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TWO CONTRASTING PROPERTIES OF SOLUTIONS FOR ONE-DIMENSIONAL STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The paper is concerned with the comparison of two solutions for a one-dimensional stochastic partial differential equation. Noting that support compactness of solutions propagates with passage of time, we define the SCP property and show that the SCP property and the strong positivity are two contrasting properties of solutions for one-dimensional SPDEs, which are due to degeneracy of the noise-term coefficient.

1. **Introduction and main results.** Let us consider the following one-dimensional stochastic partial differential equation (SPDE):

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) + b\big(u(t,x)\big) + a\big(u(t,x)\big)\dot{W}(t,x) \quad (t \geq 0 \text{ and } x \in R), \\ u(0,x) &= f(x)\end{aligned} \tag{1.1}$$

where $\Delta = \frac{\partial^2}{\partial x^2}$, $\dot{W}(t,x)$ is a space-time white noise, and $a(u)$ and $b(u)$: $R \longrightarrow R$ are continuous functions. Such SPDEs arise in various fields such as population biology, population genetics, statistical physics and so on (*cf.* [1], [8], [7], [3], [4]).

We notice that the SPDE (1.1) should be understood in the sense of Schwartz distributions. Then under a mild assumption on $a(u)$ and $b(u)$, (1.1) is equivalent to the following stochastic integral equation (SIE) (see §2 and also [7]):

$$\begin{aligned}u(t,x) = G(t)f(x) &+ \int_0^t \int_R G(t-s,x,y)\, b\big(u(s,y)\big)\, ds\, dy \\ &+ \int_0^t \int_R G(t-s,x,y)\, a\big(u(s,y)\big)\dot{W}(s,y)\, ds\, dy,\end{aligned} \tag{1.2}$$

where $G(t,x,y) = \frac{1}{\sqrt{4\pi t}} \exp -\frac{(y-x)^2}{4t}$ and $G(t)f(x) = \int_R G(t,x,y)f(y)\, dy$.

In the present paper we are concerned with the comparison of two solutions for the SPDE (1.1). Let $C(R)$ be the totality of continuous functions on R, and let $C_c^n(R)$ and $C_0^n(R)$ $(0 \leq n \leq \infty)$ be the totality of C^n functions with compact support and vanishing at infinity respectively.

For $f \in C(R)$ let

$$|f|_{(p)} = \sup_{x \in R} |e^{p|x|} f(x)| \quad (p \in R).$$

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We introduce subspaces C_{tem} and C_{rap} of $C(R)$ by

$$C_{\text{tem}} = \{f \in C(R) \mid |f|_{(-\lambda)} < \infty \text{ for every } \lambda > 0\},$$
$$C_{\text{rap}} = \{f \in C(R) \mid |f|_{(\lambda)} < \infty \text{ for every } \lambda > 0\}.$$

The topologies of these spaces are induced by norms $\{|f|_{(-\lambda)} : \lambda > 0\}$ and $\{|f|_{(\lambda)} : \lambda > 0\}$, respectively.

We denote by $C_{\#}^{+}$ the totality of nonnegative elements of $C_{\#}$ for $\# =$ tem, rap, c or 0.

It is known that if $a(u)$ and $b(u)$ are Lipschitz continuous, then for every $f \in C_{\text{tem}}$ the SPDE (1.1) has a pathwise unique C_{tem}-valued solution $u(t,\cdot) = u(t,\cdot,\omega)$, (see [7] and Theorem 2.2 below). Also, if $a(u)$ and $b(u)$ are continuous functions with a linear growth condition, for every $f \in C_{\text{tem}}$ one can construct a space-time white noise $\dot{W}(t,x)$ and a C_{tem}-valued solution $u(t,x)$ of the SPDE (1.1) associated with $\dot{W}(t,x)$ defined on a suitable probability space (see §2 for the detail and also [10]).

Now let us consider the following special case:

$$a(u) = \sqrt{|u|} \quad \text{and} \quad b(u) = 0. \tag{1.3}$$

Then for every $f \in C_{\text{tem}}^{+}$ there exists a C_{tem}^{+}-valued solution $u(t,x)$ of (1.1) that is uniquely determined in the law sense, and moreover, $X_t(dx) = u(t,x)\,dx$ defines a measure-valued branching diffusion process (MBD) on R (see [8]). For the MBD process over R^d Dawson and Iscoe [6] discovered the remarkable phenomenon that if the initial state $X_0(dx)$ has compact support, then so does $X_t(dx)$ for every $t > 0$ with probability one. Hence for the SPDE (1.1) with (1.3), if $u(0) = f \geq 0$ is a continuous function on R with compact support, the nonnegative solution $u(t,\cdot)$ also has compact support. Thus the support compactness propagates with the passage of time. Then we say the SCP *property* holds.

On the other hand C. Mueller [9] recently discussed the following case:

$$a(u) = |u|^{\alpha} \quad (\alpha \geq 1) \quad \text{and} \quad b(u) = 0. \tag{1.4}$$

He proved that if $u(0) = f \geq 0$ is a continuous function with compact support and $f(x) > 0$ for some $x \in R$, then the solution $u(t,x)$ of (1.1) satisfies

$$P\big(u(t,x) > 0 \text{ for every } x \in R \mid t < \sigma_\infty\big) = 1 \quad \text{for every } t > 0, \tag{1.5}$$

where $\sigma_\infty = \lim_{n\to\infty} \sigma_n$, $\sigma_n = \inf\{t \geq 0 \mid \sup_{x\in R} u(t,x) \geq n\}$.

We note that if $\alpha = 1$, $P(\sigma_\infty = \infty) = 1$, but if $\alpha > 1$, the solution $u(t,\cdot)$ is defined up to the explosion time σ_∞, since the explosion might occur.

In this paper we would like to assert that the SCP property and the strong positivity are two contrasting properties of solutions for one-dimensional SPDEs, which are due to degeneracy of the noise-term coefficient $a(u)$ at $u = 0$. Indeed we shall obtain sufficient conditions for the SCP property and a strong comparison of solutions generalizing the strong positivity as follows.

THEOREM 1.1. *Assume that $a(u)\colon R \longrightarrow R$ and $b(u)\colon R \longrightarrow R$ are continuous functions such that for some constant $C > 0$*

$$|a(u)| + |b(u)| \leq C(1 + |u|) \quad \textit{for } u \in R. \tag{1.6}$$

(i) Suppose that

$$a(0) = 0 \quad \textit{and} \quad b(0) \geq 0. \tag{1.7}$$

Then for every $f \in C^+_{\text{tem}}$ there exists a C^+_{tem}-valued solution $u(t,\cdot)$ of the SPDE *(1.1) associated with a space-time white noise $\dot{W}(t,x)$ defined on a probability space with filtration $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.*

(ii) Suppose further that for each $K > 0$ there exists a constant $a_K > 0$ such that

$$|a(u)| \geq a_K u^{1/2} \quad \textit{for } 0 \leq u \leq K, \tag{1.8}$$

and that for some $C > 0$,

$$|b(u)| \leq C|u| \quad \textit{for } u \in R. \tag{1.9}$$

Then the SCP *property holds, that is, if $f \in C^+_c(R)$, $P\big(u(t,\cdot) \in C^+_c(R) \text{ for every } t > 0\big) = 1$ holds for every C^+_{tem}-valued solution $u(t,\cdot)$ of the SPDE (1.1).*

REMARK 1. Under the assumptions of Theorem 1.1 we do not know the uniqueness of solutions for the SPDE (1.1), but the theorem asserts the SCP property holds for any nonnegative solution.

REMARK 2. $a(u) = |u|^\alpha$ $(0 < \alpha \leq 1/2)$ satisfies the assumptions of Theorem 1.1.

REMARK 3. Theorem 1.1 still holds even if Δ is replaced by a one-dimensional nondegenerate diffusion operator.

We next consider an SPDE associated with a generalized Fleming-Viot diffusion process over R, that is, an infinite-dimensional version of Gillespie and Sato's diffusion model in population genetics, (see [11] for the genetical motivation).

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) + \sigma\big(x, u(t,x)\big) - u(t,x)\int_R \sigma\big(y, u(t,y)\big)\,dy + \sqrt{\alpha(x)\,u(t,x)}\,\dot{W}(t,x)\\ &\quad - u(t,x)\int_R \sqrt{\alpha(y)\,u(t,y)}\,\dot{W}(t,y)\,dy \quad (t \geq 0, x \in R)\\ u(0,x) &= f(x)\end{aligned} \tag{1.10}$$

where $\alpha(x)\colon R \longrightarrow R$ is a uniformly positive bounded continuous function and $\sigma(x,u)\colon R \times R \longrightarrow R$ is a continuous function satisfying, for some constant $C > 0$,

$$|\sigma(x,u)| \leq C|u| \quad \text{for } x \in R \text{ and } u \in R. \tag{1.11}$$

Then the SPDE (1.10) is rewritten in the following way: for every $\varphi \in C_c^\infty(R)$,

(1.12)

$$\begin{aligned}\langle u(t),\varphi\rangle &= \langle f,\varphi\rangle + \int_0^t \Big\{\langle u(x),\Delta\varphi\rangle + \int_R \sigma\big(x,u(s,x)\big)\varphi(x)\,dx - \int_R \sigma\big(x,u(s,x)\big)\,dx\,\langle u(s),\varphi\rangle\Big\}\,ds \\ &\quad + M_t(\varphi),\end{aligned}$$

where $\langle u,\varphi\rangle = \int_R u(x)\varphi(x)\,dx$, and $M_t(\varphi)$ is a martingale with quadratic variation process

$$(1.13)\qquad \langle M(\varphi)\rangle_t = \int_0^t \Big(\langle u(s),\alpha\varphi^2\rangle - 2\langle u(s),\alpha\varphi\rangle\langle u(s),\varphi\rangle + \langle u(s),\alpha\rangle\langle u(s),\varphi\rangle^2\Big)\,ds.$$

If $\alpha(x) = \alpha$ is constant and $\sigma(x,u) = \sigma(x)u$, (1.10) reduces to a standard form of Fleming-Viot diffusion model incorporating selection. Then uniqueness in law of C_{tem}-valued probability density soltuions for the SPDE (1.10) is known (*cf.* [8]), for which a key point is that the integrand of (1.13) reduces to a quadratic polynomial of $u(s)$. If a non-constant $\alpha(x)$ is involved, it is a cubic polynomial of $u(s)$, which makes the uniqueness problem extremely difficult and it still remains open. However even in this case one can show that for every probability density f in C_{tem}, there exists a C_{tem}-valued probability density solution $u(t,x)$ of the SPDE (1.10) associated with a space-time white-noise $W(t,x)$ defined on a suitable probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$.

THEOREM 1.2. *The* SPDE *(1.10) has the* SCP *property, that is, if $f \in C_c^+(R)$ is a probability density, then any C_{tem}^+-valued probability density solution $u(t,\cdot)$ of the* SPDE *(1.10) satisfies*

$$P\big(u(t,\cdot) \in C_c^+(R) \text{ for every } t \geq 0\big) = 1.$$

In the case $\alpha(x) \equiv \alpha$ and $b(u) \equiv 0$, Theorem 1.2 was proved by Dawson and Hochberg [2] in the setting of measure-valued diffusion processes; however their proof is very sophisticated and it is not applicable to the present case. So we shall, in §3, present an alternative proof, which is simple and transparent.

We next obtain a strong comparison theorem for solutions of the SPDE (1.1) under a Lipschitz condition on the coefficients, which extends Mueller's result [9].

THEOREM 1.3. *Assuming that $a(u)\colon R \longrightarrow R$ and $b(u)\colon R \longrightarrow R$ are Lipschitz continuous, let $u_1(t,x)$ and $u_2(t,x)$ be two C_{tem}-valued solutions of the* SPDE *(1.1) with the initial conditions $u_1(0) = f_1 \in C_{\text{tem}}$ and $u_2(0) = f_2 \in C_{\text{tem}}$. Suppose that $f_1 \geq f_2$ and $f_1(x) > f_2(x)$ for some $x \in R$. Then $P\big(u_1(t,x) > u_2(t,x)$ for every $t > 0$ and every $x \in R\big) = 1$.*

Combining this with Theorem 2.2 we obtain

COROLLARY 1.4. *Assume further that $a(0) = 0$ and $b(0) \geq 0$ in addition to the assumptions of Theorem 1.3. Let $u(t,\cdot)$ be the unique C_{tem}-valued solution of the* SPDE *(1.1) with the initial condition $u(0) = f \in C_{\text{tem}}$. Suppose that $f \geq 0$ and $f(x) > 0$ for some $x \in R$. Then $P\big(u(t,x) > 0$ for every $t > 0$ and every $x \in R\big) = 1$.*

The proofs for Theorem 1.1 and 1.2 are essentially based on Iscoe's lemma in [6] on a non-linear differential equation with a singular boundary condition (see Lemma 3.1).

On the other hand the proof of Theorem 1.3 is a refinement of Mueller's arguments in [9]. Although it seems that a dichotomy holds between the SCP property and the strong positivity for general one-dimensional SPDEs (1.1) with $a(0) = b(0) = 0$, our results give only a partial solution for this.

In §2 we summarize several basic facts on one-dimensional SPDEs. Most of them seem to be known implicitly, but to make the paper self-contained we shall briefly give their proofs in the appendix. The proofs of Theorem 1.1 and 1.2 will be given in §3, and Theorem 1.3 will be proved in §4. In §5 we will discuss some examples.

2. **Preliminaries and basic facts.** Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete probability space with filtration, and let $\dot{W}(t,x)$ be an $\{\mathcal{F}_t\}$-space-time white noise, *i.e.* $\dot{W}(t,x)_{t\geq 0, x\in R}$ is an $\{\mathcal{F}_t\}$-adapted centered Gaussian fields with

$$E\big(\dot{W}(t,x)\dot{W}(t',x')\big) = \delta(t-t')\,\delta(x-x') \quad \text{for } t,t' \geq 0 \text{ and } x,x' \in R.$$

For an $\{\mathcal{F}_t\}$-predictable functional $\phi(t,x,\omega)$: $[0,\infty) \times R \times \Omega \longrightarrow R$ satisfying

$$\int_0^t \int_R \phi(s,x,\omega)^2\,ds\,dx < \infty \quad \text{for every } t > 0,$$

one can define a stochastic integral $\int_0^t \int_R \phi(s,x,\omega)\dot{W}(s,x)\,ds\,dx$ as an $\{\mathcal{F}_t\}$-local martingale with quadratic variation process $\int_0^t \int_R f(s,x,\omega)^2\,ds\,dx$, (see [14] for the stochastic integral).

Suppose that we are given two $\{\mathcal{F}_t\}$-predictable functionals $a(t,x,u,\omega)$ and $b(t,x,u,\omega)$: $[0,\infty) \times R \times R \times \Omega \longrightarrow R$, and an $\{\mathcal{F}_t\}$-space-time white noise $\dot{W}(t,x)$.

Let us consider the following SPDE:

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x) + b\big(t,x,u(t,x)\big) + a\big(t,x,u(t,x)\big)\,\dot{W}(t,x) \\ u(0,x) &= f(x). \end{aligned} \tag{2.1}$$

More precisely, the SPDE (2.1) should be understood in the sense of the Schwartz distributions, so that for every $\varphi \in C_c^\infty(R)$

$$\begin{aligned} \langle u(t),\varphi\rangle &= \langle f,\varphi\rangle + \int_0^t \Big(\langle u(s),\Delta\varphi\rangle + \big\langle b\big(s,\cdot,u(s,\cdot)\big),\varphi\big\rangle\Big)\,ds \\ &\quad + \int_0^t \int_R a\big(s,x,u(s,x)\big)\varphi(x)\dot{W}(s,x)\,ds\,dx. \end{aligned} \tag{2.1$'$}$$

We assume that for every $T > 0$, there exists a constant $C_T > 0$ such that

$$|a(t,x,u,\omega)| + |b(t,x,u,\omega)| \leq C_T(1+|u|) \text{ for } 0 \leq t \leq T \text{ and } (x,u) \in R\times R,\ P\text{-a.s. } \omega. \tag{2.2}$$

If an $\{\mathcal{F}_t\}$-predictable functional $u(t,\cdot) = u(t,\cdot,\omega)$ is a C_{tem}-valued (C_{rap}-valued) continuous process and satisfies the equation (2.1), we say $u(t,\cdot)$ is a C_{tem}-valued (C_{rap}-valued) solution of (2.1). Iwata discussed in [7] the equivalence of C_{tem}-valued solutions of the SPDE (2.1) and the SIE (2.3) under the assumption: $a(s,x,u,\omega)$ is bounded and $a(s,x,u,\omega) = a(u)$. We here generalize it slightly.

THEOREM 2.1. *Let* $f \in C_{\text{tem}}$. *Under the condition (2.2),* $u(t,\cdot)$ *is a* C_{tem}*-valued solution of the* SPDE *(2.1) if and only if* $u(t,\cdot)$ *is an* $\{\mathcal{F}_t\}$*-predictable and* C_{tem}*-valued continuous process that satisfies the following* SIE *(2.3):*

$$u(t,x) = G(t)f(x) + \int_0^t \int_R G(t-s,x,y)\, b\big(s,y,u(s,y)\big)\, ds\, dy + \int_0^t \int_R G(t-s,x,y)\, a\big(s,y,u(s,y)\big)\dot{W}(s,y)\, ds\, dy \tag{2.3}$$

for $t \geq 0$ *and* $x \in R$.

The following Theorem 2.2 is a modification of Iwata [7], Theorem 4.1.

THEOREM 2.2 (EXISTENCE AND UNIQUENESS THEOREM). *Suppose that functionals* $a(t,x,u,\omega)$ *and* $b(t,x,u,\omega)$ *satisfy (2.2) and the following (2.4):*

(2.4) *for every* $T > 0$ *there exists an* $L_T > 0$ *such that P-a.s.*
$|a(t,x,u_1,\omega) - a(t,x,u_2,\omega)| + |b(t,x,u_1,\omega) - b(t,x,u_2,\omega)| \leq L_T|u_1 - u_2|$
for $0 \leq t \leq T$ *and* $(x,u_1,u_2) \in R \times R \times R$.

Then for every $f \in C_{\text{tem}}$ *the* SPDE *(2.1) has a (pathwise) unique* C_{tem}*-valued solution* $u(t,x)$.

THEOREM 2.3 (NONNEGATIVITY OF SOLUTIONS). *In addition to the assumption of Theorem 2.2 assume that P-a.s.*

$$a(t,x,u,\omega) \text{ and } b(t,x,u,\omega) \text{ are continuous in } (x,u), \tag{2.5}$$

$$a(t,x,0,\omega) = 0 \text{ and } b(t,x,0,\omega) \geq 0. \tag{2.6}$$

Let $u(t,x)$ *be the* C_{tem}*-valued solution of the* SPDE *(2.1) with* $u(0) = f \in C^+_{\text{tem}}$. *Then* $P\big(u(t,\cdot) \geq 0 \text{ for every } t \geq 0\big) = 1$.

COROLLARY 2.4 (COMPARISON THEOREM). *Suppose that* $\{\mathcal{F}_t\}$*-predictable functionals* $a(t,x,u,\omega)$ *and* $b_i(t,x,u,\omega)$ $(i = 1,2)$ *satisfy the assumptions of Theorem 2.2. Let* $u_i(t,\cdot)$ *be the* C_{tem}*-valued solution of the* SPDE *(2.1) associated with the coefficients* $a(t,x,u,\omega)$ *and* $b_i(t,x,u,\omega)$ *having the initial condition* $u_i(0) = f_i \in C_{\text{tem}}$. *Suppose further that*

$$a(t,x,u,\omega) \text{ and } b_i(t,x,u,\omega) \quad (i=1,2) \text{ are continuous in } (x,u), P\text{-a.s.} \tag{2.7}$$

$$b_1(t,x,u,\omega) \geq b_2(t,x,u,\omega) \quad \text{for } t \geq 0,\ x \in R,\ u \in R,\ P\text{-a.s.} \tag{2.8}$$

$$f_1 \geq f_2. \tag{2.9}$$

Then $P\big(u_1(t,\cdot) \geq u_2(t,\cdot) \text{ for every } t \geq 0\big) = 1$.

THEOREM 2.5 (EXISTENCE OF C^+_{rap}-VALUED SOLUTIONS). *Suppose that for every* $T > 0$ *there exist* $C_T > 0$ *and* $0 < \theta < 1$ *such that for* $0 \leq t \leq T$, $x \in R$ *and* $u \in R$,

$$|a(t,x,u,\omega)| \leq C_T(|u| + |u|^\theta), \tag{2.10}$$

$$|b(t,x,u,\omega)| \leq C_T|u|,\ P\text{-a.s.} \tag{2.11}$$

Let $u(t,\cdot)$ be a C^+_{tem}-valued solution of the SPDE *(2.1) with $u(0) = f \in C^+_{\mathrm{rap}}$. Then $u(t,\cdot)$ is a C^+_{rap}-valued solution.*

THEOREM 2.6 (EXISTENCE OF C^+_{tem}-VALUED SOLUTIONS). *Let $a(u)\colon R \longrightarrow R$ and $b(u)\colon R \longrightarrow R$ be continuous functions satisfying a linear growth condition; for some $K > 0$*

$$|a(u)| + |b(u)| \leq K(1 + |u|) \quad \text{for } u \in R. \tag{2.12}$$

Suppose that

$$a(0) = 0, \quad b(0) \geq 0. \tag{2.13}$$

Then for every $f \in C^+_{\mathrm{tem}}$, there exist an $\{\mathcal{F}_t\}$-space-time white noise $\dot{W}(t,x)$ and a C^+_{tem}-valued solution $u(t,x)$ of the SPDE *(1.1) with $u(0) = f$ on a suitable probability space with filtration $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.*

The proofs of these theorems will be briefly given in the appendix.

3. **Proofs of Theorem 1.1 and 1.2.** In the case: $a(u) = |u|^{1/2}$ and $b(u) = 0$, Iscoe [6] used a solution of a simple non-linear equation with a singular boundary condition to prove the SCP property. So our strategy in proving Theorem 1.1 and 1.2 is to consider how to reduce them to the case; $a(u) = |u|^{1/2}$ and $b(u) = 0$. Indeed we will carry out it by using some comparisons for Theorem 1.1 and by introducing an MBD-like process from the solution of (1.10) for Theorem 1.2.

The following lemma is found in [6].

LEMMA 3.1. *Let $c > 0$ be a fixed constant.*

(i) For $r > 0$ there exists a unique positive solution $v(x) = v(x;r) \in C^2(-r,r)$ of the equation

$$v''(x) = cv(x)^2 \text{ for } |x| < r \tag{3.1}$$
$$\lim_{|x| \to r^+} v(x) = \infty.$$

Moreover it holds that $v(x;r) \longrightarrow 0$ as $r \longrightarrow \infty$ uniformly on each compact interval.

(ii) For any fixed $h \in C^+_c(R)$, let us consider the following equation:

$$v''(x) = cv(x)^2 - h(x) \quad \text{for } x \in R. \tag{3.2}$$

Then there exists a unique positive solution $v(x) = v_h(x) \in C^2_0(R)$ of (3.2). Moreover, if h vanishes in $(-r,r)$, it holds that

$$v_h(x) \leq v(x;r) \quad \text{for } |x| < r. \tag{3.3}$$

PROOF OF THEOREM 1.1. (i) follows from Theorem 2.6. To see (ii), let $u(t,\cdot)$ be a C^+_{tem}-valued solution with $u(0) = f \in C^+_c(R)$ of the SPDE (1.1). Let $r > 0$, $T > 0$ and

$K > 0$ be fixed and set $\sigma = \sigma_K = \inf\{t \geq 0 \mid |u(t)|_\infty \geq K\}$, where $|\cdot|_\infty$ stands for the supremum norm. Since by Theorem 2.5 $u(t,\cdot)$ is C^+_{rap}-valued continuous, it holds $\sigma_K \longrightarrow \infty$ as $K \longrightarrow \infty$, P-a.s. Recalling a_K of (1.8), for $c = a_K^2 \exp(-CT)/2$ and $h = \theta\phi$ with a $\theta > 0$ and $\phi \in C^+_c(R)$ we denote by $v_\theta(x)$ the solution of the equation (3.2). Set $w(t,x) = \exp(-Ct)\,u(t,x)$, and use Ito's formula with the C^2-function $v_\theta(x)$. Then we have

$$\begin{aligned}
&\exp(-\langle w(t\wedge\sigma), v_\theta\rangle - \theta\int_0^{t\wedge\sigma}\langle w(s),\phi\rangle\,ds - \exp(-\langle f, v_\theta\rangle)\\
&\quad= \int_0^{t\wedge\sigma}\exp\Big(-\langle w(s), v_\theta\rangle - \theta\int_0^s \langle w(\tau),\phi\rangle\,d\tau\Big)\\
&\qquad\times\Big(\langle w(s), Cv_\theta - \Delta v_\theta - \theta\phi\rangle - e^{-Cs}\big\langle b\big(u(s)\big), v_\theta\big\rangle\\
&\qquad+\frac{1}{2}e^{-2Cs}\big\langle a\big(u(s)\big)^2, v_\theta^2\big\rangle\Big)\,ds + \text{ a martingale.}
\end{aligned}\tag{3.4}$$

Note by (1.8) and (1.9) that for $0 \leq s \leq \sigma$,

$$-e^{-Cs}b\big(u(s,x)\big) \geq -Cw(s,x) \quad\text{and}\quad \frac{1}{2}e^{-2Cs}a\big(u(s,x)\big)^2 \geq cw(s,x).$$

Since $v_\theta(x)$ is a solution of the equation (3.2), it follows from (3.4) that

$$E\Big(\exp\Big(-\langle w(t\wedge\sigma), v_\theta\rangle - \theta\int_0^{t\wedge\sigma}\langle w(s),\phi\rangle\,ds\Big)\Big) \geq \exp(-\langle f, v_\theta\rangle) \quad\text{for every } 0 \leq t \leq T.$$

For the initial function f, take an $r > 0$ such that $(-r,r) \supseteq$ the support of f. Note that by (3.3) if $\phi \in C^+_c(R)$ vanishes in $(-r,r)$, $v_\theta(x) \leq v(x:r)$ for $|x| < r$, hence

$$E\Big(\exp\Big(-\theta\int_0^{t\wedge\sigma}\langle w(s),\phi\rangle\,ds\Big)\Big) \geq \exp\big(-\langle f, v(\cdot:r)\rangle\big) \quad\text{for every } 0 \leq t \leq T,\tag{3.5}$$

which implies

$$E\Big(\exp\Big(-\theta\int_0^{T\wedge\sigma}\langle w(s), I_r^c\rangle\,ds\Big)\Big) \geq \exp\big(-\langle f, v(\cdot:r)\rangle\big),\tag{3.6}$$

where $I_r^c(x) = 1$ if $|x| \geq r$ and $I_r^c(x) = 0$ otherwise. Furthermore, letting $\theta \longrightarrow \infty$ and $r \longrightarrow \infty$, by Lemma 3.1(i) we obtain

$$P\Big(\int_0^{T\wedge\sigma}\langle w(s), I_r^c\rangle\,ds = 0 \text{ for some } r > 0\Big) = 1 \quad\text{for every } T > 0 \text{ and } K > 0,\tag{3.7}$$

which concludes the latter part of Theorem 1.1.

PROOF OF THEOREM 1.2. First fix a constant γ so that

$$\sigma(x,u) + \big(\alpha(x) - \gamma\big)u \leq 0 \quad\text{for } x \in R \text{ and } u \geq 0.\tag{3.8}$$

Let $u(t,x)$ be a C_{tem}-valued continuous probability density solution of the SPDE (1.10) with $u(0) = f$ being a continuous probability density function with compact support. Then as in Theorem 2.4(i) one can show $u(t,\cdot)$ is C_{rap}-valued continuous. Setting

$$b(t) = \int_R \sigma\big(x, u(t,x)\big)\,dx + \langle u(t),\alpha\rangle - \gamma,\tag{3.9}$$

$$N_t(\varphi) = \int_0^t\int_R \sqrt{\alpha(x)\,u(s,x)}\varphi(x)\dot{W}(s,x)\,ds\,dx,\tag{3.10}$$

let us consider the following SDE:

$$dz_t = z_t b(t)\,dt + z_t\,dN_t(1),\qquad z_0 = 1. \tag{3.11}$$

Then the solution of (3.11) is given explicitly;

$$z_t = \exp\Big(N_t(1) + \int_0^t b(s)\,ds - \frac{1}{2}\int_0^t \langle u(s), \alpha\rangle\,ds\Big), \tag{3.12}$$

hence

$$P(z_t > 0 \text{ for every } t \geq 0 \text{ and } z_t \longrightarrow 0 \text{ as } t \longrightarrow \infty) = 1. \tag{3.13}$$

By (1.10), for $f \in C_0^2(R)$,

(3.14)

$$\begin{aligned}\langle u(t), \varphi\rangle &= \langle f, \varphi\rangle + \int_0^t \Big\{\langle u(s), \Delta\varphi\rangle + \int_R \sigma\big(x, u(s,x)\big)\varphi(x)\,dx - \int_R \sigma\big(x, u(s,x)\big)\,dx\langle u(s), \varphi\rangle\Big\}\,ds \\ &\quad + N_t(\varphi) - \int_0^t \langle u(s), \varphi\rangle\,dN_s(1).\end{aligned}$$

Let us define a C^+_{rap}-valued continuous process $w(t,\cdot)$ by

$$w(t,x) = z_t u(t,x). \tag{3.15}$$

Using Ito's formula together with (3.14) we have

$$\langle w(t), \varphi\rangle = \langle f, \varphi\rangle + \int_0^t \Big(\langle w(s), \Delta\varphi\rangle + z_s\langle c(s), \varphi\rangle\Big)\,ds + \int_0^t z_s\,dN_s(\varphi), \tag{3.16}$$

where $c(t,x) = \sigma\big(x, u(t,x)\big) + \big(\alpha(x) - \gamma\big)u(t,x) \leq 0$ by (3.8).

For $\varepsilon > 0$ define $\tau = \tau_\varepsilon = \inf\{t \geq 0 : z_t \geq \varepsilon\}$. Denote by $v_\theta(x)$ the solution of the equation (3.2) with $h = \theta\phi$ for $\theta > 0$ and $\phi \in C_c^+$ and $c = \frac{1}{2}\varepsilon\alpha_{\min}$ ($\alpha_{\min} = \inf_{x\in R}\alpha(x)$). By Ito's formula

(3.17)

$$\begin{aligned}\exp\Big(-\langle w(t\wedge\tau), v_\theta\rangle - \theta\int_0^{t\wedge\tau}\langle w(s), \phi\rangle\,ds\Big) - \exp(-\langle f, v_\theta\rangle) \\ = \int_0^{t\wedge\tau} \exp\Big(-\langle w(s), v_\theta\rangle - \theta\int_0^s \langle w(r), \phi\rangle\,dr\Big) C(s)\,ds + a \text{ martingale}\end{aligned}$$

where

$$\begin{aligned}C(s) &= \langle w(t), -\Delta v_\theta - \theta\phi\rangle - z_s\langle c(s), v_\theta\rangle + \frac{1}{2}z_s\langle u(s), \alpha v_\theta^2\rangle \\ &\geq \Big\langle w(s), -\Delta v_\theta - \theta\phi + \frac{1}{2}\varepsilon\alpha_{\min}v_\theta^2\Big\rangle \\ &= 0 \quad \text{for } 0 \leq s \leq \tau.\end{aligned}$$

Hence

$$\exp\Big(-\theta\int_0^{t\wedge\tau}\langle w(s), \phi\rangle\,ds\Big) \geq \exp(-\langle f, v_\theta\rangle) \quad \text{for every } t > 0. \tag{3.18}$$

By the same argument as (3.5)–(3.7) we obtain

$$P\Big(\int_0^{t\wedge\tau}\langle w(s), I_r^c\rangle\,ds = 0 \text{ for some } r > 0\Big) = 1 \quad \text{for every } \varepsilon > 0 \text{ and } t > 0. \tag{3.19}$$

Since $\tau_\varepsilon \longrightarrow \infty$ as $\varepsilon \longrightarrow 0$ holds by (3.13), it follows from (3.19) that $u(t,x)$ has compact support for every $t > 0$ with probability one, completing the proof of Theorem 1.2.

4. **Proof of Theorem 1.3.** We will prove the theorem by refining the method of Mueller [9], which is based on a large deviation estimate for stochastic integrals with respect to a space-time white noise.

To avoid technical complication we first assume that $b(u) = 0$. As stated in §2, (1.1) with $b(u) = 0$ is equivalent to the following stochastic integral equation (SIE):

$$u(t,x) = G(t)f(x) + \int_0^t \int_R G(t-s,x,y)\, a\big(u(s,y)\big)\dot{W}(s,y)\, ds\, dy. \tag{4.1}$$

The following estimate due to Mueller [9] which plays a key role in proving the theorem.

LEMMA 4.1. *Let $K > 0$ be fixed and let $b(t,y,\omega)$ be an $\{\mathcal{F}_t\}$-predictable functional such that $|b(t,x,\omega)| \le Ke^{-(T-t)|x|}$ for every $0 \le t \le T/2$ and every $x \in R$ almost surely. Then there exist $c_1 > 0$ and $c_2 > 0$ depending on K such that for every $0 < \varepsilon < 1$ and every $0 < T < 1$*

$$P\Big(\Big|\int_0^t \int_R G(t-s,x,y)\, b(s,y)\, \dot{W}(s,y)\, ds\, dy\Big| > \varepsilon e^{-(T-t)|x|}$$
$$\text{for some } 0 < t < T/2 \text{ and } x \in R\Big) \le c_1 T \varepsilon^{-24} \exp(-c_2 \varepsilon^2 T^{1/4}).$$

REMARK. If $b(t,x,\omega)$ is non-random, the stochastic integral defines a two parameter Gaussian fields, so the proof of Lemma 4.1 is quite standard. But for a random $b(t,x,\omega)$, a similar calculation is possible (see [9] for the details).

Suppose that for $i = 1,2$, $u_i(t,x)$ be the unique C_{tem}-valued solution of the equation (4.1) with $f_i \in C_{\text{tem}}$, and that $f_1 \ge f_2$ and $f_1(x) > f_2(x)$ for some $x \in R$. By Corollary 2.4 it suffices to prove the theorem assuming $f_1 - f_2$ has compact support. Set $u(t,x) = u_1(t,x) - u_2(t,x) \ge 0$ and $f = f_1 - f_2$. Then it satisfies

$$u(t,x) = G(t)f(x) + \int_0^t \int_R G(t-s,x,y)\, a\big(s,y,u(s,y)\big)\dot{W}(s,y)\, ds\, dy \quad \text{with } f \in C_c^+, \tag{4.2}$$

where $a(s,y,u,\omega)$: $[0,\infty) \times R \times R \times \Omega \longrightarrow R$ is an $\{\mathcal{F}_t\}$-predictable functional defined by

$$a(s,y,u,\omega) = a\big(u + u_2(s,y,\omega)\big) - a\big(u_2(s,y,\omega)\big). \tag{4.3}$$

By the Lipschitz continuity of $a(u)$, we have a constant $L > 0$ such that

$$|a(s,y,u,\omega)| \le L|u| \text{ for every } (s,y,u,\omega) \in [0,\infty) \times R \times R \times \Omega. \tag{4.4}$$

Let

$$N(t,x) = \int_0^t \int_R G(t-s,x,y)\, a\big(s,y,u(s,y)\big)\dot{W}(s,y)\, ds\, dy. \tag{4.5}$$

LEMMA 4.2. *Let $M > 0$ be fixed. Then there exist $c_1 > 0$ and $c_2 > 0$, depending on M and L, such that if $f(x) \le \beta I_{[-M,M]}(x)$ for every $x \in R$ with a $\beta > 0$, then for every $0 < \varepsilon < 1$ and $0 < T < 1$*

$$P\Big(|N(t,x)| > \varepsilon\beta e^{-(T-t)|x|} \text{ for some } 0 < t < \frac{T}{2} \text{ and some } x \in R\Big)$$
$$\le c_1 \varepsilon^{-24} \exp(-c_2 \varepsilon^2 T^{-\frac{1}{4}}).$$

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PROOF. As is easily checked,

(4.6)
$$G(t)I_{[-M,M]}(x) \le C_M e^{-(T-t)|x|} \quad \text{for every } 0 < t < T \le 1 \text{ and } x \in R, \text{ with } C_M = e^{M+1}.$$

Let

$$\sigma = \inf\{t \ge 0 \mid u(t,x) \ge \beta(C_M + 1)e^{-(T-t)|x|} \text{ for some } x \in R\}$$
$$= \infty \quad \text{if } \{\cdot\} = \emptyset,$$

and set

$$M(t,x) = \int_0^t \int_R G(t-s,x,y)\, I(s<\sigma)\, a\big(s,y,u(s,y)\big)\dot{W}(s,y)\,ds\,dy. \tag{4.7}$$

We claim that if $0 < \varepsilon < 1$,

(4.8)
$$P\Big(|N(t,x)| > \varepsilon\beta e^{-(T-t)|x|} \text{ for some } 0 < t < T/2 \text{ and some } x \in R\Big)$$
$$\le P\Big(|M(t,x)| > \varepsilon\beta e^{-(T-t)|x|} \text{ for some } 0 < t < T/2 \text{ and some } x \in R\Big).$$

Suppose that

$$|M(t,x)| \le \varepsilon\beta e^{-(T-t)|x|} \quad \text{for every } 0 < t < T/2 \text{ and } x \in R. \tag{4.9}$$

Since $M(t,x) = N(t,x)$ for every $x \in R$ and $0 \le t < \sigma$, by (4.6) and (4.9) it holds

$$u(t,x) \le \beta(C_M + \varepsilon)e^{-(T-t)|x|} \quad \text{for every } x \in R \text{ and } 0 \le t < \sigma \wedge T/2.$$

Since $u(t,\cdot)$ is C^+_{rap}-valued continuous by Theorem 2.4, this implies $\sigma \wedge T/2 < \sigma$. Hence it follows $T/2 < \sigma$, which yields

$$|N(t,x)| \le \varepsilon\beta e^{-(T-t)|x|} \quad \text{for every } 0 < t < T/2 \text{ and } x \in R. \tag{4.10}$$

Thus we obtain (4.8).

Finally, noting by (4.4) that $b(s,y,\omega) = \beta^{-1}I(s<\sigma)\,a\big(s,y,u(s,y)\big)$ satisfies

$$|b(s,y,w)| \le L(C_M + 1)e^{-(T-t)|y|},$$

Lemma 4.1 is applicable for $\beta^{-1}M(t,x)$, completing the proof of Lemma 4.2.

We are now in position to prove Theorem 1.3. Choose $a < b$ and $\beta > 0$ such that $f(x) > \beta I_{(a,b)}(x)$ for every $x \in R$. Fix an arbitrary $M > 0$ such as $[-M/2, M/2] \supseteq (a,b)$ and $t > 0$.

STEP 1. As easily checked, there is an $m_0 = m_0(t,a,b,M)$ such that if $m \ge m_0$ and $(c,d) \supseteq (a,b)$

$$G(s)I_{(c,d)}(x) \ge \frac{1}{4} I_{(c-M/m,\,d+M/m)}(x) \quad \text{for every } \frac{t}{2m} \le s \le \frac{t}{m} \text{ and } x \in R. \tag{4.11}$$

STEP 2. We claim that if $[-M,M] \supseteq (c,d) \supseteq (a,b)$ and $f(x) \geq \alpha I_{(c,d)}(x)$ with an $\alpha > 0$, then for every $m \geq m_0$

$$P\Big(u(s,x) \geq \frac{\alpha}{8} I_{(c-M/m,d+M/m)}(x) \text{ for every } \frac{t}{2m} \leq s \leq \frac{t}{m} \text{ and } x \in R\Big) \geq 1 - c_1 16^{24} \exp(-c_2 16^{-2} t^{1/4} m^{1/4}), \tag{4.12}$$

where c_1 and c_2 are the constants in Lemma 4.2, which depend on M and L.

To prove (4.12), by Theorem 2.2, it suffices to show it assuming that

$$2\alpha I_{[-M,M]}(x) \geq f(x) \geq \alpha I_{(c,d)}(x).$$

By Lemma 4.2, for every $0 < \varepsilon < 1$ and $m \geq m_0$

$$P\Big(|N(s,x)| > 2\varepsilon\alpha e^{-(T-s)|x|} \text{ for some } 0 \leq s \leq \frac{t}{m} \text{ and some } x \in R\Big) \leq c_1 \varepsilon^{-24} \exp(-c^2 \varepsilon_2 t^{-1/4} m^{1/4}). \tag{4.13}$$

Since $u(t,x) = G(t)f(x) + N(t,x)$, it follows from (4.11) and (4.13) that

$$\begin{aligned}
&P\Big(u(s,x) < \frac{\alpha}{8} I_{(c-M/m,d+M/m)}(x) \text{ for some } \frac{t}{2m} \leq s \leq \frac{t}{m} \text{ and } x \in R\Big) \\
&\quad \leq P\Big(N(s,x) < -\frac{\alpha}{8} \text{ for some } \frac{t}{2m} \leq s \leq \frac{t}{m} \text{ and some } c - \frac{M}{m} \leq x \leq d + \frac{M}{m}\Big) \\
&\quad \leq P\Big(|N(s,x)| > \frac{\alpha}{8} \text{ for some } \frac{t}{2m} \leq s \leq \frac{t}{m} \text{ and some } x \in R\Big) \\
&\quad \leq c_1 16^{24} \exp(-c_2 16^{-2} t^{-1/4} m^{1/4}).
\end{aligned}$$

Thus we get (4.12).

STEP 3. Let us define the events A_k and B_k:

$$A_k = \Big[u(s,x) \geq \beta 8^{-k} I_{(a-Mk/m,b+Mk/m)}(x) \text{ for every } s \in \Big[\frac{2k+1}{2m}t, \frac{k+1}{m}t\Big] \text{ and } x \in R\Big]$$

and

$$B_k = \Big[u(s,x) \geq \beta 8^{-k} I_{(a-Mk/m,b+Mk/m)}(x) \text{ for every } s \in \Big[\frac{k}{m}t, \frac{2k+1}{2m}t\Big] \text{ and } x \in R\Big].$$

We set

$$c(m) = c_1 16^{24} \exp(-c_2 16^{-2} t^{-1/4} m^{1/4}). \tag{4.14}$$

Note that on the event A_{k-1}

$$u\Big(\frac{kt}{m}, x\Big) \geq \beta 8^{-k+1} I_{(a-M(k-1)/m,b+M(k-1)/m)}(x) \quad \text{for } x \in R.$$

By the Markov property, (4.12), (4.14) and Corollary 2.4

$$P(A_k \mid \mathcal{F}_{kt/m}) \geq 1 - c(m) \quad P\text{-a.s. on } A_{k-1} \quad (1 \leq k \leq m/2),$$

which yields

$$P(A_k \mid A_{k-1} \cap \cdots \cap A_0) \geq 1 - c(m) \quad \text{for } 1 \leq k \leq m/2. \tag{4.15}$$

In the same way we have

$$P(B_k \mid B_{k-1} \cap \cdots \cap B_0) \geq 1 - c(m) \quad \text{for } 1 \leq k \leq m/2. \tag{4.16}$$

Noting that

$$\begin{aligned} P\Big(\bigcap_{0\leq k\leq m/2} A_k \cap \bigcap_{0\leq k\leq m/2} B_k\Big) &\geq 1 - \Big(1 - P\Big(\bigcap_{0\leq k\leq m/2} A_k\Big)\Big) - \Big(1 - P\Big(\bigcap_{0\leq k\leq m/2} B_k\Big)\Big) \\ &\geq \big(1 - c(m)\big)^{m/2} P(A_0) + \big(1 - c(m)\big)^{m/2} P(B_0) - 1. \end{aligned}$$

and

$$\lim_{m\to\infty} P(A_0) = \lim_{m\to\infty} P(B_0) = 1 \text{ and } \lim_{m\to\infty} \big(1 - c(m)\big)^{m/2} = 1,$$

we obtain for every $t > 0$ and $M > 0$

$$\begin{aligned} P\Big(u(s,x) > 0 \text{ for every } \frac{t}{4} \leq s \leq \frac{t}{2} \text{ and every } a - \frac{M}{4} \leq x \leq b + \frac{M}{4}\Big) &\geq \lim_{m\to\infty} P\Big(\bigcap_{0\leq k\leq m/2} A_k \cap \bigcap_{0\leq k\leq m/2} B_k\Big) \\ &= 1 \end{aligned}$$

which concludes Theorem 1.3 in the case $b(u) \equiv 0$.

Even if $b(u)$ is not identically zero but a general Lipschitz continuous function, the proof is essentially unchanged. For two C_{tem}-valued solutions $u_1(t,x)$ and $u_2(t,x)$ of (1.1), $u(t,x) = u_1(t,x) - u_2(t,x) \geq 0$ satisfies

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + b\big(t,x,u(t,x)\big) + a\big(t,x,u(t,x)\big)\dot{W}(t,x) \tag{4.17}$$

where $a(s,y,u,\omega)$ and $b(s,y,u,\omega)$: $[0,\infty) \times R \times R \times \Omega \longrightarrow R$ are $\{\mathcal{F}_t\}$-predictable functionals defined by

$$a(s,y,u,\omega) = a\big(u + u_2(s,y,\omega)\big) - a\big(u_2(s,y,\omega)\big) \tag{4.18}$$

$$b(s,y,u,\omega) = b\big(u + u_2(s,y,\omega)\big) - b\big(u_2(s,y,\omega)\big). \tag{4.19}$$

By the Lipschitz continuity of $a(u)$ and $b(u)$, we have a constant $L > 0$ such that

$$|a(s,y,u,\omega)| \leq L|u| \quad \text{for every } (s,y,u,\omega) \in [0,\infty) \times R \times R \times \Omega \tag{4.20}$$

$$|b(s,y,u,\omega) \leq L|u| \quad \text{for every } (s,y,u,\omega) \in [0,\infty) \times R \times R \times \Omega. \tag{4.21}$$

Setting $w(t,x) = e^{Lt}u(t,x)$, we have the following equation.

$$\begin{aligned} w(t,x) = G(t)f(x) &+ \int_0^t \int_R G(t-s,x,y)\Big(Lw(s,y) + e^{Ls} b\big(s,y,e^{-Ls}w(s,y)\big)\Big)\,ds\,dy \\ &+ \int_0^t \int_R G(t-s,x,y) e^{Ls} a\big(s,y,e^{-Ls}w(s,y)\big)\dot{W}(s,y)\,ds\,dy. \end{aligned} \tag{4.22}$$

Note that by (4.18) $Lw(s,y)+e^{Ls}b\bigl(s,y,e^{-Ls}w(s,y)\bigr)\geq 0$ holds and by (4.17) and (4.19)

$$\int_0^t\int_R G(t-s,x,y)e^{Ls}a\bigl(s,y,e^{-Ls}w(s,y)\bigr)\dot{W}(s,x)\,ds\,dy$$

has the same estimate as $N(t,x)$. Accordingly the arguments in the case $b(u)=0$ are still valid for general $b(u)$, completing the proof of Theorem 1.3.

5. **Examples.** We here present two examples of one-dimensional SPDEs occurring in population genetics.

EXAMPLE 1 (GENETICAL DIFFUSION MODEL WITH RANDOM SELECTION). Let us consider the following SPDE:

$$\frac{\partial u(t,x)}{\partial t}=\Delta u(t,x)+u(t,x)\bigl(1-u(t,x)\bigr)\dot{W}(t,x)\quad (t\geq 0,x\in R) \tag{5.1}$$
$$0\leq u(0,x)\leq 1$$

The SPDE (5.1) describes a continuum limit in space of a genetical diffusion model incorporating random selection, where Δ means one-dimensional nearest neighbour migration. Since the coefficient $a(u)=u(1-u)$ is Lipschitz continuous in $0\leq u\leq 1$, by Theorem 2.2 and 2.3, for every $u(0)\in C(R\longrightarrow[0,1])$ the SPDE (5.1) has a unique $C(R\longrightarrow[0,1])$-valued solution $u(t,x)$.

Furthermore, by Corollary 1.4, if f is neither identically 0 nor identically $1, 0<u(t,x)<1$ holds for every $t>0$ and $x\in R$ with probability one.

EXAMPLE 2. We next consider the following SPDE:

$$\frac{\partial u(t,x)}{\partial t}=\Delta u(t,x)+\sqrt{u(t,x)\bigl(1-u(t,x)\bigr)}\,\dot{W}(t,x)\quad (t\geq 0,x\in R) \tag{5.2}$$
$$u(0,x)=f(x).$$

The SPDE (5.2) describes a continuum limit of one-dimensional stepping stone model in population genetics. By Theorem 2.5, for every $u(0,\cdot)=f\in C(R\longrightarrow[0,1])$ there exists a space-time white noise $\dot{W}(t,x)$ and a $C(R\longrightarrow[0,1])$-valued solution $u(t,x)$ of (5.2) on a suitable probability space. Furthermore one can prove the uniqueness of solutions in the law sense by using a duality technique, (*cf.* [12]). In this case the coefficient is not Lipschitz continuous, so that Theorem 1.3 is not applicable. Also the assumption of Theorem 1.1 is not satisfied. It seems a somewhat subtle problem to see whether the SCP property does hold or not for the SPDE (5.2).

6. **Appendix.** The theorems stated in §2 do not seem to be novel, since one can prove them by repeating quite standard arguments in the stochastic calculus. However it would be convenient to present their proofs briefly for selfcontainedness.

We first prepare several lemmas which will be frequently used in the proofs of Theorems 2.1–2.6.

LEMMA 6.1. *Let $\phi(t,x,\omega)\colon [0,\infty)\times R\times\Omega \longrightarrow R$ be a $\{\mathcal{F}_t\}$-predictable functional. Then for each $p>0$ there exists an absolute constant $C_p>0$ such that*

(6.1)
$$E\Big(\Big(\int_0^t\int_R \phi(s,x,\omega)\dot W(s,x)\,ds\,dx\Big)^{2p}\Big)\le C_pE\Big(\Big(\int_0^t\int_R \phi(s,x,\omega)^2\,ds\,dy\Big)^{p}\Big)\quad \text{for } t>0,$$

whenever the stochastic integral is well-defined.

LEMMA 6.2. *(i) There exists a constant $C>0$ such that*

$$(6.2)\quad \int_0^{t\vee t'}\int_R\big(G(t-s,x,y)-G(t'-s,x'y)\big)^2\,ds\,dy \le C(|t-t'|^{1/2}+|x-x'|)\quad \text{for } t,t'\ge 0 \text{ and } x,x'\in R,$$

where $G(t,x,y)=0$ for $t\le 0$.

(ii) For every $\lambda\in R$ and $T>0$,

$$(6.3)\qquad \sup_{0\le t\le T}\sup_{x\in R} e^{-\lambda|x|}\int_R G(t,x,y)e^{\lambda|y|}\,dy<\infty.$$

LEMMA 6.3. *(i) Let $X(t,x)_{0\le t\le 1,x\in R}$ be a two parameter process. Suppose that for every $\lambda>0$ there exist $p>0$, $\gamma>2$ and $C_\lambda>0$ such that*

$$(6.4)\qquad E\big(|X(t,x)-X(t',x')|^{2p}\big)\le C_\lambda(|t-t'|^\gamma+|x-x'|^\gamma)e^{\lambda|x|}$$

for $0\le t,t'\le 1$ and $x\in R$, $x'\in R$ with $|x-x'|\le 1$.

Then $X(t,\cdot)$ has a C_{tem}-valued continuous version.

(ii) Let $X_n(t,\cdot)_{0\le t\le 1,n\ge 1}$ be a sequence of C_{tem}-valued continuous processes. Suppose that for every $\lambda>0$ there exist $p>0$, $\gamma>2$ and $C_\lambda>0$ such that

$$(6.5)\qquad E\big(|X_n(t,x)-X_n(t',x')|^{2p}\big)\le C_\lambda(|t-t'|^\gamma+|x-x'|^\gamma)$$

for $0\le t,t'\le 1$, $x\in R$, $x'\in R$ with $|x-x'|\le 1$, and $n\ge 1$.

Then the sequence of probability distributions on $C([0,1]\longrightarrow C_{\mathrm{tem}})$ induced by $X_n(t,\cdot)$ is tight.

(iii) Let $X(t,\cdot)$ be a C_{tem}-valued continuous process. Suppose that for every $\lambda>0$ there exist $p>0$, $\gamma>2$ and $C_\lambda>0$ such that

$$(6.6)\qquad E\big(|X(t,x)-X(t',x')|^{2p}\big)\le C_\lambda(|t-t'|^\gamma+|x-x'|^\gamma)e^{-\lambda|x|}$$

for $0\le t,t'\le 1$, $x\in R$, $x'\in R$ with $|x-x'|\le 1$. Then $X(t,\cdot)$ is C_{rap}-valued continuous P-a.s.

LEMMA 6.4. *Let $\lambda>0$, and $U(t,x)\colon [0,T]\times R\longrightarrow R_+$ be a measurable function satisfying*

$$(6.7)\qquad \sup_{0\le t\le T}\int_R e^{-\lambda|x|}U(t,x)\,dx<\infty.$$

(i) Suppose for some $C > 0$

(6.8) $$U(t,x) \leq C \int_0^t \int_R (t-s)^{-1/2} G(t-s,x,y) U(s,y)\, ds\, dy \quad \text{for } 0 \leq t \leq T \text{ and } x \in R.$$

Then $U(t,x) = 0$ *for every* $0 \leq t \leq T$ *and* $x \in R$.

(ii) Suppose for a measurable $V(t,x)\colon [0,T] \times R \longrightarrow R_+$ *and some* $C > 0$

(6.9)
$$U(t,x) \leq C \int_0^t \int_R (t-s)^{-1/2} G(t-s,x,y) U(s,y)\, ds\, dy + V(t,x) \quad \text{for } 0 \leq t \leq T \text{ and } x \in R.$$

If $V(t,x)$ *satisfies*

(6.10) $$\sup_{0 \leq t \leq T} \int_R e^{\lambda|x|} V(t,x)\, dx < \infty,$$

then

(6.11) $$\sup_{0 \leq t \leq T} \int_R e^{\lambda|x|} U(t,x) < \infty.$$

Lemma 6.1 follows from the martingale inequality, and Lemma 6.2 can be proved by straightforward calculations. Lemma 6.3 is a variant of Kolmogorov-Totoki's theorem (*cf.* [13]), and Lemma 6.4 is a sort of Gronwall's inequality, so their proofs are omitted.

PROOF OF THEOREM 2.1. Let

$$C^2_{\mathrm{rap}} = \{\varphi \in C^2(R) \mid \varphi, \varphi' \text{ and } \varphi'' \text{ are in } C_{\mathrm{rap}}\}$$

equipped with the topology induced by a family of norms $\{|\cdot|_{(\lambda,2)} : \lambda > 0\}$:

$$|\varphi|_{(\lambda,2)} = |\varphi|_{(\lambda)} + |\varphi'|_{(\lambda)} + |\varphi''|_{(\lambda)},$$

and for $T > 0$ let $D^2_{\mathrm{rap}}(T) = \big\{ f \in C^{1,2}\big([0,T) \times R\big) \mid f(t,\cdot)$ is C^2_{rap}-valued continuous and $\frac{\partial f}{\partial t}(t,\cdot)$ is C_{rap}-valued continuous in $0 \leq t < T \big\}$.

1°. Notice that $C^\infty_c(R)$ is dense in C^2_{rap}.

2°. It is not difficult to show that the equation (2.1)′ holds for every $\varphi \in C^2_{\mathrm{rap}}$. (Use 1°, C_{tem}-valued continuity of $u(t)$, and a truncation method by stopping times.)

3°. Next we show that for every $\phi \in D_{\mathrm{rap}}(T)$ and $0 < t < T$. Using 2°, we have

(6.12)
$$\begin{aligned} \langle u(t), \phi(t) \rangle = \langle f, \phi(0) \rangle &+ \int_0^t \Big(\Big\langle u(s), \Big(\frac{\partial}{\partial s} + \Delta\Big)\phi(s) \Big\rangle + \Big\langle b\big(s,\cdot,u(s,\cdot)\big), \phi(s) \Big\rangle \Big)\, ds \\ &+ \int_0^t \int_R a\big(s,x,u(s,x)\big) \phi(s,x) \dot W(s,x)\, ds\, dx. \end{aligned}$$

Let $\Delta = \{t_0 = 0 < t_1 < \cdots < t_N = t\}$ with $|\Delta| = \max_{1 \leq i \leq N} |t_i - t_{i-1}|$, and define functions $\pi_\Delta(s)$ and $\bar\pi_\Delta(s)$ by $\pi_\Delta(s) = t_{i-1}$ and $\bar\pi_\Delta(s) = t_i$ for $t_{i-1} \leq s < t_i$.

(6.13)
$$\begin{aligned} &\langle u(t), \phi(t) \rangle - \langle f, \phi(0) \rangle \\ &\quad = \sum_{i=1}^N \Big(\big\langle u(t_i), \phi(t_i) - \phi(t_{i-1}) \big\rangle + \big\langle u(t_i) - u(t_{i-1}), \phi(t_{i-1}) \big\rangle \Big) \\ &\quad = \int_0^t \Big(\Big\langle u\big(\bar\pi_\Delta(s)\big), \frac{\partial \phi}{\partial s}(s) \Big\rangle + \Big\langle u(s), \Delta\phi\big(\pi_\Delta(s)\big) \Big\rangle + \Big\langle b\big(s,\cdot,u(s,\cdot)\big), \phi\big(\pi_\Delta(s)\big) \Big\rangle \Big)\, ds \\ &\qquad + \int_0^t \int_R a\big(s,x,u(s,x)\big) \phi\big(\pi_\Delta(s), x\big) \dot W(s,x)\, ds\, dx. \end{aligned}$$

It is easy to see that the first term of the r.h.s. of (6.13) converges as $|\Delta| \longrightarrow 0$ to the second term of the r.h.s. of (6.12), so it suffices to show that the second term of the r.h.s. of (6.13) converges in probability as $|\Delta| \longrightarrow 0$ to the third term of the r.h.s. of (6.12). For this let any $\lambda > 0$ be fixed, and for $M > 0$ let $\tau_M = \inf\{s \geq 0 \mid |u(s)|_{(-\lambda)} \geq M\}$. Note that $\tau_M \longrightarrow \infty$ as $M \longrightarrow \infty$ by the continuity of $u(s)$, and that

(6.14)

$$\begin{aligned} E\Big(\Big(\int_0^{t\wedge\tau_M}\int_R a\big(s,x,u(s,x)\big)\phi\big(\pi_\Delta(s),x\big)\dot W(s,x)\,ds\,dx \\ -\int_0^{t\wedge\tau_M}\int_R a\big(s,x,u(s,x)\big)\phi(s,x)\dot W(s,x)\,ds\,dx\Big)^2\Big) \\ = E\Big(\int_0^{t\wedge\tau_M}\int_R a\big(s,x,u(s,x)\big)^2\Big(\phi\big(\pi_\Delta(s),x\big)-\phi(s,x)\Big)^2\,ds\,dx\Big) \\ \leq C_T^2(1+M)^2\sup_{0\leq s\leq t}\Big|\frac{\partial\phi}{\partial s}(s)\Big|_{(2\lambda)}^2 t/\lambda|\Delta|^2, \end{aligned}$$

which vanishes as $|\Delta| \longrightarrow 0$. Hence the second term of the r.h.s. of (6.13) converges in probability to the third term of the r.h.s. of (6.12) as $|\Delta| \longrightarrow 0$; thus we have shown that (6.12) holds for every $\phi \in D^2_{\mathrm{rap}}(T)$.

4°. Let $\phi^a_T(t,x) = G(T-t,a,x)$. Then $\phi^a_T \in D^2_{\mathrm{rap}}(T)$, so by 3° and

$$\frac{\partial\phi}{\partial t}(t,x)+\Delta\phi(t,x)=0 \quad \text{for } 0\leq t<T \text{ and } x\in R,$$

we get

$$\begin{aligned} G(T-t)\,u(t)(a) = G(T)f(a)+\int_0^t\int_R G(T-s,a,x)\,b\big(s,x,u(s,x)\big)\,ds \\ +\int_0^t\int_R G(T-s,a,x)\,a\big(s,x,u(s,x)\big)\dot W(s,x)\,ds\,dx \quad \text{for } a\in R. \end{aligned}$$

Hence letting $t \longrightarrow T$, we see that $u(t,x)$ satisfies the SIE (2.3).

5°. Conversely, let $u(t)$ be an $\{\mathcal{F}_t\}$-predictable and C_{tem}-valued continuous process which satisfies the SIE (2.3). To see that $u(t)$ satisfies (2.1)′ insert (2.3) into $\int_0^t\langle u(s),\Delta\varphi\rangle\,ds$. Then a key part is to show

$$\begin{aligned} \int_0^t\int_R\Big(\int_0^s\int_R G(s-r,x,y)\,a\big(r,y,u(r,y)\big)\dot W(r,y)\,dr\,dy\Big)\Delta\varphi(x)\,ds\,dx \\ = \int_0^t\int_R\Big(\int_r^t G(s-r)(\Delta\varphi)(y)\,ds\Big)a\big(s,y,u(s,y)\big)\dot W(r,y)\,dr\,dy \\ = \int_0^t\int_R\big(G(t-r)\varphi(y)-\varphi(y)\big)\,a\big(r,y,u(r,y)\big)\dot W(r,y)\,dr\,dy, \end{aligned}$$

which is justified using a stochastic Fubini theorem (*cf.* Lemma 2.4 in [7]). Thus we can prove Theorem 2.1.

PROOF OF THEOREM 2.2. 1°. Consider Picard's iterative approximation $\{u_n(t,x)\}$ for the equation (1.2); $u_0(t,x) = G(t)f(x)$ and

$$\begin{aligned} u_{n+1}(t,x) = G(t)f(x)+\int_0^t\int_R G(t-s,x,y)\,b\big(s,y,u_n(s,y)\big)\,ds\,dy \\ +\int_0^t\int_R G(t-s,x,y)\,a\big(s,y,u_n(s,y)\big)\dot W(s,y)\,ds\,dy. \end{aligned} \tag{6.15}$$

Using Lemma 6.1 and 6.2 together with Hölder's inequality, one can easily show that $\sup_{0\le t\le T}\int_R e^{-\lambda|x|}E|u_n(t,x)|^{2p}\,dx<\infty$ for every $\lambda>0$, $p>0$, $T>0$ and $n\ge 0$, and

$$\text{(6.16)}\quad \int_R e^{-\lambda|x|}E|u_{n+1}(t,x)|^{2p}\,dx\le C_{p,T,\lambda}\Big(1+\int_0^t\int_R (t-s)^{1/2}e^{-\lambda|x|}E|u_n(s,y)|^{2p}\,ds\,dx\Big).$$

It follows from (6.16) that for every $\lambda>0$, $p>0$ and $T>0$

$$\text{(6.17)}\qquad \sup_{n\ge 1}\sup_{0\le t\le T}\int_R e^{-\lambda|x|}E|u_n(t,x)|^{2p}\,dx<\infty.$$

Also, by (2.4) and Hölder's inequality,

(6.18)

$$\begin{aligned}&\int_R e^{-\lambda|x|}E|u_{n+1}(t,x)-u_n(t,x)|^{2p}\,dx\\&\quad\le C_{p,T,\lambda}\int_0^t\int_R (t-s)^{-1/2}e^{-\lambda|x|}E|u_n(s,x)-u_{n-1}(s,x)|^{2p}\,ds\,dx\quad\text{for } 0\le t\le T,\end{aligned}$$

so that there exists an $\{\mathcal{F}_t\}$-predictable functional $u(t,x,\omega)$ such that

$$\text{(6.19)}\qquad \sup_{0\le t\le T}\int_R e^{-\lambda|x|}E\big(|u(t,x)|^{2p}\big)\,dx<\infty\quad\text{for every }\lambda>0,\ p>0\text{ and }T>0,$$

$$\lim_{n\to\infty}\sup_{0\le t\le T}\int_R e^{-\lambda|x|}E|u_n(t,x)-u(t,x)|^{2p}\,dx=0\quad\text{for }\lambda>0,\ p>0\text{ and }T>0,$$

and $u(t,x)$ satisfies

$$\text{(6.20)}\qquad \begin{aligned}u(t,x)=G(t)f(x)&+\int_0^t\int_R G(t-s,x,y)\,b\big(s,y,u(s,y)\big)\,ds\,dy\\&+\int_0^t\int_R G(t-s,x,y)\,a\big(s,y,u(s,y)\big)\dot W(s,y)\,ds\,dy\end{aligned}$$

P-a.s. for almost every $t\ge 0$ and $x\in R$.

2°. Let

$$X(t,x)=\int_0^t\int_R G(t-s,x,y)\,a\big(s,y,u(s,y)\big)\dot W(s,y)\,ds\,dy.$$

Using Hölder's inequality, Lemma 6.1 and (2.2), we see that for $p>1$ and $q=p/(p-1)$,

$$\begin{aligned}&E\big(|X(t,x)-X(t',x')|^{2p}\big)\\&\quad\le C_{p,T}\Big(\int_0^{t\vee t'}\int_R\big(G(t-s,x,y)-G(t'-s,x',y)\big)^2\,ds\,dy\Big)^{p/q}\\&\qquad\times\int_0^{t\vee t'}\int_R\big(G(t-s,x,y)-G(t'-s,x,y)\big)^2\Big(1+E\big(|u(s,y)|^{2p}\big)\Big)\,ds\,dy,\end{aligned}$$

so, by Lemma 6.2 and (6.19) one can check the moment condition (6.4) for $X(t,x)$.

On the other hand it is easy to see that $Y(t,x)=\int_0^t\int_R G(t-s,x,y)\,b\big(s,y,u(s,y)\big)\,ds\,dy$ is C_{tem}-valued continuous. Hence $u(t,\cdot)$ indeed has a C_{tem}-valued continuous version which is a C_{tem}-valued solution of the SPDE (2.1).

3°. We next show the uniqueness of C_{tem}-valued solutions of (2.1). Let $u(t,\cdot)$ and $v(t,\cdot)$ be two C_{tem}-valued solutions of (2.1), and for an arbitrarily fixed $\lambda>0$ and $n\ge 1$ set

$$\sigma_n=\inf\{t\ge 0\mid |u(t)|_{(-\lambda/3)}\ge n\text{ or }|v(t)|_{(-\lambda/3)}\ge n\}.$$

Note that $w_n(t,\cdot) = \big(u(t,\cdot) - v(t,\cdot)\big)I(t < \sigma_n)$ satisfies $|w_n(t,x)| \le 2ne^{\lambda|x|/3}$ and

(6.21)

$$w_n(t,x) = I(t<\sigma_n)\int_0^t \int_R G(t-s,x,y)I(s<\sigma_n)\Big(b\big(s,y,u(s,y)\big) - b\big(s,y,v(s,y)\big)\Big)\,ds\,dy$$
$$+ I(t<\sigma)\int_0^t \int_R G(t-s,x,y)I(s<\sigma_n)$$
$$\Big(a\big(s,y,u(s,y)\big) - a\big(s,y,v(s,y)\big)\Big)\dot W(s,y)\,ds\,dy.$$

Using the condition (2.4) and a similar argument to get (6.18), we have $C_{\lambda,T} > 0$ such that

(6.22)

$$\int_R e^{-\lambda|x|}E|w_n(t,x)|^2\,dx \le C_{\lambda,T}\int_0^t ds\,(t-s)^{-1/2}\int_R e^{-\lambda|x|}E|w_n(s,x)|^2\,dx \quad \text{for } 0 \le t \le T,$$

hence by Lemma 6.4, $P\big(u(t,\cdot) = v(t,\cdot) \text{ for } 0 \le t < \sigma_n\big) = 1$ holds. Finally, since $u(t,\cdot)$ and $v(t,\cdot)$ are C_{tem}-valued continuous, $\sigma_n \longrightarrow \infty$ as $n \longrightarrow \infty$ P-a.e., and $u(t,\cdot) = v(t,\cdot)$ holds for every $t \ge 0$ with probability one. Thus the proof of Theorem 2.2 is complete.

PROOF OF THEOREM 2.3. 1°. For $\varepsilon > 0$, choose a nonnegative and symmetric C^∞-function $\rho_\varepsilon(x)$ defined on R satisfying

$$\rho_\varepsilon(x) = 0 \quad \text{for } |x| \ge \varepsilon \text{ and } \int_R \rho_\varepsilon^2(x) = 1.$$

Define a spatially correlated noise $\dot W_x^\varepsilon(t)$ $(x \in R)$ by

$$\dot W_x^\varepsilon(t) = \int_R \rho_\varepsilon(x-y)\dot W(t,y)\,dy. \tag{6.23}$$

Note that for each $x \in R$, $W_x^\varepsilon(t) = \int_0^t \dot W_x^\varepsilon(s)\,ds$ is a one-dimensional Brownian motion.

2°. Setting $\Delta_\varepsilon = \big(G(\varepsilon) - I\big)/\varepsilon$ for $\varepsilon > 0$, consider the following equation:

(6.24)

$$u_\varepsilon(t,x) = f(x) + \int_0^t \Big(\Delta_\varepsilon u_\varepsilon(s,x) + b\big(s,x,u_\varepsilon(s,x)\big)\Big)\,ds + \int_0^t a\big(s,x,u_\varepsilon(s,x)\big)\,dW_x^\varepsilon(s) \quad (x \in R),$$

where the last term is a standard one-dimensional stochastic integral.

Assume that for every $T > 0$ there exists a $C_T > 0$ such that

$$|a(t,x,u,\omega) - a(t,x',u',\omega) + |b(t,x,u,\omega) - b(t,x',u',\omega)| \le C_T(|u-u'| + |x-x'|) \tag{6.25}$$

for $0 \le t \le T$, $x, x', u, u' \in R$, P-a.s.

Then one can show that for every $f \in C_{\mathrm{tem}}$, (6.24) has a unique C_{tem}-valued solution $u_\varepsilon(t,x)$.

3°. We may assume $u(0) = f$ is bounded; otherwise it can be reduced to this case by a standard approximation procedure.

4°. We claim

$$P\big(u_\varepsilon(t,\cdot) \ge 0 \text{ for every } t \ge 0\big) = 1. \tag{6.26}$$

Noting that for each $x \in R$, $u_\varepsilon(t,x)$ is a semi-martingale, apply Ito's formula with a function $\varphi(u) = -\min\{u,0\}$, being approximated by smooth functions as in a one-dimensional case (*cf.* [5], p. 437). Since $b(s,x,u,\omega) \geq -L_T|u|$ by (2.4) and (2.6), for $0 \leq t \leq T$ and $x \in R$ we see

$$
\begin{aligned}
E\Big(\varphi\big(u_\varepsilon(t,x)\big)\Big) &= -\int_0^t E\Big(I\big(u_\varepsilon(s,x) \leq 0\big)\Big(\Delta_\varepsilon u_\varepsilon(s,x) + b\big(s,x,u_\varepsilon(s,x)\big)\Big)\Big)\,ds \\
&\leq (L_T + 1/\varepsilon)\int_0^t E\Big(\varphi\big(u_\varepsilon(s,x)\big)\Big)\,ds \\
&\quad + (1/\varepsilon)\int_0^t\int_R G(\varepsilon,x,y)\,E\Big(\varphi\big(u_\varepsilon(s,y)\big)\Big)\,ds\,dy.
\end{aligned} \tag{6.27}
$$

Hence, by Gronwall's lemma for $\sup_{x\in R} E(\varphi(u_\varepsilon(t,x))$, $E(\varphi(u_\varepsilon(t,x)) = 0$ for every $t \geq 0$ and $x \in R$, which yields (6.26).

5°. Let

$$
\begin{aligned}
G_\varepsilon(t) &= \exp t\Delta_\varepsilon = e^{-t/\varepsilon}\sum_{n=0}^{\infty}\frac{(t/\varepsilon)^n}{n!}G(n\varepsilon) \equiv e^{-t/\varepsilon}I + R_\varepsilon(t), \\
R_\varepsilon(t,x,y) &= e^{-t/\varepsilon}\sum_{n=1}^{\infty}\frac{(t/\varepsilon)^n}{n!}G(n\varepsilon,x,y).
\end{aligned} \tag{6.28}
$$

We use the following estimates, which are checked by elementary calculations.

LEMMA 6.6. *(i)*

$$\int_R R_\varepsilon(t,x,y)^2\,dy \leq \sqrt{\frac{3}{8\pi}}\,t^{-1/2},$$

(ii) For some $\alpha > 0$ and $\beta > 0$

$$\int_R |R_\varepsilon(t,x,y) - G(t,x,y)|\,dy \leq e^{-t/\varepsilon} + \alpha(\varepsilon/t)^{1/3} \quad \text{if } 0 < \varepsilon/t \leq \beta.$$

(iii)

$$\lim_{\varepsilon\to 0}\int_0^t\int_R\big(R_\varepsilon(s,x,y) - G(s,x,y)\big)^2\,ds\,dy = 0 \quad \text{for } t > 0 \text{ and } x \in R.$$

6°. Let $u(t,x)$ be the unique C_{tem}-valued solution of the SPDE (2.1). $u(t,x)$ and $u_\varepsilon(t,x)$ satisfy

$$
\begin{aligned}
u(t,x) &= G(t)f(x) + \int_0^t\int_R G(t-s,x,y)\,b\big(s,y,u(s,y)\big)\,ds\,dy \\
&\quad + \int_0^t\int_R G(t-s,x,y)\,a\big(s,y,u(s,y)\big)\dot{W}(s,y)\,ds\,dy.
\end{aligned} \tag{6.29}
$$

and

$$
\begin{aligned}
u_\varepsilon(t,x) &= G_\varepsilon(t)f(x) + \int_0^t\int_R G_\varepsilon(t-s,x,dy)\,b\big(s,y,u_\varepsilon(s,y)\big)\,ds \\
&\quad + \int_0^t e^{-(t-s)/\varepsilon}a\big(s,x,u_\varepsilon(s,x)\big)\,dW_x^\varepsilon(s) \\
&\quad + \int_0^t\int_R R_\varepsilon(t-s,x,y)\,a\big(s,y,u_\varepsilon(s,y)\big)\,dW_y^\varepsilon(s)\,dy.
\end{aligned} \tag{6.30}
$$

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Since f is bounded, it follows from (6.29), (6.30) and Lemma 6.6 that for every $T > 0$,

$$\sup_{0<\varepsilon\le 1}\ \sup_{0\le t\le T}\ \sup_{x\in R} E\big(|u_\varepsilon(t,x)|^2\big) < \infty, \tag{6.31}$$

$$\sup_{0\le t\le T}\ \sup_{x\in R} E\big(|u(t,x)|^2\big) < \infty. \tag{6.32}$$

7°. By (6.29), (6.30) and Lemma 6.1, for some $C > 0$

(6.33)

$$\begin{aligned}
&E\big(|u_\varepsilon(t,x) - u(t,x)|^2\big)\\
&\le C\Big\{ |G_\varepsilon(t)f(x) - G(t)f(x)|^2 + E\Big(\Big|\int_0^t e^{-(t-s)/\varepsilon} b\big(s,x,u_\varepsilon(s,x)\big)\,ds\Big|^2\Big)\\
&\quad + E\Big(\Big|\int_0^t\int_R R_\varepsilon(t-s,x,y)\Big(b\big(s,y,u_\varepsilon(s,y)\big) - b\big(s,y,u(s,y)\big)\Big)\,ds\,dy\Big|^2\Big)\\
&\quad + E\Big(\Big|\int_0^t\int_R \big(R_\varepsilon(t-s,x,y) - G(t-s,x,y)\big) b\big(s,y,u(s,y)\big)\,ds\,dy\Big|^2\Big)\\
&\quad + \int_0^t e^{-2(t-s)/\varepsilon} E\Big(a\big(s,x,u_\varepsilon(s,x)\big)^2\Big)\,ds\\
&\quad + \int_0^t\int_R E\Big(\Big|\int_R R_\varepsilon(t-s,x,z)\Big(a\big(s,z,u_\varepsilon(s,z)\big) - a\big(s,z,u(s,z)\big)\Big)\rho_\varepsilon(y-z)\,dz\Big|^2\Big)\,ds\,dy\\
&\quad + \int_0^t\int_R E\Big(\Big|\int_R R_\varepsilon(t-s,x,z)\Big(a\big(s,z,u(s,z)\big) - a\big(s,y,u(s,y)\big)\Big)\rho_\varepsilon(y-z)\,dz\Big|^2\Big)\,ds\,dy\\
&\quad + \int_0^t\int_R \Big(\int_R R_\varepsilon(t-s,x,z)\rho_\varepsilon(y-z)\,dz - G(t-s,x,y)\Big)^2 E\Big(a\big(s,y,u(s,y)\big)^2\Big)\,ds\,dy\Big\}\\
&\equiv \sum_{k=1}^{8} J_k(\varepsilon,t,x).
\end{aligned}$$

Using (2.2), (6.25), (6.30), (6.31) and Lemma 6.6, for every $T > 0$ we have a $C_T > 0$ such that

$$J_3(\varepsilon,t,x) + J_6(\varepsilon,t,x) \le C_T \int_0^t (t-s)^{-1/2} \sup_{y\in R} E\big(|u_\varepsilon(s,y) - u(s,y)|^2\big)\,ds \quad \text{for } 0\le t\le T \text{ and } x\in R, \tag{6.34}$$

and

$$\lim_{\varepsilon\to 0}\ \sup_{0\le t\le T}\ \sup_{x\in R} J_k(\varepsilon,t,x) = 0 \quad \text{for } 1\le k\le 8 \text{ with } k\ne 3,6. \tag{6.35}$$

Thus, setting $U(\varepsilon,t) = \sup_{x\in R} E\big(|u_\varepsilon(t,x) - u(t,x)|^2\big)$, we have an $H(\varepsilon,t)$ such that

$$U(\varepsilon,t) \le C_T \int_0^t (t-s)^{-1/2} U(\varepsilon,s)\,ds + H(\varepsilon,t) \quad (0\le t\le T), \tag{6.36}$$

$$\lim_{\varepsilon\to 0}\ \sup_{0\le t\le T} H(\varepsilon,t) = 0. \tag{6.37}$$

Hence, by Gronwall's lemma we get

$$\lim_{\varepsilon\to 0}\ \sup_{0\le t\le T}\ \sup_{x\in R} E\big(|u_\varepsilon(t.x) - u(t,x)|^2\big) = 0 \quad \text{for every } T > 0. \tag{6.38}$$

8°. By (6.26) and (6.38) the nonnegativity of $u(t,x)$ is inherited from that of $u_\varepsilon(t,x)$. Finally, it is a routine task to relax the assumption (6.25) to the continuity condition (2.5), so it is omitted. Thus the proof of Theorem 2.3 is complete.

PROOF OF THEOREM 2.5. 1°. By (2.11),

(6.39)
$$E\big(u(t,x)\big) = G(t)f(x) + \int_0^t \int_R G(t-s,x,y)\,E\Big(b\big(s,y,u(s,y)\big)\Big)\,ds\,dy \le G(t)f(x) + C_T\int_0^t\int_R G(t-s,x,y)E\big(u(s,y)\big)\,ds\,dy \quad \text{for } 0\le t\le T \text{ and } x\in R.$$

Since $f\in C^+_{\text{rap}}$, it follows from Lemma 6.2 and 6.4 that for every $\lambda>0$ and $T>0$,

$$\sup_{0\le t\le T}\int_R e^{\lambda|x|}E\big(u(t,x)\big)\,dx<\infty. \tag{6.40}$$

2°. Using Lemma 6.1 and Hölder's inequality, for every $T>0$ and $p>0$ we have a $C_{T,p}>0$ satisfying

(6.41)
$$E\big(u(t,x)^{2p}\big)\le C_{T,p}\Big\{\big(G(t)f(x)\big)^{2p} + \int_0^t\int_R (t-s)^{-1/2}G(t-s,x,y)\Big(E\big(u(s,y)^{2p}\big)+E\big(u(s,y)^{2p\theta}\big)\Big)\,ds\,dy\Big\}$$

for $0\le t\le T$ and $x\in R$.

3°. Suppose that for $p>0$, $\lambda>0$ and $T>0$,

$$\sup_{0\le t\le T}\int_R e^{\lambda|x|}E\big(u(t,x)^{2p\theta}\big)\,dx<\infty. \tag{6.42}$$

Then Lemma 6.4 is applicable for (6.40); hence it follows

$$\sup_{0\le t\le T}\int_R e^{\lambda|x|}E\big(u(t,x)^{2p}\big)\,dx<\infty. \tag{6.43}$$

Accordingly, by an induction argument starting at (6.43) and Hölder's inequality together with (6.41) and (6.42), (6.43) holds for every $\lambda>0$, $p>0$ and $T>0$.

4°. Using (6.43) and Hölder's inequality, one can check the moment condition (6.6) for

$$X(t,x)=\int_0^t\int_R G(t-s,x,y)\,a\big(s,y,u(s,y)\big)\dot W(s,y)\,ds\,dy;$$

hence $X(t,\cdot)$ is C_{rap}-valued continuous.

Moreover, by (6.42) and (2.11) $Y(t,\cdot)=\int_0^t\int_R G(t-s,\cdot,y)\,b\big(s,y,u(s,y)\big)\,ds\,dy$ is C_{rap}-valued continuous; hence $u(t,\cdot)$ is C_{rap}-valued continuous in $t\ge 0$ P-a.s., completing the proof of Theorem 2.5.

PROOF OF THEOREM 2.6. 1°. Choose two sequences of Lipschitz continuous functions $\{a_n(u)\}$ and $\{b_n(u)\}$ such that

$$|a_n(u)|+|b_n(u)|\le K(1+|u|)\quad\text{for } u\in R \text{ and } n\ge 1, \tag{6.44}$$

$$a_n(0)=0,\quad b_n(0)\ge 0\quad\text{for } n\ge 1 \tag{6.45}$$

and that $\{a_n(u)\}$ and $\{b_n(u)\}$ converge to $a(u)$ and $b(u)$ uniformly in $u \in R$ as $n \longrightarrow \infty$. Then by Theorem 2.3 there exists a unique C^+_{tem}-valued solution $u_n(t,\cdot)$ of the SPDE (1.1) with $a_n(u)$ and $b_n(u)$ for each $n \geq 1$. Using (6.44) and (6.45) and the same arguments as Theorem 2.2 together with Lemma 6.3, one can check the moment condition (6.5), so that the family of probability distributions on $C([0,T \longrightarrow C^+_{\text{tem}})$ induced by $\{u_n(t,\cdot)\}$ is tight.

2°. It is a routine task to see that any limit point of the family is realized as a C_{tem}-valued solution of the SPDE (1.1) with $a(u)$ and $b(u)$, thus we complete the proof of Theorem 2.6.

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