{ij}]/(t_{ij} - t_{ji}))$ by formal completion at the point $\tilde{x}$ (with coordinates $t_{ij} = 0$).

Step 3. Let $\overline{\kappa(x)}$ be an algebraic closure of $\kappa(x)$, write $\overline{W} = W(\overline{\kappa(x)})$, and let $\overline{\Lambda} = \overline{W} \otimes_{W(\kappa(\mathfrak{p}))} \widehat{\mathcal{O}}_{\mathfrak{p}}$. For the rest of the proof we fix compatible embeddings $\overline{\Lambda} \hookrightarrow \mathbb{C}$ and $F \hookrightarrow \mathbb{C}$. Using this, we obtain a moduli point $x_{\mathbb{C}}^{\mathrm{can}} \in Z_{\mathbb{C}}$ of the canonical lifting of x. Write (X, λ, θ) for the triplet corresponding to the point x and $(X^{\mathrm{can}}, \lambda^{\mathrm{can}}, \theta^{\mathrm{can}})$ for its canonical lifting.

Similar to the discussion in Section 2.6 (cf. 2.11), write $\mathfrak{A} = \mathfrak{A}_x \widehat{\otimes} \overline{\Lambda} = \mathrm{Spf}(A)$, with $A = \overline{\Lambda}[\![q_{ij} - 1]\!]/(q_{ij} - q_{ji})$. Our assumption that $\mathcal{Z}$ is formally linear at x implies that $\mathfrak{Z} = \mathfrak{Z}_x \widehat{\otimes} \overline{\Lambda}$ can be described as $\mathfrak{Z} = \mathrm{Spf}(A/\mathfrak{a}) \hookrightarrow \mathfrak{A}$, where $\mathfrak{a} \subseteq A$ is an ideal generated by elements of the form $(\prod_{ij} q_{ij}^{m_{ij}}) - 1$, with $m_{ij} \in \mathbb{Z}_p$.

Let K be the quotient field of $\overline{\Lambda}$. We have an isomorphism

$$K[\![\tau_{ij}]\!]/(\tau_{ij} - \tau_{ji}) \xrightarrow{\sim} K[\![q_{ij} - 1]\!]/(q_{ij} - q_{ji}),$$

given by $\tau_{ij} \mapsto \log(q_{ij})$. The element $\sum_{ij} m_{ij}\tau_{ij}$ maps to $\sum_{ij} m_{ij} \log(q_{ij}) = \log(\prod_{ij} q_{ij}^{m_{ij}})$, which, up to a unit in $K[\![q_{ij}-1]\!]/(q_{ij}-q_{ji})$, is equal to $(\prod_{ij} q_{ij}^{m_{ij}})-1$. Using the chosen embedding of $\overline{\Lambda}$ into $\mathbb{C}$, we obtain a ring homomorphism

$$A = \overline{\Lambda}[\![q_{ij} - 1]\!]/(q_{ij} - q_{ji}) \to A_{\mathbb{C}} := \mathbb{C}[\![\tau_{ij}]\!]/(\tau_{ij} - \tau_{ji})$$

https://doi.org/10.1023/A:1000411631772 Published online by Cambridge University Press

such that the ideal $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \cdot A_{\mathbb{C}}$ is generated by elements of the form $\sum_{ij} m_{ij}\tau_{ij}$, where the coefficients m_{ij} (as above) are now viewed as elements of $\mathbb{C}$, via the chosen embedding $\mathbb{Z}_p \subseteq \overline{\Lambda} \hookrightarrow \mathbb{C}$.

In the ring A, let us consider the ideal $\mathfrak{n}$ generated by the elements $q_{ij} - 1$, and the maximal ideal $\mathfrak{m} = (\mathfrak{n}, \mathfrak{p})$. The universal polarized deformation $\mathfrak{X} \to \mathfrak{A}$ is algebraizable, i.e., it is the formal completion along the closed fibre ($=$ $\mathfrak{m}$-adic completion) of an abelian scheme $\mathcal{X} \to \mathrm{Spec}(A)$. Write $\mathfrak{X}' \to \mathfrak{A}'$ for the formal completion of $\mathcal{X}/\mathrm{Spec}(A)$ along the canonical lifting of x. So: $\mathfrak{A}' = \mathrm{Spf}(A)$, where A is now considered with its $\mathfrak{n}$-adic topology, and $\mathfrak{X}'$ is the formal completion of $\mathcal{X}$ along the closed subscheme $\mathcal{X} \otimes_{A,\mathrm{ev}_0} \overline{\Lambda}$ corresponding to the canonical lifting of x. (Here $\mathrm{ev}_0\colon A \to \overline{\Lambda}$ is the map 'evaluation at the origin' given by $q_{ij} \mapsto 1$.)

We claim that the modules $H = \mathrm{H}^1_{\mathrm{DR}}(\mathfrak{X}/\mathfrak{A})$ and $H' = \mathrm{H}^1_{\mathrm{DR}}(\mathfrak{X}'/\mathfrak{A}')$ are isomorphic as A-modules (*not* as topological A-modules), and that the notion of horizontality is the same for the two modules. This follows from the fact that H' and $\Omega^{\mathrm{cont}}_{A/\overline{\Lambda}}$ (continuous forms w.r.t. the $\mathfrak{n}$-adic topology) are finitely generated over A and A is $\mathfrak{m}$-adically complete.

Pulling back the formal abelian scheme $\mathfrak{X}' \to \mathfrak{A}'$ via the continuous homomorphism $A \to A_{\mathbb{C}}$ ($\mathfrak{n}$-adic topology on A) thus yields a formal abelian scheme $\mathfrak{X}_{\mathbb{C}} \to \mathfrak{A}_{\mathbb{C}} := \mathrm{Spf}(A_{\mathbb{C}})$ for which we have a description of the de Rham cohomology in terms of the elements τ_{ij}. Namely, there is a horizontal basis $a_1, \ldots, a_g, c_1, \ldots, c_g$ for the $A_{\mathbb{C}}$-module $H_{\mathbb{C}} = \mathrm{H}^1_{\mathrm{DR}}(\mathfrak{X}_{\mathbb{C}}/\mathfrak{A}_{\mathbb{C}})$ such that the Hodge flag $\mathcal{F}^1$ is spanned by the elements $c_j + \sum_i \tau_{ij} a_i$. The basis $a_1, \ldots, a_g, c_1, \ldots, c_g$ is symplectic for the alternating bilinear form on $H_{\mathbb{C}}$ associated to the (pull-back of the) polarization λ on $\mathfrak{X}$.

Write $\mathcal{X}_{\mathbb{C}} \to \mathrm{Spec}(A_{\mathbb{C}})$ for the pull-back of $\mathcal{X} \to \mathrm{Spec}(A)$ via $A \to A_{\mathbb{C}}$; its formal completion along the closed fibre is $\mathfrak{X}_{\mathbb{C}} \to \mathfrak{A}_{\mathbb{C}}$. The corresponding morphism $\mathrm{Spec}(A_{\mathbb{C}}) \to \mathsf{A}_{g,1,n} \otimes \mathbb{C}$ sends the point with coordinates $\tau_{ij} = 0$ to the point $x^{\mathrm{can}}_{\mathbb{C}}$, and it follows from the given description of the de Rham cohomology that $\mathfrak{X}_{\mathbb{C}} \to \mathfrak{A}_{\mathbb{C}}$ is the universal deformation of $X^{\mathrm{can}}_{\mathbb{C}}$. We thus have an isomorphism of $\mathrm{Spf}(A_{\mathbb{C}})$ with the formal completion of $\mathsf{A}_{g,1,n} \otimes \mathbb{C}$ at $x^{\mathrm{can}}_{\mathbb{C}}$. The comparison isomorphism between de Rham and singular cohomology gives an isomorphism $H_{\mathbb{C}} \otimes_{A_{\mathbb{C}},\mathrm{ev}_0} \mathbb{C} \xrightarrow{\sim} \mathrm{H}^1(X^{\mathrm{can}}, \mathbb{C})$. Under this isomorphism the subspace $\mathrm{Span}(a_1, \ldots, a_g) \otimes_{A_{\mathbb{C}},\mathrm{ev}_0} \mathbb{C}$ obtained from the 'unit subcrystal' (cf. 2.6), is mapped to $\mathrm{H}^{0,1}(X^{\mathrm{can}}, \mathbb{C})$ and the Hodge flag $\mathrm{Span}(c_1, \ldots, c_g) \otimes_{A_{\mathbb{C}},\mathrm{ev}_0} \mathbb{C}$ maps to $\mathrm{H}^{1,0}(X^{\mathrm{can}}, \mathbb{C})$.

Step 4. The assumption that $\mathcal{Z}$ is formally linear at x implies that the point x lies in the locus where the structure morphism $\mathcal{Z}_{\overline{\Lambda}} \to \mathrm{Spec}(\overline{\Lambda})$ is smooth. Since this is an open locus, the same is true for the point x^{can}, and by the results of [13, Chap. IV, Sect. 17] it follows that $x^{\mathrm{can}}_{\mathbb{C}}$ is a non-singular point of $Z_{\mathbb{C}}$. We claim that the isomorphism of $\mathrm{Spf}(A_{\mathbb{C}})$ with the formal completion of $\mathsf{A}_{g,1,n} \otimes \mathbb{C}$ at $x^{\mathrm{can}}_{\mathbb{C}}$ restricts to an isomorphism

$$\mathfrak{Z}_{\mathbb{C}} := \mathrm{Spf}(A_{\mathbb{C}}/\mathfrak{a}_{\mathbb{C}}) \xrightarrow{\sim} (Z_{\mathbb{C}})_{/\{x^{\mathrm{can}}_{\mathbb{C}}\}} \subseteq (\mathsf{A}_{g,1,n} \otimes \mathbb{C})_{/\{x^{\mathrm{can}}_{\mathbb{C}}\}}. \tag{1}$$

First we remark that the composite morphism

$$\mathrm{Spec}(A_{\mathbb{C}}/\mathfrak{a}_{\mathbb{C}}) \hookrightarrow \mathrm{Spec}(A_{\mathbb{C}}) \to \mathsf{A}_{g,1,n} \otimes \mathbb{C}$$

factors through $Z_{\mathbb{C}}$. The closed formal subscheme $\mathrm{Spf}(A_{\mathbb{C}}/\mathfrak{a}_{\mathbb{C}}) \hookrightarrow (\mathsf{A}_{g,1,n} \otimes \mathbb{C})_{/\{x_{\mathbb{C}}^{\mathrm{can}}\}}$ is therefore contained in $(Z_{\mathbb{C}})_{/\{x_{\mathbb{C}}^{\mathrm{can}}\}}$. The point $x_{\mathbb{C}}^{\mathrm{can}} \in Z_{\mathbb{C}}$ being non-singular, the claim then follows from the fact that the dimensions of $A_{\mathbb{C}}/\mathfrak{a}_{\mathbb{C}}$ and $Z_{\mathbb{C}}$ are equal.

Combining (1) with the description of the de Rham cohomology of $\mathfrak{X}_{\mathbb{C}}/\mathfrak{A}_{\mathbb{C}}$ as in Step 3 and the remarks in Step 2, we conclude that $Z_{\mathbb{C}}$ is formally linear at the point $x_{\mathbb{C}}^{\mathrm{can}}$, in the sense explained in 4.3. (Recall that $\mathfrak{a}_{\mathbb{C}}$ is generated *linear* forms $\sum_{ij} m_{ij}\tau_{ij}$.) By Corollary 5.5 of Part I (see above), this implies that $Z_{\mathbb{C}}$ is of Hodge type, which, after Step 1, was what we had to prove. □

5. Oort's conjecture

The following conjecture was formulated by Oort (cf. [23]). Notice that one could formulate the conjecture for general Shimura varieties (in which case we would use the terminology 'special points' rather than 'CM-points'). Here, however, we restrict our attention to moduli spaces of abelian varieties.

CONJECTURE 5.1 (Oort). *Let $Z \hookrightarrow \mathsf{A}_{g,1,n} \otimes \mathbb{C}$ be an irreducible algebraic subvariety such that the CM-points on Z are dense for the Zariski topology. Then Z is a subvariety of Hodge type, as defined in Section* 1.

Remark 5.2. In [1, Chap. X, Sect. 4], a number of problems are suggested, the first of which is equivalent to the above conjecture for $\dim(Z) = 1$. Notice that in Part I, we gave a counterexample to loc. cit., Problems 2 and 3.

Let Z be a subvariety as in the conjecture. Then Z is defined over a number field, since it is the Zariski closure of a set of points which are rational over $\overline{\mathbb{Q}}$ (even over the union $\mathbb{Q}^{\mathrm{CM}} \subset \overline{\mathbb{Q}}$ of all CM-subfields). It follows from the results of the previous sections (in particular Corollary 3.7 and Theorem 4.5) that the conjecture is equivalent to the following statement.

CONJECTURE 5.3 (Variant of 5.1). *Let F be a number field, and let $Z \hookrightarrow \mathsf{A}_{g,1,n} \otimes F$ be an irreducible algebraic subvariety such that the CM-points on Z are dense for the Zariski topology. Then there is a collection T of CM-points $t \in Z(K_t)$ (K_t a number field containing F) and a prime number p such that the collection T is Zariski dense in Z, and such that every X_t is canonical at a prime $\mathfrak{q}_t$ of K_t lying over p.*

Given an abelian variety X_t of CM-type over a number field K_t we know that X_t is canonical at infinitely many primes of K_t. However, it is not true that for any infinite collection of CM-points $t \in \mathsf{A}_g(K_t)$ there is always a prime number p such that infinitely many of the abelian varieties X_t are canonical at a prime $\mathfrak{q}_t$ above p. For example, let $\mathcal{E}_1, \mathcal{E}_2, \ldots,$ be elliptic curves of CM-type such that the conductor of $\mathrm{End}(\mathcal{E}_i)$ (see 3.6) is divisible by the first i rational prime numbers. By [24, Lem. 2.2] we conclude that for every given prime number p, there are only finitely many $\mathcal{E}_i$ which are canonical at a prime above p. Of course this does not provide a counterexample to the conjecture: the Zariski closure of our collection of CM-points $\mathcal{E}_i$ is the whole moduli space $\mathsf{A}_{1,1,n}$ (assuming that the $\mathcal{E}_i$ were equipped with a level n structure), which is certainly of Hodge type.

5.4. In the proof of the next result we will use the Galois representation on the ℓ-torsion and on the Tate-ℓ-module of an abelian variety X defined over a number field. General theory about this can be found in [31], [26] and in the letters of Serre to Ribet and Tate [28], [29]; part of the material of these letters is given in Chi's paper [4]. Here we only record some facts needed further on.

Let X be an abelian variety over a number field F, and write $\rho_\ell \colon \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}(T_\ell X)$ and $\overline{\rho}_\ell \colon \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{Aut}(X(\overline{\mathbb{Q}})[\ell])$ for the Galois representation on its Tate-ℓ-module and its ℓ-torsion respectively. We write G_ℓ for the algebraic envelope of the image of ρ_ℓ. Its connected component of the identity is a reductive algebraic group over $\mathbb{Q}_\ell$ containing the group $\mathbb{G}_{\mathrm{m}} \cdot \mathrm{Id}$ of homotheties. Its Lie algebra does not change if we replace F by a finite extension, but G_ℓ itself may be non-connected and may become smaller after such an extension.

Choose an embedding $\sigma \colon F \to \mathbb{C}$, and write $V = \mathrm{H}_1(X_\sigma(\mathbb{C}), \mathbb{Q})$ and $V_\ell = T_\ell X \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. There is a natural comparison isomorphism $V_\ell \cong V \otimes \mathbb{Q}_\ell$, and, by the results of Borovoi, Deligne and Piatetski-Shapiro (see [9]), G_ℓ^0 is an algebraic subgroup of $\mathrm{MT}(X_\sigma) \otimes \mathbb{Q}_\ell$. The Mumford–Tate conjecture (stated in [20] and in a more refined version in [30]) asserts that the two groups are equal.

It is known that the representation of $\mathrm{MT}(X_\sigma)$ on V is defined by miniscule weights. By this we mean the following: first we write $\mathrm{MT}(X_\sigma)_{\overline{\mathbb{Q}}}$ as the almost direct product of its center Z and a number of simple factors $M_1, \ldots, M_r$. Then every irreducible submodule $W \subseteq V_{\overline{\mathbb{Q}}}$ decomposes as a tensor product $W \cong \chi \otimes W_1(\varpi_1) \otimes \cdots \otimes W_r(\varpi_r)$, where χ is a character of Z, and where $W_i(\varpi_i)$ is an irreducible M_i-module with highest weight ϖ_i (with respect to a chosen Borel subgroup of M_i). The representation V is said to be defined by miniscule weights if all weights ϖ_i are miniscule in the sense of [3, Chap. VIII, Sect. 7, No. 3].

An immediate consequence of the Mumford–Tate conjecture would be that the representation of G_ℓ on V_ℓ is also defined by miniscule weights. So far this has not been proved in general[1]. However, Zarhin proved it under the additional assumption that X has ordinary reduction at a set of places of density 1; see [33, Th. 4.2].

[1] This has recently been proven by Pink, see R. Pink, *ℓ-adic monodromy groups, cocharacters, and the Mumford-Tate conjecture*, J. reine angew. Math. 495 (1998), 187–237.

LEMMA 5.5. *Let K be an algebraically closed field of characteristic zero, let G be a reductive algebraic group over K and let V be a finite-dimensional representation of G which is defined by miniscule weights. Write $V \cong W_1^{d_1} \oplus \cdots \oplus W_m^{d_m}$, where $W_1, \ldots, W_m$ are mutually non-isomorphic irreducible representations of G. If ϖ is a weight of W_i then it has multiplicity d_i in the representation V. The total number of different weights that occur in the representation V is therefore equal to* $\dim(W_1) + \cdots + \dim(W_m)$.

Proof. Suppose $\mathfrak{g}$ is a simple Lie algebra over K, ϖ is a miniscule weight of $\mathfrak{g}$ (with respect to a chosen Cartan subalgebra) and W is an irreducible $\mathfrak{g}$-module with highest weight ϖ. The lemma follows directly from the following two facts, proven in [3, Chap. VIII, Sect. 7, No. 3]: (i) all weights of W have multiplicity 1, (ii) the Weyl group acts transitively on the set of weights of W. □

5.6. Let X be defined over the number field F, and let v be a finite place of F such that X has good reduction at v. If $\ell \nmid v$ then ρ_ℓ is unramified at v. By the choice of a place $\bar{v}$ of $\overline{\mathbb{Q}}$ extending v we get a well-determined action of a Frobenius element $\rho_\ell(\mathrm{Fr}_{\bar{v}}) \in \mathrm{Aut}(V_\ell X)$. Alternatively, X having good reduction at v means that it extends to an abelian scheme $\mathcal{X}_v$ over $\mathrm{Spec}(\mathcal{O}_v)$, whose special fibre X_v is an abelian variety over the finite field $\kappa(v)$. Let π_v be the Frobenius endomorphism of X_v, which acts on the Tate module $T_\ell X_v$. Via the choice of the place $\bar{v}$ we get an isomorphism $T_\ell X \cong T_\ell X_v$, and the action of π_v on $T_\ell X$ obtained in this manner is given by the element $\rho_\ell(\mathrm{Fr}_{\bar{v}})$.

Associated to X_v there is an algebraic torus T_v over $\mathbb{Q}$, called a Frobenius torus. As a module under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, its character group $X^*(T_v)$ is isomorphic to $\Gamma_v/\mathrm{Tors}(\Gamma_v)$, where $\Gamma_v \subset \overline{\mathbb{Q}}^*$ is the subgroup generated by the eigenvalues of $\rho_\ell(\mathrm{Fr}_{\bar{v}})$. This description determines T_v up to isomorphism. The choice of a place $\bar{v}$ as above induces an injective homomorphism $T_v \otimes \mathbb{Q}_\ell \hookrightarrow G_\ell$. For more on Frobenius tori we refer to [4, Sect. 3].

The following facts were proved by Serre (see [4]): (i) the rank of G_ℓ does not depend on ℓ, (ii) we can replace F by a finite extension such that all groups G_ℓ and all Frobenius tori T_v (for places v of good reduction) are connected, and (iii) after replacing F by such an extension, there is a Zariski open and dense subset $U \subseteq G_\ell$ such that if $\ell \nmid v$ and $\rho_\ell(\mathrm{Fr}_{\bar{v}}) \in U(\mathbb{Q}_\ell)$, then $T_v \otimes \mathbb{Q}_\ell$ is a maximal torus of G_ℓ (the set of places v for which this holds thus has density 1).

THEOREM 5.7. *Let (X, λ, θ) be a principally polarized abelian variety with a Jacobi level n structure, defined over a number field F. Suppose that for some finite field extension $F \subseteq F'$, the set*

$$\mathcal{P}^\circ(F') = \{\text{finite places } v \text{ of } F' \mid X \otimes F' \text{ has good and ordinary reduction at } v\}$$

has Dirichlet density 1. For each $v \in \mathcal{P}^\circ(F)$ with residue characteristic p_v not dividing n, let $(X_v, \lambda_v, \theta_v)$ be the reduction at v, and let $x_v^{\mathrm{can}} \in \mathsf{A}_{g,1,n} \otimes \mathbb{Q}$ be the moduli point of its canonical lifting. Define $Z \subseteq \mathsf{A}_{g,1,n} \otimes \mathbb{Q}$ as the Zariski closure *of the set* $\{x_v^{\mathrm{can}} \mid v \in \mathcal{P}^\circ(F), p_v \nmid n\}$*. Then* Z *is a union of subvarieties of Hodge type; more precisely:*

$$Z_{\overline{\mathbb{Q}}} = \bigcup_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} S^{(\sigma)} \cup \{s_1, \ldots, s_r\},$$

where $S \hookrightarrow \mathsf{A}_{g,1,n} \otimes \overline{\mathbb{Q}}$ *is the smallest subvariety of Hodge type containing the moduli point of* $(X, \lambda, \theta) \otimes_F \overline{\mathbb{Q}}$ *for some embedding* $F \hookrightarrow \overline{\mathbb{Q}}$*, and where* $s_1, \ldots, s_r$ *(*$r \in \mathbb{Z}_{\geqslant 0}$*) are CM-points.*

Proof. Suppose F' is a finite extension of F such that $\mathcal{P}^\circ(F')$ has Dirichlet density 1. Let F'' be a Galois extension of F' of degree d, and write $\mathcal{Q}$ for the set of primes of F' which split completely in F''/F'. Using the Čebotarev density theorem we see that the set $\mathcal{P}^\circ(F') \cap \mathcal{Q}$ has Dirichlet density $1/d$, which means that the function

$$\sum_{\mathfrak{p} \in \mathcal{P}^\circ(F') \cap \mathcal{Q}} N(\mathfrak{p})^{-s} + 1/d \cdot \log(s-1)$$

extends to a holomorphic function on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geqslant 1\}$. If $\mathcal{R}$ is the set of primes of F'' lying over $\mathcal{P}^\circ(F') \cap \mathcal{Q}$, then it follows that the function

$$\sum_{\mathfrak{p}' \in \mathcal{R}} N(\mathfrak{p}')^{-s} + \log(s-1)$$

also extends to a holomorphic function of s for $\mathrm{Re}(s) \geqslant 1$. Clearly, $\mathcal{R} \subseteq \mathcal{P}^\circ(F'')$, and almost all elements of $\mathcal{P}^\circ(F'') \setminus \mathcal{R}$ have degree at least 2 over $\mathbb{Q}$. It readily follows from this that $\mathcal{P}^\circ(F'')$ has Dirichlet density 1.

The preceding remarks show that, in proving the theorem, we may replace F by a finite extension. We claim that, after such an extension, there exists an infinite subset $\mathcal{P}' \subseteq \mathcal{P}^\circ(F)$ and a prime $\mathfrak{p}$ of $\mathcal{O}_F$ such that each of the abelian varieties X_v^{can} with $v \in \mathcal{P}'$ is canonical at some prime $\mathfrak{q}$ above $\mathfrak{p}$. Before we prove this, let us show how the result would follow from it.

So, suppose we have such a set $\mathcal{P}'$, and write $Z' \subseteq Z$ for the Zariski closure of the corresponding set of CM-points $\{x_v^{\mathrm{can}} \mid v \in \mathcal{P}', p_v \nmid n\}$. It follows from Theorem 4.2 that almost all points x_v^{can} lie on

$$\cup_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} S^{(\sigma)} \hookrightarrow \mathsf{A}_{g,1,n},$$

so $Z'_{\overline{\mathbb{Q}}} \subseteq Z_{\overline{\mathbb{Q}}}$ is contained in the union of $\cup_{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} S^{(\sigma)}$ and a finite number of CM-points. On the other hand, from 3.7 and 4.5 we see that all irreducible components of Z' are of Hodge type. Therefore, we are done if we show that the moduli point x_F of (X, λ, θ) lies on Z'. This we can see as follows.

First we may replace $\mathcal{P}'$ by an infinite subset such that its Zariski closure Z' is irreducible over $\mathbb{Q}$. Over some open part $U = \mathrm{Spec}(\mathbb{Z}[1/N])$ of $\mathrm{Spec}(\mathbb{Z})$, the point x_F extends to a section $x\colon \mathrm{Spec}(\mathcal{O}_F[1/N]) \to \mathsf{A}_{g,1,n}$. We define $\mathcal{Z}'$ as the Zariski closure of Z' inside $\mathsf{A}_{g,1,n} \otimes \mathbb{Z}[1/N]$. Then we have an infinite collection $\mathcal{P}'' = \{v \in \mathrm{Spec}(\mathcal{O}_F[1/N]) \mid v \in \mathcal{P}', p_v \nmid N\}$ such that (by construction) every

$$x_v\colon \mathrm{Spec}(\kappa(v)) \to \mathsf{A}_{g,1,n} \otimes \mathbb{Z}[1/N]$$

with $v \in \mathcal{P}''$ factors through $\mathcal{Z}'$. Because the collection $\mathcal{P}''$ is dense in $\mathrm{Spec}(\mathcal{O}_F[1/N])$ it then follows that x factors through $\mathcal{Z}'$, which means that x_F is a point of Z'. We conclude that the theorem follows if we can construct a set $\mathcal{P}'$ as above.

From now on we use the notations and results discussed in 5.4 up to 5.6 above. We replace F by a finite extension such that $\mathcal{P}^\circ(F)$ has Dirichlet density 1 and such that the groups G_ℓ and the Frobenius tori T_v are connected (for all ℓ and all places v where X has good reduction). This implies that all endomorphisms of $X \otimes \overline{\mathbb{Q}}$ and $X_v \otimes \overline{\kappa(v)}$ are defined over F and $\kappa(v)$, respectively. We write $\mathfrak{f} = \mathfrak{f}(X)$ for the conductor of the endomorphism ring $\mathrm{End}(X)$, i.e., the index of $\mathrm{End}(X)$ in a maximal order of $\mathrm{End}^0(X)$, and if X has good reduction at a place v of F then we simply write $\mathfrak{f}_v$ for the conductor of $\mathrm{End}(X_v)$.

Suppose ℓ is a prime number and v is an element of $\mathcal{P}^\circ(F)$ such that $\ell \nmid \mathfrak{f}_v$ and such that ℓ splits completely in the field $\mathbb{Q}(\pi_v)^{\mathrm{norm}} \subset \overline{\mathbb{Q}}$ generated by the eigenvalues of π_v. We claim that under these assumptions X_v^{can} (which is defined over some number field $K \supseteq F$) is canonical at all primes λ of K above ℓ (where we take K large enough such that X_v^{can} has good reduction at all primes of K). In fact, the assumption that ℓ splits completely in $\mathbb{Q}(\pi_v)^{\mathrm{norm}}$ implies that the reduction Y_λ of X_v^{can} modulo λ is ordinary (using [32, Lem. 5]) and since ℓ does not divide the conductor of $\mathrm{End}(X_v^{\mathrm{can}})$, the endomorphism rings of X_v^{can} and Y_λ are the same (see Lemma 2.8), so X_v^{can} is the canonical lifting of Y_λ. Therefore, we are done if we show that there are primes ℓ such that the set

$$\mathcal{P}^\circ(\ell) = \{v \in \mathcal{P}^\circ(F) \mid \ell \nmid \mathfrak{f}_v \text{ and } \ell \text{ splits completely in the field } \mathbb{Q}(\pi_v)^{\mathrm{norm}}\}$$

is infinite.

Suppose X' is an abelian variety which is F-isogenous to X, say by an isogeny $f\colon X \to X'$ of degree d. For a place v where X and X' have good reductions X_v and X'_v, the associated fields $\mathbb{Q}(\pi(X_v))^{\mathrm{norm}}$ and $\mathbb{Q}(\pi(X'_v))^{\mathrm{norm}}$ are naturally isomorphic and there is an isogeny $f_v\colon X_v \to X'_v$ of degree d, cf. [12, Chap. I, Prop. 2.7]. It follows that for all ℓ, the sets $\mathcal{P}^\circ(\ell)$ associated to X and X' differ only by finitely many elements. We may therefore assume that $X = Y_1^{m_1} \times \cdots \times Y_r^{m_r}$, where $Y_1, \ldots, Y_r$ are mutually non-isogenous simple abelian varieties over F and $m_1, \ldots, m_r$ are positive integers.

Choose a place v of F and a place $\bar{v}$ of $\overline{\mathbb{Q}}$ extending v, such that X has good reduction at v and such that $T_v \otimes \mathbb{Q}_\ell \subseteq G_\ell$ is a maximal torus for every $\ell \neq p_v$. Choose a prime p with $p \neq p_v$ which splits completely in $\mathrm{End}^0(X)$, i.e., $\mathrm{End}^0(X) \otimes \mathbb{Q}_p$ is a product of matrix algebras over $\mathbb{Q}_p$. Let Y be one of

https://doi.org/10.1023/A:1000411631772 Published online by Cambridge University Press

the simple factors Y_i, let E be the center of $\mathrm{End}^0(Y)$, and let $\mathfrak{p}_1, \dots, \mathfrak{p}_e$ be the primes of $\mathcal{O}_E$ above p. By the choices we have made, $\mathrm{End}^0(Y) \otimes \mathbb{Q}_p \cong \mathrm{M}_d(\mathbb{Q}_p)^e$ with $e = [E : \mathbb{Q}]$. The representation ρ_p of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ on $V_pY = T_pY \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ decomposes as

$$V_pY \cong \mathbb{V}_1^d \oplus \cdots \oplus \mathbb{V}_e^d,$$

where $\mathbb{V}_1, \dots, \mathbb{V}_e$ are mutually non-isomorphic and absolutely irreducible G_p-modules such that E acts on $\mathbb{V}_i$ through its completion $E_{\mathfrak{p}_i} \cong \mathbb{Q}_p$. Write $\tilde{\rho}_p$ for the representation of G_p on $\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_e$.

Let $P_v(t) = \det(t \cdot \mathrm{Id} - \tilde{\rho}_p(\mathrm{Fr}_{\bar{v}}) \mid \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_e)$ be the characteristic polynomial of $\tilde{\rho}_p(\mathrm{Fr}_{\bar{v}})$, which has coefficients in $\mathbb{Z}$. We define $\delta_Y \in \mathbb{Z}$ as the discriminant of the polynomial $P_v(t)$, and we put $\delta_X = \delta_{Y_1} \cdots \delta_{Y_r}$. We observe that $\delta_X \neq 0$. In fact, the eigenvalues of $\tilde{\rho}_p(\mathrm{Fr}_{\bar{v}})$ are the elements $\varpi(\pi_v) \in \overline{\mathbb{Q}}_p$, where ϖ runs through the set of weights (counted with multiplicities) of $\tilde{\rho}_p$ with respect to the maximal torus $T_v \otimes \mathbb{Q}_p$. Since T_v is generated by the element π_v, we have $\varpi(\pi_v) = \varpi'(\pi_v)$ if and only if $\varpi = \varpi'$. By Zarhin's result [33, Th. 4.2], the representation ρ_p is defined by miniscule weights, and using Lemma 5.5 we conclude that all eigenvalues of $\tilde{\rho}_p(\mathrm{Fr}_{\bar{v}})$ have multiplicity 1, hence $\delta_X \neq 0$.

Next we consider prime numbers ℓ satisfying the following conditions: (i) ℓ splits completely in $\mathrm{End}^0(X_v)$, (ii) $\ell \nmid \delta_X \cdot \mathfrak{f}(X)$. We claim that for every such ℓ the set $\mathcal{P}^\circ(\ell)$ is infinite. To see this, consider the representation $\overline{\rho}_\ell$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ on $X(\overline{\mathbb{Q}})[\ell]$, and write $\gamma = \overline{\rho}_\ell(\mathrm{Fr}_{\bar{v}}) \in \mathrm{Aut}(X(\overline{\mathbb{Q}})[\ell]) \cong \mathrm{GL}_{2g}(\mathbb{F}_\ell)$, where $g = \dim(X)$.

As above, let Y be one of the simple factors of X, and write γ_Y for the restriction of γ to $Y(\overline{\mathbb{Q}})[\ell]$. The assumption that $\ell \nmid \mathfrak{f}(X)$ implies that $\mathrm{End}(Y) \otimes \mathbb{Z}_\ell$ is a maximal order of $\mathrm{End}(Y) \otimes \mathbb{Q}_\ell$, which by (i) is isomorphic to $\mathrm{M}_d(\mathbb{Q}_\ell)^e$. Thus $\mathrm{End}(Y) \otimes \mathbb{Z}_\ell \cong \mathrm{M}_d(\mathbb{Z}_\ell)^e$, and we conclude from this that there exists a decomposition of the Tate module (as a $\mathbb{Z}_\ell$-module with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$)

$$T_\ell Y = \mathbb{T}_1^d \oplus \cdots \oplus \mathbb{T}_e^d,$$

where $\mathbb{T}_i$ is a free $\mathbb{Z}_\ell$-module of rank $2 \cdot \dim(Y)/ed$ and $\mathbb{T}_i \otimes \mathbb{Q}_\ell$ is an absolutely irreducible representation of G_ℓ. The above argument shows that all eigenvalues of $\mathrm{Fr}_{\bar{v}}$ on $(\mathbb{T}_1 \oplus \cdots \oplus \mathbb{T}_e) \otimes \overline{\mathbb{Q}}_\ell$ have multiplicity 1. Moreover, assumption (i) on ℓ implies that all these eigenvalues lie in $\mathbb{Z}_\ell$. Now

$$Y(\overline{\mathbb{Q}})[\ell] \cong \big((\mathbb{T}_1/\ell\mathbb{T}_1) \oplus \cdots \oplus (\mathbb{T}_e/\ell\mathbb{T}_e)\big)^d$$

and we assumed that $\ell \nmid \delta_X$, so the $2\dim(Y)/d$ different eigenvalues of $\mathrm{Fr}_{\bar{v}}$ are also different modulo ℓ. It follows that $\gamma_Y \in \mathrm{Aut}(Y(\overline{\mathbb{Q}})[\ell]) \cong \mathrm{GL}_{2\dim(Y)}(\mathbb{F}_\ell)$ is diagonalizable over $\mathbb{F}_\ell$ with eigenvalues all of multiplicity d.

Now we start working backwards. Suppose w is a place in the set $\mathcal{P}^\circ(F)$, and $\bar{w}$ is an extension of w to $\overline{\mathbb{Q}}$, such that $\ell \neq p_w$, and $\overline{\rho}_\ell(\mathrm{Fr}_{\bar{w}}) = \gamma$. Since $\mathrm{Aut}\big(X(\overline{\mathbb{Q}})[\ell]\big)$

https://doi.org/10.1023/A:1000411631772 Published online by Cambridge University Press

is finite and $\mathcal{P}^\circ(F)$ has density 1, there are infinitely many such places w, by the Čebotarev density theorem. Therefore, the proof is finished if we show that all such w lie in the set $\mathcal{P}^\circ(\ell)$, i.e., $\ell \nmid \mathfrak{f}_w$ and ℓ splits completely in the field $\mathbb{Q}(\pi_w)^{\mathrm{norm}}$.

First we reduce to the case that $X = Y$ is absolutely simple. Since $\mathbb{Q}(\pi_w)^{\mathrm{norm}}$ is the compositum of the fields $\mathbb{Q}(\pi_{Y_i,w})^{\mathrm{norm}}$ associated to the simple factors Y_i, the prime ℓ splits completely in $\mathbb{Q}(\pi_w)^{\mathrm{norm}}$ if and only if it splits completely in each of the fields $\mathbb{Q}(\pi_{Y_i,w})^{\mathrm{norm}}$. Also, we claim that $\ell \mid \mathfrak{f}_w$ if and only if ℓ divides one of the factors $\mathfrak{f}_w(Y_i)$. In the 'if' direction this is clear, so let us assume that ℓ does not divide any of the factors $\mathfrak{f}_w(Y_i)$. Suppose $A \subseteq Y_{i,w}$ and $A' \subseteq Y_{j,w}$ are simple factors which are isogenous. Then $\mathrm{End}^0(A) = \mathrm{End}^0(A')$ is a CM-field (since A and A' are ordinary) and, by assumption, both orders $\mathrm{End}(A)$ and $\mathrm{End}(A')$ are maximal at ℓ. Using Tate's theorem that $\mathrm{Hom}(A, A') \otimes \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Gal}}(T_\ell A, T_\ell A')$ we conclude that A and A' are prime-to-ℓ isogenous. From this remark and our assumption that X is the product of the factors $Y_i^{m_i}$ it then follows that ℓ does not divide $\mathfrak{f}_w = \mathfrak{f}_w(X)$. From now on we may therefore assume that $X = Y$ is absolutely simple.

The characteristic polynomial $P_w(t)$ of the action of $\mathrm{Fr}_{\bar{w}}$ on $\mathbb{T}_1 \oplus \cdots \oplus \mathbb{T}_e$ has coefficients in $\mathbb{Z}$. Modulo ℓ it is a product of linear factors, and all zeroes have multiplicity 1. By Hensel's lemma, $P_w(t) = (t - \alpha_1) \cdots (t - \alpha_u)$ in $\mathbb{Z}_\ell[t]$, with all $\alpha_i \in \mathbb{Z}_\ell$ different and $u = 2 \cdot \dim(Y)/d$. Let $Y_w^{(1)}, \ldots, Y_w^{(s)}$ be the simple factors of the reduction Y_w, and let $\pi_w^{(i)}$ be the Frobenius automorphism of $Y_w^{(i)}$. Then ℓ splits completely in each of the fields $\mathbb{Q}(\pi_w^{(i)})^{\mathrm{norm}} \subset \overline{\mathbb{Q}}$ generated by the eigenvalues of $\pi_w^{(i)}$, hence it splits completely in $\mathbb{Q}(\pi_w)^{\mathrm{norm}}$. Finally, the eigenvalues α_i of $\mathrm{Fr}_{\bar{w}}$ on $\mathbb{T}_1 \oplus \cdots \oplus \mathbb{T}_e$ are all different and $T_\ell Y \cong (\mathbb{T}_1 \oplus \cdots \oplus \mathbb{T}_e)^d$, so we get

$$\begin{aligned} \mathrm{End}(Y_w) \otimes \mathbb{Z}_\ell &\cong \mathrm{End}_{\mathrm{Fr}_{\bar{w}}}(T_\ell Y) \cong \mathrm{M}_d(\mathbb{Z}_\ell^u) \hookrightarrow \mathrm{M}_d(\mathbb{Q}_\ell^u) \\ &\cong \mathrm{End}_{\mathrm{Fr}_{\bar{w}}}(V_\ell Y) \cong \mathrm{End}(Y_w) \otimes \mathbb{Q}_\ell, \end{aligned}$$

and we see that $\ell \nmid \mathfrak{f}_w$. This finishes the proof. □

Remark 5.8. (i) It was conjectured by Serre in [29] that $\mathcal{P}^\circ(F)$ has density 1 for all abelian varieties over a number field F, where F should be taken large enough such that the groups G_ℓ are connected.

(ii) It is easy to see that in the statement of the theorem, we must allow a finite number of 'exceptional' CM-points $s_1, \ldots, s_r$.

6. The canonical lifting of a moduli point

6.1. So far we only considered canonical liftings of ordinary abelian varieties over a finite field. However, we can associate a moduli point $x^{\mathrm{can}} \in \mathsf{A}_{g,1,n} \otimes \mathbb{Q}$ to any point $x \in (\mathsf{A}_{g,1,n} \otimes \mathbb{F}_p)^\circ$. Namely, write $\kappa(x)$ for the residue field of x, and let $\overline{\kappa(x)}$ be an algebraic closure. Since x is an ordinary moduli point, we have a canonical lifting of $s_x\colon \mathrm{Spec}(\overline{\kappa(x)}) \to \mathsf{A}_{g,1,n}$ to a section $s\colon \mathrm{Spec}(W(\overline{\kappa(x)})) \to \mathsf{A}_{g,1,n}$, and we define x^{can} as the image under s of the generic point of $\mathrm{Spec}(W(\overline{\kappa(x)}))$. The point x^{can} is easily seen to be independent of any choices.

In order to understand the behaviour of the canonical lifting under specialization, we have to generalize our notion of canonical lifting to abelian schemes over a perfect ring. This is done as follows.

Let R be a domain of characteristic $p > 0$ with fraction field K. Write K^{perf} for the perfect closure of K, and let R^{perf} be the integral closure of R in K^{perf}, which is a perfect closure of R. The ring $W(R^{\mathrm{perf}})$ of Witt vectors is a domain, complete and separated for the p-adic topology, and $W(R^{\mathrm{perf}})/p \cong R^{\mathrm{perf}}$. Suppose $X_0 \to \mathrm{Spec}(R)$ is an ordinary abelian scheme. By extending scalars we get an ordinary abelian scheme X over $\mathrm{Spec}(R^{\mathrm{perf}})$, and because R^{perf} is a perfect ring of characteristic p, the p-divisible group $X[p^\infty]$ is the direct sum $X[p^\infty] = X[p^\infty]_\mu \oplus X[p^\infty]_{\text{ét}}$ of a toroidal and an étale part. These summands each have a unique lifting to a p-divisible group, say G_μ and $G_{\text{ét}}$ respectively, over $\mathrm{Spec}(W(R^{\mathrm{perf}}))$, using [13, IV.18.3.4] and Cartier duality. Applying a theorem of Serre and Tate (see [15]), we get a lifting X^{can} of X over $\mathrm{Spec}(W(R^{\mathrm{perf}}))$ whose p-divisible group is $G_\mu \oplus G_{\text{ét}}$.

This construction is functorial in the obvious sense. For example, if $\mathfrak{m} \subset R$ is a maximal ideal then the quotient homomorphism $R \twoheadrightarrow \kappa = R/\mathfrak{m}$ naturally extends to a homomorphism $R^{\mathrm{perf}} \twoheadrightarrow \kappa^{\mathrm{perf}}$ and we get a canonical map $W(R^{\mathrm{perf}}) \to W(\kappa^{\mathrm{perf}})$. It is clear from the construction that $X^{\mathrm{can}} \otimes_{W(R^{\mathrm{perf}})} W(\kappa^{\mathrm{perf}})$ is the canonical lifting of $X_0 \otimes_R \kappa$. Likewise, $X^{\mathrm{can}} \otimes_{W(R^{\mathrm{perf}})} W(K^{\mathrm{perf}})$ is the canonical lifting of $X_0 \otimes_R K$.

LEMMA 6.2. *Let x, y be points of $(\mathsf{A}_{g,1,n} \otimes \mathbb{F}_p)^\circ$ such that x specializes to y. Then x^{can} specializes to y^{can}.*

Proof. (See also [25, Proof of Lemma 1.3].) Let $\mathcal{O}_y$ be the local ring of $\mathsf{A}_{g,1,n} \otimes \mathbb{F}_p$ at y, and let $\mathfrak{p}_x \subset \mathcal{O}_y$ be the prime ideal corresponding to the point x. Let $R = \mathcal{O}_y/\mathfrak{p}_x$, then we have an ordinary abelian scheme X over R^{perf}, and, as just explained, we can form a canonical lifting X^{can} of X over $\mathrm{Spec}(W(R^{\mathrm{perf}}))$. The lemma readily follows from the functoriality of this construction (as explained above). $\square$

LEMMA 6.3. *Let W be a p-adically complete and separated domain such that $p \in W$ is prime. Let I be an index set, and let $\{\mathfrak{p}_\alpha \subset W \mid \alpha \in I\}$ be a collection of prime ideals such that $p \notin \mathfrak{p}_\alpha$. Assume that the intersection of the ideals $\mathfrak{q}_\alpha = (\sqrt{p + \mathfrak{p}_\alpha} \bmod p)$ in W/p is the zero ideal. Then the set $\{\mathfrak{p}_\alpha \mid \alpha \in I\}$ is Zariski dense in $\mathrm{Spec}(W)$.*

Proof. If $f \in \cap_{\alpha\in I}\mathfrak{p}_\alpha$ then $(f \bmod p) \in \cap_{\alpha\in I}\mathfrak{q}_\alpha = (0)$, hence $f = p \cdot f'$ for some $f' \in W$. Since $p \notin \mathfrak{p}_\alpha$ we have $f' \in \cap_{\alpha\in I}\mathfrak{p}_\alpha$, and by induction we then see that $f \in p^n \cdot W$ for every n. As W is p-adically separated, this implies $f = 0$. $\square$

PROPOSITION 6.4. *Let $x \in (\mathsf{A}_{g,1,n} \otimes \mathbb{F}_p)^\circ$, and define $Z \hookrightarrow \mathsf{A}_{g,1,n} \otimes \mathbb{Q}$ as the Zariski closure of its canonical lifting x^{can}. Then Z is a subvariety of Hodge type.* [1]

Proof. Let $Y \hookrightarrow \mathsf{A}_{g,1,n} \otimes \mathbb{F}_p$ be the Zariski closure of x, and consider the set $\mathcal{Y}$ of closed ordinary points of Y. If $Z' \hookrightarrow \mathsf{A}_{g,1,n} \otimes \mathbb{Q}$ is the Zariski closure of the set $\{y^{\mathrm{can}} \mid y \in \mathcal{Y}\}$ then by Lemma 6.2 we have $Z' \subseteq Z$. First we show that Z and Z' are in fact equal.

Let $U = \mathrm{Spec}(B) \subset (\mathsf{A}_{g,1,n} \otimes \mathbb{F}_p)^\circ$ be an affine open subscheme with $x \in U$. Write $C = U \cap Y = \mathrm{Spec}(B/J)$, then C is irreducible and $x \in C$. The ring $R = B/J$ is a domain of finite type over $\mathbb{F}_p$. As above, let R^{perf} be a perfect closure of R, let $W(R^{\mathrm{perf}})$ be its ring of Witt vectors, and let $s^{\mathrm{can}}\colon \mathrm{Spec}(W(R^{\mathrm{perf}})) \to \mathsf{A}_{g,1,n} \otimes \mathbb{Z}_p$ be the canonical lifting of $s\colon \mathrm{Spec}(R^{\mathrm{perf}}) \to (\mathsf{A}_{g,1,n} \otimes \mathbb{F}_p)^\circ$. If $\mathfrak{m} \subset R^{\mathrm{perf}}$ is a maximal ideal with quotient field $k = R^{\mathrm{perf}}/\mathfrak{m}$, then the morphism $g\colon \mathrm{Spec}(k) \to \mathrm{Spec}(R^{\mathrm{perf}})$ lifts to $W(g)\colon \mathrm{Spec}(W(k)) \to \mathrm{Spec}(W(R^{\mathrm{perf}}))$, and $s^{\mathrm{can}} \circ W(g)$ is the canonical lifting of $s \circ g$.

Let $\{\mathfrak{m}_\alpha \mid \alpha \in I\}$ be the set of maximal ideals of R^{perf}. For each $\alpha \in I$ the kernel of $W(R^{\mathrm{perf}}) \to W(R^{\mathrm{perf}}/\mathfrak{m}_\alpha)$ is a prime ideal $\mathfrak{p}_\alpha \subset W(R^{\mathrm{perf}})$. Clearly, the collection $\{\mathfrak{p}_\alpha \mid \alpha \in I\}$ satisfies the assumptions of the previous lemma, and therefore it is Zariski dense in $\mathrm{Spec}(W(R^{\mathrm{perf}}))$. By construction, every $\mathfrak{p}_\alpha$ maps into Z' under s^{can}. It follows that x^{can} also maps into Z', hence $Z = Z'$.

We thus have an irreducible algebraic subvariety $Z \hookrightarrow \mathsf{A}_{g,1,n} \otimes \mathbb{Q}$ with a dense collection of CM-points (namely the points y^{can}) which are all canonical at some prime in characteristic p. Applying Corollary 3.8 we conclude that the model $\mathcal{Z}$ of Z over $\mathbb{Z}_p$ is formally linear at some of its ordinary points, and by Theorem 4.5 we conclude that Z is of Hodge type. $\square$

Our final results are joint work with A. J. de Jong and F. Oort. The results were announced in [23], where also a sketch of the arguments was given. We keep the above notations, i.e., we fix an integer $n \geqslant 3$ and we consider an ordinary (but not necessarily closed) moduli point $x \in \mathsf{A}_{g,1,n} \otimes \mathbb{F}_p$. The problem that we are interested in is to compare

$$\mathrm{tr.deg.}_{\mathbb{F}_p}\kappa(x) = \dim(\{x\}^{\mathrm{Zar}}) \quad \text{and} \quad \mathrm{tr.deg.}_{\mathbb{Q}}\kappa(x^{\mathrm{can}}) = \dim(\{x^{\mathrm{can}}\}^{\mathrm{Zar}}).$$

We have an inequality $\mathrm{tr.deg.}_{\mathbb{F}_p}\kappa(x) \leqslant \mathrm{tr.deg.}_{\mathbb{Q}}\kappa(x^{\mathrm{can}})$. Our result shows that in general the two numbers are not equal. Before we state the precise result, we introduce some notations and we formulate a lemma.

Let R be a ring such that n is invertible in R. Given a morphism $f\colon S \to \mathsf{A}_{g,1,n} \otimes \mathrm{Spec}(R)$ of schemes over R, we simply write X_S for the corresponding abelian scheme over S, if it is clear which morphism f we take. Let $\bar{s}$ be a geometric point of S, and let ℓ be a prime number which is invertible in R. The polarization on

[1] This result was obtained independently by M. Nori (unpublished).

https://doi.org/10.1023/A:1000411631772 Published online by Cambridge University Press

$X_{\bar{s}}$ induces a non-degenerate alternating bilinear form φ_ℓ on $T_\ell X_{\bar{s}}$, and the image of the monodromy representation

$$\rho_S\colon \pi_1(S, \bar{s}) \longrightarrow \mathrm{Aut}(T_\ell X_{\bar{s}})$$

is an ℓ-adic Lie subgroup $\mathcal{G}_\ell = \mathcal{G}_\ell(S)$ of $\mathrm{CSp}(T_\ell X_{\bar{s}}, \varphi_\ell)$. Via the choice of a symplectic basis for $T_\ell X_{\bar{s}}$ we can identify $\mathcal{G}_\ell$ with a subgroup of $\mathrm{CSp}_{2g}(\mathbb{Z}_\ell)$. If S is connected then, up to conjugation, the group $\mathcal{G}_\ell(S)$ is independent both of the chosen basis and the choice of the base point $\bar{s}$.

If x is a point of $\mathsf{A}_{g,1,n}$ then we write $\mathcal{G}_\ell(x)$ for $\mathcal{G}_\ell(\mathrm{Spec}(\kappa(x)))$. Write $S = \{x\}^{\mathrm{Zar}}$ for the Zariski closure of $\{x\}$ inside $\mathsf{A}_{g,1,n}$, then the monodromy representation $\rho_{\mathrm{Spec}(\kappa(x))}$ factors through ρ_S, hence $\mathcal{G}_\ell(x) = \mathcal{G}_\ell(\{x\}^{\mathrm{Zar}})$. From this we see that if x specializes to a point y, then $\mathcal{G}_\ell(y)$ is conjugated to a subgroup of $\mathcal{G}_\ell(x)$.

LEMMA 6.5. *Given a positive integer g and two different prime numbers p and ℓ, not dividing n, there exists an irreducible curve $C \subset \mathsf{A}_{g,1,n} \otimes \overline{\mathbb{F}}_p$ such that C meets the ordinary locus $(\mathsf{A}_{g,1,n} \otimes \overline{\mathbb{F}}_p)^\circ$ and $\mathcal{G}_\ell(C) = \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$.*

Proof. Choose a primitive nth root of unity in $\overline{\mathbb{F}}_p$. We will construct C as a subvariety of the moduli space $\mathsf{A} = \mathsf{A}_{g,1,(n)} \otimes_{\mathbb{Z}[\zeta_n, 1/n]} \overline{\mathbb{F}}_p$ of abelian varieties with a symplectic level n structure, which can be identified with an irreducible component of $\mathsf{A}_{g,1,n} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p$.

First we remark that, for some fixed, sufficiently large integer m, it suffices to construct an irreducible curve C which intersects the ordinary locus and for which $\mathcal{G}_\ell(C)$ maps surjectively to $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell^m)$. (In fact, for $\ell \neq 2$ we can take $m = 2$; for $\ell = 2$ we take $m = 3$. We omit the proof of this fact; a similar statement can be found in [30, Chap. IV, 3.4]).

Consider the Galois covering

$$g\colon \mathsf{A}' = \mathsf{A}_{g,1,(\ell^m n)} \otimes \overline{\mathbb{F}}_p \to \mathsf{A} = \mathsf{A}_{g,1,(n)} \otimes \overline{\mathbb{F}}_p,$$

which has Galois group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell^m)$. Write $d = g(g+1)/2$, which of course is the dimension of A. By first choosing an embedding $\mathsf{A} \hookrightarrow \mathbb{P}^{\mathsf{N}}$ and then projecting from a sufficiently general linear subvariety of codimension $d+1$, we can find an affine open subscheme $\mathsf{U} \subset \mathsf{A}$ for which there exists a finite morphism $f\colon \mathsf{U} \to \mathbb{A}^{\mathsf{d}}$. Write U' for the inverse image of U in A'.

Starting from the morphism $f \circ g\colon \mathsf{U}' \to \mathbb{A}^d$ and applying [14, Th. 6.3] $d-1$ times, we find a line $L \subset \mathbb{A}^d$ such that $(f \circ g)^{-1}(L)$ is an irreducible curve in U'. Let $C \subset \mathsf{A}$ and $C' \subset \mathsf{A}'$ be the Zariski closure of $f^{-1}(L)$ and $(f \circ g)^{-1}(L)$, respectively. The diagram

$$\begin{tikzcd} C' \arrow[r, hook] \arrow[d, "g_{|C'}"'] & \mathsf{A}' \arrow[d, "g"] \\ C \arrow[r, hook] & \mathsf{A} \end{tikzcd}$$

is Cartesian and C and C' are irreducible curves. It follows that $g_{|C'}: C' \to C$ is a Galois covering with group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell^m)$. By what was said before, this implies that C has the required properties. $\square$

THEOREM 6.6 (A. J. de Jong, B. M., F. Oort). *Given a prime number p not dividing n and an integer $g \geqslant 1$, there exists a field k of characteristic p and a k-valued ordinary moduli point $x \in \mathsf{A}^{\circ}_{g,1,n}(k)$ such that*

$$\mathrm{tr.deg.}_{\mathbb{F}_p}\kappa(x) = 1 \quad \text{and} \quad \mathrm{tr.deg.}_{\mathbb{Q}}\kappa(x^{\mathrm{can}}) = \frac{g(g+1)}{2}.$$

Proof. We take a curve $C \subset \mathsf{A}_{g,1,n} \otimes \overline{\mathbb{F}}_p$ as in the lemma. For $x \in \mathsf{A}_{g,1,n} \otimes \mathbb{F}_p$ we take the generic point of C, which is an ordinary moduli point. Clearly, $\mathrm{tr.deg.}_{\mathbb{F}_p}\kappa(x) = 1$.

By Proposition 6.4, the Zariski closure Z of the point $x^{\mathrm{can}} \in \mathsf{A}_{g,1,n} \otimes \mathbb{Q}$ is a subvariety of Hodge type. We are done if we show that it is equal to $\mathsf{A}_{g,1,n} \otimes \mathbb{Q}$. To see this we use that, by construction, the monodromy representation of Z has a 'large' image.

Write $\mathcal{Z}$ for the Zariski closure of Z over $\mathrm{Spec}(\mathbb{Z}_p)$, and let $\bar{\eta}$ be a geometric point of $\mathcal{Z}$ which factors through the generic point η. Then η specializes to x, and as $\mathcal{G}_\ell(x) \supseteq \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$ (by construction of C) we have $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell) \subseteq \mathcal{G}_\ell(\eta) = \mathcal{G}_\ell(Z)$.

Next we choose a number field F such that there exists an F-rational point $z \in Z(F)$. If $\bar{z}$ is a geometric point factoring through z then we have a homomorphism $z_*: \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \pi_1(Z, \bar{z})$, which is a section on $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ of the natural map $\pi_1(Z, \bar{z}) \to \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Let M be the generic Mumford–Tate group on Z, and write $\overline{Z} = Z \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. The homomorphisms $\rho_{\overline{Z}}: \pi_1(\overline{Z}, \bar{z}) \hookrightarrow \pi_1(Z, \bar{z}) \to \mathrm{CSp}(T_\ell X_{\bar{z}}, \varphi_\ell)$ and $\rho \circ z_*: \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{CSp}(T_\ell X_{\bar{z}}, \varphi_\ell)$ both factor through $M(\mathbb{Q}_\ell)$. We conclude that there is a subgroup of finite index $\pi \subseteq \pi_1(Z, \bar{z})$ such that $\rho(\pi) \subset M(\mathbb{Q}_\ell) \subseteq \mathrm{CSp}_{2g}(\mathbb{Q}_\ell)$. Since $M \otimes \mathbb{Q}_\ell$ is an algebraic subgroup of $\mathrm{CSp}_{2g} \otimes \mathbb{Q}_\ell$ with $\mathbb{G}_{\mathrm{m}} \cdot \mathrm{Id} \subset M$, and since $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell) \subseteq \mathcal{G}_\ell(Z)$ we conclude that $M = \mathrm{CSp}_{2g} \otimes \mathbb{Q}$ and $Z = \mathsf{A}_{g,1,n}$. This finishes the proof. $\square$

https://doi.org/10.1023/A:1000411631772 Published online by Cambridge University Press

Acknowledgements

This paper contains the results of the third and fourth chapter of my Utrecht Ph.D. thesis [18]. It is a great pleasure to thank my advisors, F. Oort and A. J. de Jong, for their constant support, their friendship, and for everything they have taught me. I thank Harvard University for its hospitality during my stay as a Visiting Fellow in the first half of 1994, when part of the research for this paper was done. Also I would like to thank all those who have shown interest in my work—in particular I thank B. Edixhoven, R. Noot and T. Zink for their valuable comments.

References

1. André, Y.: *G-functions and Geometry*, Aspects of Mathematics, Vol. E13, Vieweg, Braunschweig, 1989.
2. André, Y.: Mumford–Tate groups of mixed Hodge structures and the theorem of the fixed part, *Compositio Math.* 82 (1992), 1–24.
3. Bourbaki, N.: *Groupes et Algèbres de Lie, Chapitres 7 et 8 (Nouveau Tirage)*, Éléments de mathématique, Masson, Paris, 1990.
4. Chi, W.: ℓ-adic and λ-adic representations associated to abelian varieties defined over number fields, *American J. of Math.* 114 (1992), 315–353.
5. de Jong, A. J.: *A Result on Formal Linearity*, Preprint nr. 911, University of Utrecht, April 1995.
6. de Jong, A. J. and Noot, R.: Jacobians with complex multiplication, in *Arithmetic Algebraic Geometry*, G. van der Geer, F. Oort and J. Steenbrink (eds), Progress in Math., Vol. 89, Birkhäuser, Boston, 1991, pp. 177–192.
7. Deligne, P.: Travaux de Shimura, in *Séminaire Bourbaki*, Exposé 389, Février 1971, pp. 123–165.
8. Deligne, P.: Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques, in *Automorphic Forms, Representations, and L-functions, Part 2*, A. Borel and W. Casselman (eds), Proc. of Symp. in Pure Math., Vol. XXXIII, American Mathematical Society, 1979, pp. 247–290.
9. Deligne, P.: Hodge cycles on abelian varieties (Notes by J. S. Milne), in *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics 900, Springer-Verlag, Berlin, 1982, pp. 9–100.
10. Deligne, P. and Illusie, L.: Cristaux ordinaires et coordonnées canoniques (with an appendix by N. Katz), Exposé V in *Surfaces Algébriques*, J. Giraud, L. Illusie, and M. Raynaud (eds), Lecture Notes in Mathematics 868, Springer-Verlag, Berlin, 1981, pp. 80–137.
11. Demazure, M., Grothendieck, A., et al.: *SGA3; Schémas en Groupes I*, Lecture Notes in Mathematics 151, Springer-Verlag, Berlin, 1970.
12. Faltings, G. and Chai, C.-L.: *Degeneration of Abelian Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 22, Springer-Verlag, Berlin, 1990.
13. Grothendieck, A. and Dieudonné, J.: Éléments de géométrie algébrique, *Publ. Math. de l'I.H.E.S.*, 4, 8, 11, 17, 20, 24 and 32 (1960–67).
14. Jouanolou, J. P.: *Théorèmes de Bertini et Applications*, Progress in Math. Vol. 42, Birkhäuser, Boston, 1983.
15. Katz, N. M.: Serre–Tate local moduli, Exposé $\mathrm{V}^{\mathrm{bis}}$ in *Surfaces Algébriques*, J. Giraud, L. Illusie, and M. Raynaud (eds), Lecture Notes in Mathematics 868, Springer-Verlag, Berlin, 1981, pp. 138–202.
16. Messing, W.: *The Crystals Associated to Barsotti–Tate Groups: with Applications to Abelian Schemes*, Lecture Notes in Mathematics 264, Springer-Verlag, Berlin, 1972.
17. Milne, J. S.: Canonical models of (mixed) Shimura varieties and automorphic vector bundles, in *Automorphic forms, Shimura varieties, and L-functions*, L. Clozel and J. S. Milne (eds), Persp. in Math. Vol. 10(I), Academic Press, Inc., 1990, pp. 283–414.
18. Moonen, B.: Special points and linearity properties of Shimura varieties, PhD thesis, University of Utrecht, 1995.

19. Moonen, B.: Linearity properties of Shimura varieties, I, the *Journal of Algebraic Geometry* Vol. 7, No. 3 July 1998, pp. 539–567.
20. Mumford, D.: Families of abelian varieties, in *Algebraic Groups and Discontinuous Subgroups*, A. Borel and G. D. Mostow (eds), Proc. of Symp. in Pure Math. Vol. 9, American Mathematical Society, 1966, pp. 347–351.
21. Noot, R.: Hodge classes, Tate classes, and local moduli of abelian varieties, PhD thesis, University of Utrecht, 1992.
22. Noot, R.: Models of Shimura varieties in mixed characteristic, *J. Alg. Geom.* 5 (1996), pp. 187–207.
23. Oort, F.: *Canonical Liftings and Dense Sets of CM-Points*, in *Arithmetic Geometry*, F. Catanese (ed.), Symposia Mathematica, Vol. XXXVII, Cambridge University Press, 1997, pp. 228–234.
24. Oort, F.: Endomorphism algebras of abelian varieties, in *Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata*, Vol. II, H. Hijikata et al. (eds), Tokyo, 1988, Kinokuniya Company, Ltd., pp. 469–502.
25. Oort, F. and Sekiguchi, T.: The canonical lifting of an ordinary Jacobian variety need not be a Jacobian variety, *J. Math. Soc. Japan* 38 (1986), 427–437.
26. Ribet, K. A.: Galois action on division points of abelian varieties with many real multiplications, *American J. of Math.* 98 (1976), 751–804.
27. Schlessinger, M.: Functors of Artin rings, *Trans. Amer. Math. Soc.* 130 (1968), 208–222.
28. Serre, J.-P.: Letters to Ken Ribet, 1 January and 29 January, 1981.
29. Serre, J.-P.: Letter to John Tate, 2 January, 1985.
30. Serre, J.-P.: Représentations l-adiques, in *Kyoto Symposium on Algebraic Number Theory*, Japan Soc. for the promotion of science, 1977, pp. 177–193. (Œuvres 112).
31. Serre, J.-P.: *Abelian ℓ-Adic Representations and Elliptic Curves*, Advanced Book Classics, Addison-Wesley Publishing Company, Inc., 1989.
32. Tate, J.: Classes d'isogénie des variétés abéliennes sur un corps fini (d'après T. Honda), in *Séminaire Bourbaki, Exposé 352*, Novembre 1968, pp. 95–110.
33. Zarhin, Yu.: Weights of simple Lie algebras in the cohomology of algebraic varieties, *Math. USSR Izvestia* 24 (1985), 245–281.