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REGULAR-NILPOTENT GROUPS OF AUTOMORPHISMS

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Abstract. Using the definition of *regular p-group* given by M. Hall [**1**], a new class of finite groups called *regular-nilpotent* has been defined. The action of these groups as automorphisms of compact Riemann surfaces has been investigated. It is proved that a necessary and sufficient condition for a Fuchsian group to cover a regular-nilpotent group is that its orbit genus be zero and its periods satisfy the *least common multiple condition*, first defined by Harvey [**2**] and Maclachlan [**4**].

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1. Introduction. This paper is another sequel to [**6**] in which I used the notion of p-localization defined in [**3**] to obtain the best possible bound 16 $(g-1)$ for the order of a finite nilpotent group acting as the group of automorphisms of a Riemann surface of genus $g \geq 2$. Marshall Hall [**1**] has defined a particular class of p-groups called *regular p-groups*. These p-groups have the property that, for any two elements a, b and any integer of the form $n = p^\alpha$, the identity $(ab)^n = a^n b^n S_1^n S_2^n \cdots S_t^n$ is satisfied. Here $S_1, S_2, \ldots, S_t$ are appropriate elements from the commutator subgroup of the group generated by a and b. In this paper, we define yet another class of finite groups called *regular-nilpotent*: namely, the nilpotent groups all of whose p-Sylow subgroups are regular p-groups. After establishing the existence of such finite groups, we then look for necessary and sufficient conditions under which this particular type of nilpotent group is covered by a co-compact Fuchsian group. We also generalize a property, described in [**8**], of 3-groups of automorphisms of Riemann surfaces to any other odd prime. More precisely, we shall show that for any odd prime $p \geq 3$, the smallest p-group covered by the Fuchsian group $(0; p, p, p^2)$ is an irregular group C_p wr C_p of order p^{p+1}. Here wr denotes wreath product.

2. Notation and terminology. Let P be a finite p-group whose order is p^n, where p is a prime. If P is of class less than p and $n = p^\alpha$, $P = \langle a_1, a_2, \ldots, a_r \rangle$, then we have

$$(a_1 a_2 \cdots a_r)^n = a_1^n a_2^n \cdots a_r^n S_1^n S_2^n \cdots S_t^n,$$

where $S_1, S_2, \ldots, S_t$ are elements of the commutator subgroup of the group P

Part (A). Known facts about regular p-groups.
1. Every p-group of class less than p is regular.
2. Every p-group of order at most p^p is regular.

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3. The group P is regular if every subgroup generated by two elements in P is regular.

4. Every subgroup and every factor group of a regular p-group are also regular.

5. If P is a finite p-group, P is regular if and only if for any a, b in P we have $(ab)^p = a^p b^p S^p$, where $S \in [P_1, P_1]$ and $P_1 = \langle a, b\rangle$.

6. If P is regular with $n = p^\alpha$, then the following conditions are satisfied.

(i) $[a^n, b] = 1$ if and only if $[a, b]^n = 1$, where $a \in P, b \in P$.

(ii) If $[a^n, b] = 1$, then $[a, b^n] = 1$.

(iii) A commutator S involving an element u has order at most that of u modulo the centre of P.

(iv) The order of the elements $a_1 a_2 \cdots a_r$ is at most $\max\{|a_i| : 1 \le i \le r\}$.

7. If in a regular p-group we have $a^n = b^n = 1$, then ab has order at most n.

Part (B). Well-known facts about Fuchsian groups and p-localization.

1. A Fuchsian group is a discrete subgroup Γ of orientation preserving isometries of the upper half-plane D with hyperbolic structure. Moreover, if D/Γ is a compact surface, then Γ has the following presentation:

$$\Gamma(S) = \left\langle x_1, x_2, \ldots, x_s, a_1, b_1, \ldots, a_g b_g \,\middle|\, x_i^{m_i}(i = 1, \ldots, s), \prod_{i=1}^{s} x_i \prod_{j=1}^{g} [a_j, b_j] \right\rangle$$

where $S = (g : m_1, m_2, \ldots, m_s)$ is the signature of the group Γ. The integers $m_1, m_2, \ldots, m_s$ are the periods of Γ and g is its orbit genus. The groups of automorphisms of compact Riemann surfaces are the quotient groups of Fuchsian groups. Every Fuchsian group has an associated fundamental region whose hyperbolic area $\mu(\Gamma)$ depends only on the group itself. Suppose that Γ has the signature S defined above. Then

$$\mu(\Gamma) = 2\pi\left[(2g - 2) + \sum_{k=1}^{s}\left(1 - m_k^{-1}\right)\right].$$

2. If Γ_1 is a subgroup of finite index in the group $\Gamma(S)$, where S is a non-degenerate signature, then there is a signature S_1 such that $\Gamma_1 \cong \Gamma(S_1)$ and we have the following Riemann–Hurwitz index formula

$$[\Gamma : \Gamma_1] = \mu(\Gamma(S_1))/\mu(\Gamma(S)).$$

3. It is well known that a compact Riemann surface of genus $g \ge 2$ can be represented as the quotient group D/Γ, where Γ is a Fuchsian group with signature $(g; -)$ called the *surface group* of genus g. Here – denotes the empty set of periods.

4. A finite group G acts as the group of automorphisms of a given surface group if and only if there is a Fuchsian group Π and a homomorphism Φ from Π onto G having Γ as its kernel. Such a homomorphism will be smooth; (that is, has torsion-free kernel). Also Π admits G as a smooth factor group. Moreover, the homomorphism $\Phi : \Pi \to G$ is smooth if and only if it preserves the periods of the Fuchsian group Π.

5. Suppose that $S = (g; m_1, m_2, \ldots, m_s)$ is the signature of the Fuchsian group Γ. Let α_i be the largest number such that $p^{\alpha_i} \mid m_i (i = 1, \ldots, s)$. The signature

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$S_p = (g; p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_s})$ is called the *p-localization signature* of S. We have

$$\Gamma(S_p) = \left\langle x_i', \ldots, x_s', a_1', b_1', \ldots, a_g', b_g' \middle| x_1'^{p^{\alpha_1}}, \ldots, x_s'^{p^{\alpha_s}}, \prod_{i=1}^{s} x_i' \prod^{g_{j=1}} \left(a_j' b_j' a_j'^{-1} b_j'^{-1}\right) \right\rangle.$$

6. A signature S is called *p-local* if every period of S is already a power of the same prime p so that $S = S_p$. We also call the homomorphism $l_p : \Gamma(S) \to \Gamma(S_p)$, obtained by extending the function defined on the generating sets by

$$x_i \to x_i', \quad a_j \to a_j', \quad b_k \to b_k' \quad (i = 1, 2, \ldots, s; j, k = 1, \ldots, g),$$

the *p-localization homomorphism*.

7. Each smooth homomorphism $\Phi : \Gamma(S) \to G = G_{p_1} \times G_{p_2} \times \cdots G_{p_s}$ from $\Gamma(S)$ onto the finite nilpotent group G that can be written as the direct product of its Sylow subgroups $G_{pi} (i = 1, 2, \ldots, s)$ determines a set of homomorphisms of the form $\Phi_{p_i} : \Gamma(S_{p_i}) \to G_{p_i}$ such that if $y \in \Gamma(S)$ and $g_i = \Phi_{p_i}(l_{p_i}(y))$, then we have $\Phi(y) = g_1 g_2 \cdots g_s$. Therefore, we can obtain all possible smooth homomorphisms from the Sylow p-subgroups of G.

3. Regular-nilpotent groups of automorphisms.

We shall now introduce and study the action of a class of finite nilpotent groups that we call *regular-nilpotent* as automorphisms of a compact Riemann surface of genus $g \geq 2$.

Definition 3.1. P is said to be a *regular p-group* if for any pair of elements a, b in P and $n = p^\alpha$ we have

$$(ab)^n = a^n b^n S_1^n S_2^n \cdots S_t^n,$$

where the S_i are elements of the commutator subgroup $[P_1, P_1]$ of the group $P_1 = \langle a, b \rangle$.

Definition 3.2. A finite group G is called *regular-nilpotent* if it is nilpotent and all of its Sylow subgroups are regular p-groups.

Theorem 3.1. *Let $S_p = (g; p^{\alpha_1} p^{\alpha_2}, \ldots, p^{\alpha_s})$ be a p-local signature. The Fuchsian group $\Gamma(S_p)$ covers a regular p-group G_p if S_p has orbit genus $g = 0$ and its periods are all equal; that is, $\alpha_1 = \alpha_2 = \cdots = \alpha_s = \alpha$.*

Proof. By Corollary 6.7 of **[3]**, $\Gamma(S_p)$ is residually a finite p-group, since S_p is a p-local signature. Since $g = 0$, every nilpotent automorphism group covered by $\Gamma(S_p)$ is a finite p-group by Theorem 2.1.1 of **[6]**. Hence we have

$$\Gamma(S_p) = \left\langle x_1, x_2, \ldots, x_s \middle| x_1^{p^{\alpha_1}} = x_1^{p^{\alpha_2}} = \cdots = x_s^{p_s} = x_1 x_2 \cdots x_s = 1 \right\rangle.$$

Now suppose that $\alpha = \max\{x_1, x_2, \ldots, \alpha_s\}$. On the one hand we have the inequalities $p^{\alpha_j} \leq p^\alpha (j = 1, \ldots, s)$. On the other hand, if we let x_N be an element of order p^α, then from the long relation of $\Gamma(S_p)$ given above we obtain

$$\left(x_N^{-1}\right)^{p^\alpha} = (x_{N+1} x_{N+2} \cdots x_s x_1 x_2 \cdots x_{N-1})^{p^\alpha} = 1.$$

However, if G_p is a regular p-group, then by Property 6(iv) of § 2 Part A we have

$$|x_{N+1}x_{N+2}\cdots x_1x_1x_2\cdots x_{N-1}| = p^{\alpha} \leq |x_j| = p^{\alpha_j} \quad (j \neq N),$$

where $|y|$ denotes the order of the element y in G. Hence $p^{\alpha_j} = p^{\alpha}(j \neq N)$. The result now follows.

We note that the result is not true in general for signatures S_p with non-zero orbit genus. Moreover, since $\Gamma(S_p)$ covers the regular p-group G_p smoothly, the signature S_p cannot be degenerate. See [**3**]. This implies that $s \geq 2$. For $s = 2$, S_p is still a non-degenerate signature, since here we have $m_1 = m_2$.

The next theorem generalizes the observation in [**7**, p. 240] on 3-groups of automorphisms. This explains why the smallest 3-group of automorphisms must have order 81 and must be $\mathbf{Z}_3$ wr $\mathbf{Z}_3$. Note that here, however, we are merely dealing with the Fuchsian groups (p, p, p^2), which do not necessarily give an upper bound for the rest of the p-groups of automorphisms. □

THEOREM 3.2. *Let p be an odd prime. Consider the Fuchsian group $\Gamma = \Gamma(0; p, p, p^2)$. The smallest nilpotent group G covered by Γ has the following properties.*

(i) *$G \cong$ Sylow p-subgroup of $\mathbf{S}_{p^2} \cong C_p$ wr C_p.*

(ii) *$G = \langle x, y | x^p = y^p = (xy)^{p^2} = [x, [x, y^n]] = 1 \quad (n = 1, 2, \ldots, p-1)$.*

(iii) *The order of G is given by $|G| = p^{p+1}$.*

(iv) *G is not a regular p-group.*

Proof. First, we shall investigate the p-subgroup of the symmetric group $\mathbf{S}_{p^2}$ on p^2 letters. We shall show that this group is generated by two elements of order p whose product has order p^2. The Sylow subgroups of $\mathbf{S}_{p^2}$ are easily constructed by means of the wreath product. Observe that the factors of $(p^2)!$ divisible by p are $p, 2p, 3p, \ldots, (p-1)p, p^2$. Hence $(p^2)!$ is divisible by $p^{p-1} \times p^2 = p^{p+1}$ and this is the highest power of p dividing $(p^2)!$

Now $\mathbf{S}_{p^2}$ has a subgroup which is the direct product of the cyclic groups generated by the p-cycles $a_1 = (1, 2, \ldots, p)$, $a_2 = (p+1, p+2, \ldots, 2p)$, $a_3 = (2p+1, \ldots, 3p), \ldots, a_p = (p^2 - p + 1, \ldots, p^2)$. Consider another element of order p; for example

$$b = (1, p+1, 2p+1, \ldots, P^2 - p + 1)(2, p+2, 2p+2, \ldots, p^2 - p + 2)\cdots \\ \times (p, 2p, 3p, \ldots, p^2).$$

It can be checked that

$$a_{k+1} = ba_kb^{-1} = b^2a_{k-1}b^{-2} = \cdots = b^ka_1b^{-k}.$$

Therefore, we have

$$a_k = b^{k-1}a_1b^{1-k} \quad (k = 2, \ldots, p),$$
$$G = \langle a_1, a_2, \ldots, a_p, b\rangle = \langle a_1, b\rangle \cong C_p \text{ wr } C_p$$

and the last group is a p-Sylow subgroup of $\mathbf{S}_{p^2}$. Moreover $|G| = p^{p+1}$.

To find a presentation for this group we consider the relations $a_1^p = b^p = 1$ and note that

$$a_1b = (1, p+2, 2p+2, \ldots, p(p-1)+2, 2, p+3, \ldots, \ldots, p^2)$$

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is a p^2-cycle. To get the extra relation we note that

$$a_{i+1}a_{j+1} = a_{j+1}a_{i+1} \Rightarrow (b^i a_1 b^{-i})(b^j a_1 b^{-j}) = (b^j a_1 b^{-j})(a^i a_1 b^{-i}).$$

Setting $j - i = k$, the relation above becomes

$$a_1 b^{-k} a_1 b^k = b^{-k} a_1 b^k a_1 \Rightarrow [a_1, [a_1, b^k]] = 1.$$

By §2, Item 7 of Part (A), the order of $a_1 b$ is at most p and so this group is irregular. □

THEOREM 3.3. *A Fuchsian group* $\Gamma(0; m_1, m_2, \ldots, m_s)$ *with orbit genus zero can cover a regular nilpotent group* $G = G_{p_1} \times G_{p_2} \times \cdots \times G_{p_k}$ *if and only if its periods satisfy*

$$\text{l.c.m}\,\{m_1, m_2, \ldots, \hat{m}_j, m_{j+1}, \ldots, m_k\} = \text{l.c.m}\,\{m_1, m_2, \ldots, m_k\}.$$

Here $\hat{m}_j$ *means that* m_j *should be omitted from the list.*

Proof. We use an idea from Section 5 of [**3**] to localize the given signatures $S_{p_i} = (0; p_i^{\alpha_{i1}}, p_i^{\alpha_{i2}}, \ldots, p_i^{\alpha_{is}})$. Then we use our Theorem 3.1 to show that for each of the localized signatures S_{p_i} we must have $\alpha_{i1} = \alpha_{i2} = \cdots = \alpha_{is} = \alpha_i\,(i = 1, 2, \ldots, k)$. From this we deduce that S can be localized into the p-localization signatures $S_{p_1} = (0; p_1^{\alpha_1}, \ldots, p_1^{\alpha_1})$, $S_{p_2} = (0; p_2^{\alpha_2}, \ldots, p_2^{\alpha_2}), \ldots, S_{p_k} = (0; p_k^{\alpha_k}, \ldots, p_k^{\alpha_k})$.

Obviously, the signatures above each might have a different number of periods, but each must contain at least two periods in order to be a non-degenerate signature. Hence, if $p_i | \prod_j m_j$, then the same power $p_i^{\alpha_i}$ of the prime p_i must divide at least two of the periods m_i. We can now conclude that

$$\begin{aligned}\text{l.c.m}\,\{m_1, m_2, \ldots, \hat{m}_j, m_{j+1}, \ldots, m_k\} &= \text{l.c.m}\,\{m_1, m_2, \ldots, m_k\}.\\ &= p_1^{\alpha_1} \times p_2^{\alpha_2} \times \cdots \times p_k^{\alpha_k}.\end{aligned}$$

The proof is complete, since the argument can easily be reversed.

REMARK. The least common multiple condition has earlier been used in [**2**] and [**4**].

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