ARITHMETICAL IDENTITIES AND HECKE'S FUNCTIONAL EQUATION

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1. Introduction

We consider a subclass of the Dirichlet series studied by Chandrasekharan and Narasimhan in (**1**). Our objective is to generalize some identities due to Landau (**3**) concerning $r_2(n)$, the number of representations of the positive integer n as the sum of 2 squares. We shall also give a slight extension of Theorem III in (**1**).

The subclass of Dirichlet series is given in the following definition.

Definition. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two sequences of positive numbers tending to ∞, and $\{a(n)\}$ and $\{b(n)\}$ two sequences of complex numbers not identically zero. Consider the functions ϕ and ψ representable as Dirichlet series

$$\phi(s) = \sum_{n=1}^{\infty} a(n)\lambda_n^{-s}, \quad \psi(s) = \sum_{n=1}^{\infty} b(n)\mu_n^{-s}, \quad s = \sigma + it,$$

with finite abscissae of absolute convergence σ_a and σ_a^*, respectively. If $r>0$, we say ϕ and ψ satisfy the functional equation

$$\Gamma(s)\phi(s) = \Gamma(r-s)\psi(r-s)$$

if ϕ has an analytic continuation in the s-plane such that

(i) ϕ is holomorphic everywhere except for a possible simple pole at $s = r$ with residue ρ;

(ii) $\phi(s) = O(\exp K\,|\,s\,|)$, as $|\,s\,|$ tends to ∞, where $K>0$.

(iii) $\phi(s) = \dfrac{\Gamma(r-s)\psi(r-s)}{\Gamma(s)}, \quad (\sigma < r - \sigma_a^*)$.

(iv) $\sup\limits_{0 \leqq h \leqq 1} \left| \sum\limits_{k^2 \leqq \mu_n \leqq (k+h)^2} b(n)\mu_n^{\frac{1}{2}-\sigma} \right| = o(1),$

for $\sigma > \sigma_a^*$, as k tends to ∞.

From the functional equation we note that $\phi(\psi)$ has a simple pole at $s = r$ if and only if $\psi(\phi)$ is holomorphic at $s = 0$ and $\psi(0)(\phi(0)) \neq 0$.

The following theorem of Chandrasekharan and Narasimhan ((**1**), pp. 6, 14) is the basis for our study.

Theorem 1. *Let J_ν denote the Bessel function of order ν. For $x>0$ and $q>2\sigma_a^*-r-\frac{1}{2}$,*

$$\frac{1}{\Gamma(q+1)}\sum_{\lambda_n\leqq x}{}' a(n)(x-\lambda_n)^q=\sum_{n=1}^{\infty}(x/\mu_n)^{\frac{1}{2}(r+q)}b(n)J_{r+q}(2\sqrt{\mu_n x})+\frac{\phi(0)x^q}{\Gamma(q+1)}+\frac{\Gamma(r)\rho x^{r+q}}{\Gamma(q+r+1)},\tag{1.1}$$

where the dash ′ on the summation sign on the left indicates that if $q=0$ and $x=\lambda_n$, $a(n)$ is to be multiplied by $\frac{1}{2}$. The series on the right hand side converges uniformly on any compact interval in $x>0$ where the left hand side is continuous. If $q=0$, the series converges boundedly on any compact interval in $x>0$.

2. Some properties of J_ν

For $\nu>0$,

$$\lim_{x\to 0}\left(\frac{2}{x}\right)^\nu J_\nu(x)\Gamma(\nu+1)=1.\tag{2.1}$$

For arbitrary ν,

$$\frac{d}{dx}\{x^{-\nu}J_\nu(x)\}=-x^{-\nu}J_{\nu+1}(x).\tag{2.2}$$

For $\nu>0$ and $a>0$,

$$\int_0^\infty J_\nu(ax)J_{\nu+1}(x)dx=\begin{cases}a^\nu, & (a<1)\\ \frac{1}{2}, & (a=1)\\ 0, & (a>1).\end{cases}\tag{2.3}$$

For arbitrary ν, as x tends to ∞,

$$J_\nu(x)=c_1x^{-\frac{1}{2}}e^{ix}+c_2x^{-\frac{1}{2}}e^{-ix}+O(x^{-\frac{3}{2}}),\tag{2.4}$$

where c_1 and c_2 are constants.

All of these results can be found in (5). (2.1) follows from equation (8), p. 40, (2.2) is given on p. 45, (2.3) on p. 406, and (2.4) on p. 199.

3. Summary of results

Throughout the sequel we shall assume that $2\sigma_a-r<\frac{3}{2}$ and $2\sigma_a^*-r<\frac{3}{2}$, so that (1.1) is valid for $q=0$.

Define $a(0)=-\phi(0)$ for $\lambda_0=0$ and $b(0)=\Gamma(r)\rho$ for $\mu_0=0$. Since by (2.1),

$$\lim_{\mu\to 0}(x/\mu)^{\frac{1}{2}r}b(0)J_r(2\sqrt{\mu x})=\frac{x^r\rho}{r},\tag{3.1}$$

identity (1.1) in the case $q=0$ may be written as

$$\sum_{0\leqq\lambda_n\leqq x}{}' a(n)=\sum_{n=0}^{\infty}(x/\mu_n)^{\frac{1}{2}r}b(n)J_r(2\sqrt{\mu_n x}),\tag{3.2}$$

where the term for $n=0$ in the series on the right is given by (3.1).

The following theorem is a generalization of a theorem of Landau ((3), Satz 523).

Theorem 2. *Let ρ^* denote the residue of ψ at $s = r$ and put $\lambda = \rho^*/r$. For $y>0$ let*

$$A(y) = \sum_{0 \leqq \mu_n \leqq y} b(n),\ R(y) = A(y)-\lambda y^r,\ Q(y) = \int_0^y R(t)dt.$$

Then, for $x \geqq 0$, $y>0$,

$$\sum_{0 \leqq \mu_n \leqq y} (x/\mu_n)^{\frac{1}{2}r} b(n) J_r(2\sqrt{\mu_n x}) = \sum_{\lambda_n \leqq x}{}' a(n) + (x/y)^{\frac{1}{2}(r+1)} Q(y) J_{r+1}(2\sqrt{yx})$$
$$+(x/y)^{\frac{1}{2}r} R(y) J_r(2\sqrt{yx}) + \lambda r x^{\frac{1}{2}r} \int_0^y t^{\frac{1}{2}r-1} J_r(2\sqrt{tx})dt$$
$$-\sum_{n=1}^{\infty} (x/\lambda_n)^{\frac{1}{2}(r+1)} a(n) \int_{2\sqrt{yx}}^{\infty} J_{r+1}(t\sqrt{\lambda_n/x}) J_{r+2}(t)dt. \tag{3.3}$$

Corollary 3. *If $q = 0$ and $R(y) = O(y^{\frac{1}{2}r})$, the infinite series of* (1.1) *is boundedly convergent on any compact interval in $x \geqq 0$.*

Thus Corollary 3 gives a slight extension to Theorem 1 in some cases. In particular, if $a(n) = \tau(n)$, Ramanujan's arithmetical function, a result of Hardy (2) is extended. The hypotheses of Corollary 3 are satisfied by those Dirichlet series attached to entire modular forms of dimension $-r$, $r>0$, and level N which vanish at all of the rational cusps of the fundamental region, for in such a case Rankin (4) has shown that $R(y) = O(y^{\frac{1}{2}r-\frac{1}{10}})$.

The following generalizes another result of Landau ((3), Satz 559).

Theorem 4. *Let $0<\alpha<\beta$. Let $f(x)$ be real and of bounded variation on $\alpha \leqq x \leqq \beta$. Then*

$$\sum_{\alpha \leqq \lambda_n \leqq \beta}{}' a(n) \frac{f(\lambda_n+0)+f(\lambda_n-0)}{2} = \sum_{n=0}^{\infty} b(n) \int_{\alpha}^{\beta} f(x)(x/\mu_n)^{\frac{1}{2}(r-1)} J_{r-1}(2\sqrt{\mu_n x})dx.$$

If the infinite series of (3.2) is boundedly convergent on $[0, \varepsilon]$, $\varepsilon>0$, the above is valid for $0 \leqq \alpha \leqq \beta$. If

(1) $\lambda_n = \alpha = 0$, the factor of $a(0)$ should read $f(\alpha+0)$;

(2) $\lambda_n = \alpha>0$, the factor of $a(n)$ should read $\frac{1}{2}f(\lambda_n+0)$;

(3) $\lambda_n = \beta$, the factor of $a(n)$ should read $\frac{1}{2}f(\lambda_n-0)$.

4. Proof of results

Proof of Theorem 2. By (2.2) and integration by parts,

$$-x^{\frac{1}{2}r} \int_{t=0}^{t=y} R(t) d\{t^{-\frac{1}{2}r} J_r(2\sqrt{tx})\}$$
$$= (x/y)^{\frac{1}{2}(r+1)} Q(y) J_{r+1}(2\sqrt{yx}) + x^{\frac{1}{2}(r+2)} \int_0^y Q(t) t^{-\frac{1}{2}(r+2)} J_{r+2}(2\sqrt{tx})dt. \tag{4.1}$$

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Denote the second term on the right hand side of (4.1) by $S(x, y)$. Using (1.1), (2.4) and the fact that $J_\nu(x) = O(x^\nu)$ as x tends to 0, we have for fixed x and y by the Weierstrass M-test,

$$S(x, y) = x^{\frac{1}{2}(r+2)} \int_0^y t^{-\frac{1}{2}(r+2)} J_{r+2}(2\sqrt{tx}) \sum_{n=1}^{\infty} (t/\lambda_n)^{\frac{1}{2}(r+1)} a(n) J_{r+1}(2\sqrt{\lambda_n t}) dt$$

$$= O\left(x^{r+2} \int_0^y t^{\frac{1}{2}(r+\frac{1}{2})} \sum_{n=1}^{\infty} |a(n)| \lambda_n^{-\frac{1}{2}r-\frac{3}{4}} dt\right) = O(1),$$

as $2\sigma_a - r < \frac{3}{2}$. Hence, we may invert the order of summation and integration to find

$$S(x, y) = \sum_{n=1}^{\infty} (x/\lambda_n)^{\frac{1}{2}(r+1)} a(n) \int_0^{2\sqrt{yx}} J_{r+1}(t\sqrt{\lambda_n/x}) J_{r+2}(t) dt$$

$$= \sum_{\lambda_n \leq x}' a(n) - \sum_{n=1}^{\infty} (x/\lambda_n)^{\frac{1}{2}(r+1)} a(n) \int_{2\sqrt{yx}}^{\infty} J_{r+1}(t\sqrt{\lambda_n/x}) J_{r+2}(t) dt, \tag{4.2}$$

upon an application of (2.3).

On the other hand,

$$-x^{\frac{1}{2}r} \int_{t=0}^{t=y} R(t) d\{t^{-\frac{1}{2}r} J_r(2\sqrt{tx})\}$$

$$= -x^{\frac{1}{2}r} \sum_{0 \leq \mu_n \leq y} b(n) \int_{t=\mu_n}^{t=y} d\{t^{-\frac{1}{2}r} J_r(2\sqrt{tx})\} + \lambda x^{\frac{1}{2}r} \int_{t=0}^{t=y} t^r d\{t^{-\frac{1}{2}r} J_r(2\sqrt{tx})\}$$

$$= -(x/y)^{\frac{1}{2}r} A(y) J_r(2\sqrt{yx}) + \sum_{0 \leq \mu_n \leq y} (x/\mu_n)^{\frac{1}{2}r} b(n) J_r(2\sqrt{\mu_n x})$$

$$+ \lambda (xy)^{\frac{1}{2}r} J_r(2\sqrt{yx}) - r\lambda x^{\frac{1}{2}r} \int_0^y t^{\frac{1}{2}r-1} J_r(2\sqrt{tx}) dt, \tag{4.3}$$

upon an integration by parts. Combining (4.1), (4.2) and (4.3), we obtain (3.3).

Proof of Corollary 3. In view of Theorem 1 it is sufficient to assume $0 \leqq x \leqq \varepsilon$, $\varepsilon > 0$. We show that each of the terms on the right hand side of (3.3) is $O(1)$ for $0 \leqq x \leqq \varepsilon$, where $\varepsilon > 0$ is fixed. The first term is clearly $O(1)$. Using (1.1) and (2.4), one easily shows that

$$Q(y) = O(y^{\frac{1}{2}r+\frac{1}{4}}).$$

Since $J_\nu(x)$, $\nu > 0$, is bounded on $[0, \infty)$, it follows that

$$(x/y)^{\frac{1}{2}(r+1)} Q(y) J_{r+1}(2\sqrt{yx}) = O(y^{-\frac{1}{4}}) = O(1).$$

By the boundedness of J_r and our assumption on $R(y)$ it follows easily that

$$(x/y)^{\frac{1}{2}r} R(y) J_r(2\sqrt{yx}) = O(1).$$

The fourth term is present only if ψ is not entire. In such a case $r<\frac{3}{2}$, since $\frac{3}{2}>2\sigma_a^*-r\geqq 2r-r=r$. By the use of (2.4) it is easily seen that

$$\int_0^\infty t^{\frac{1}{2}r-1}J_r(2\sqrt{tx})dt$$

converges if $r<\frac{3}{2}$. If follows that the fourth term is $O(1)$. Without loss of generality assume that $\varepsilon>0$ is small enough so that $\lambda_1/\varepsilon>1$. By an application of (2.4) we have

$$I=\int_{2\sqrt{yx}}^\infty J_{r+1}(t\sqrt{\lambda_n/x})J_{r+2}(t)dt=(x/\lambda_n)^{\frac{1}{4}}\int_{2\sqrt{yx}}^\infty t^{-1}\{c_1^2e^{it(\sqrt{\lambda_n/x}+1)}$$
$$+c_1c_2e^{it(\sqrt{\lambda_n/x}-1)}+c_1c_2e^{-it(\sqrt{\lambda_n/x}-1)}+c_2^2e^{-it(\sqrt{\lambda_n/x}+1)}\}dt$$
$$+O\{(x/\lambda_n)^{\frac{1}{4}}\}+O\{(x/\lambda_n)^{\frac{3}{4}}\}.$$

We examine one of the four integrals, since the others are treated in exactly the same manner. Upon an integration by parts,

$$\int_{2\sqrt{yx}}^\infty t^{-1}e^{it(\sqrt{\lambda_n/x}-1)}dt=-\frac{(yx)^{-\frac{1}{2}}}{2i(\sqrt{\lambda_n/x}-1)}$$
$$+\frac{1}{i(\sqrt{\lambda_n/x}-1)}\int_{2\sqrt{yx}}^\infty t^{-2}e^{it(\sqrt{\lambda_n/x}-1)}dt=\frac{y^{-\frac{1}{2}}}{2i(\sqrt{\lambda_n}-\sqrt{x})}+O(1).$$

Therefore,

$$I=O(\lambda_n^{-\frac{1}{4}}).$$

Hence,

$$\sum_{n=1}^\infty (x/\lambda_n)^{\frac{1}{2}(r+1)}a(n)\int_{2\sqrt{yx}}^\infty J_{r+1}(t\sqrt{\lambda_n/x})J_{r+2}(t)dt$$
$$=O(x^{\frac{1}{2}(r+1)}\sum_{n=1}^\infty |a(n)|\lambda_n^{-\frac{1}{2}r-\frac{3}{4}})=O(1).$$

By (3.3) the proof is complete.

Proof of Theorem 4. The proof follows along the same lines as Landau's for the case $a(n)=r_2(n)$ with only obvious changes being necessary. We remark that when $\alpha=0$, the bounded convergence of the series of Bessel functions in (3.3) on $[0, \varepsilon]$, $\varepsilon>0$, is necessary for the proof. A sufficient condition (satisfied by $r_2(n)$) for such convergence is given in Corollary 3.

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