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Linear Maps on C^*-Algebras Preserving the Set of Operators that are Invertible in $\mathcal{A}/\mathcal{I}$

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Abstract. For C^*-algebras $\mathcal{A}$ of real rank zero, we describe linear maps ϕ on $\mathcal{A}$ that are surjective up to ideals $\mathcal{I}$, and $\pi(A)$ is invertible in $\mathcal{A}/\mathcal{I}$ if and only if $\pi(\phi(A))$ is invertible in $\mathcal{A}/\mathcal{I}$, where $A \in \mathcal{A}$ and $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is the quotient map. We also consider similar linear maps preserving zero products on the Calkin algebra.

1 Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. A linear mapping $\phi\colon \mathcal{A} \to \mathcal{B}$ is called unital if $\phi(I) = I$ and is called a Jordan homomorphism if $\phi(A^2) = \phi(A)^2$ for every $A \in \mathcal{A}$, or equivalently $\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A)$ for every $A, B \in \mathcal{A}$. $\mathcal{A}$ is said to have real rank zero [3] if the set of Hermitian elements of $\mathcal{A}$ with finite spectra is dense in the set of all Hermitian elements of $\mathcal{A}$. For a unital Banach algebra $\mathcal{A}$, the radical of $\mathcal{A}$ is defined as

$$\operatorname{rad}\mathcal{A} = \{A \in \mathcal{A} \mid I + AB \text{ is invertible for every } B \in \mathcal{A}\}.$$

$\mathcal{A}$ is called to be semi-simple if $\operatorname{rad}\mathcal{A} = \{0\}$. Let $\mathcal{I}$ be a closed 2-sided ideal of $\mathcal{A}$ and $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ be the canonical homomorphism onto the quotient algebra. We say that $\phi\colon \mathcal{A} \to \mathcal{A}$ is surjective up to $\mathcal{I}$ if for every $A \in \mathcal{A}$ there exists $A' \in \mathcal{A}$ such that $A - \phi(A') \in \mathcal{I}$, that is, $\mathcal{A} = \phi(\mathcal{A}) + \mathcal{I}$. It is clear that if ϕ is surjective, then ϕ is surjective up to $\mathcal{I}$. We say that a linear map $\phi\colon \mathcal{A} \to \mathcal{A}$ preserves the set $\{A \in \mathcal{A} \mid \pi(A) \text{ is invertible}\}$ in both directions if the following holds: $\pi(A)$ is invertible in $\mathcal{A}/\mathcal{I}$ if and only if $\pi(\phi(A))$ is invertible in $\mathcal{A}/\mathcal{I}$.

This is a kind of so-called linear preserver problem. Linear preserver problems concern characterizing linear maps on matrix spaces that leave invariant certain functions, subsets, or relations, etc. These problems represent one of the most extensively investigated research areas in matrix theory over the past several decades. Recently, interest in similar questions on operator algebras over infinite dimensional spaces has been growing.

One of the most famous is Kaplansky's problem [9]. Let ϕ be a surjective linear map between two semi-simple Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Suppose that $\sigma(\phi(A)) = \sigma(A)$ for all $A \in \mathcal{A}$, where $\sigma(A)$ denotes the spectrum of $A \in \mathcal{A}$. Is it true that ϕ is a Jordan isomorphism? This problem was proved first in the finite dimensional

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case. Dieudonné [5] and Marcus and Purves [10] proved that every unital invertibility preserving linear map on a complex matrix algebra is either multiplicative or antimultiplicative, that is, $\phi(AB) = \phi(B)\phi(A)$ for every A and B. This result was extended to the algebra of bounded linear operators on a Banach space by Sourour [14] and to von Neumann algebras by Aupetit [2]. The case of nonunital invertibility preserving mappings can be reduced to the unital case by considering the mapping $A \mapsto \phi(I)^{-1}\phi(A)$.

Recently, Mbekhta [11] characterized linear maps $\phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ that are surjective up to compact operators and preserving the set of Fredholm operators, that is, operators in $\mathcal{B}(\mathcal{H})$ such that their images in the Calkin algebra $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ are invertible in both directions. Here, $\mathcal{H}$ is an infinite dimensional separable complex Hilbert space.

Motivated by these results, we generalize the results of [11]. In this note we describe linear maps ϕ on C^*-algebras $\mathcal{A}$ that are surjective up to $\mathcal{I}$ and $\pi(A)$ is invertible in $\mathcal{A}/\mathcal{I}$ if and only if $\pi(\phi(A))$ is invertible in $\mathcal{A}/\mathcal{I}$, where $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is the quotient map. We also consider similar linear maps preserving zero products on the Calkin algebra. We have to mention that key ideas in Section 2 come from [11].

2 Maps Preserving the Set of Operators that are Invertible in $\mathcal{A}/\mathcal{I}$

The following lemma is in the proof of [8, Proposition 2.1], but we list it for completeness.

Lemma 2.1 *Let $\mathcal{A}$ be a unital C^*-algebra and A be an element of $\mathcal{A}$. If $A + B$ is invertible for every invertible B in $\mathcal{A}$, then $A = 0$.*

Proof Let B be an arbitrary element of $\mathcal{A}$. Choose a positive real number t such that $t\|B\| < 1$. Let

$$C = (tI + A)(tB - I)^{-1}.$$

Then C is invertible, since $tI + A$ and $tB - I$ are invertible. Hence $A + C$ is invertible. Noting that

$$(A + C)(tB - I) = t(I + AB),$$

we have that $I + AB$ is invertible. Therefore $A \in \operatorname{rad}\mathcal{A}$. Since every C^*-algebra is semi-simple, it follows that $A = 0$. ■

Lemma 2.2 *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{I}$ be a closed 2-sided ideal of $\mathcal{A}$. Then for an element K of $\mathcal{A}$, we have*

$$K \in \mathcal{I} \iff \pi(T + K) \text{ is invertible for every } T \in \mathcal{A} \text{ such that } \pi(T) \text{ is invertible.}$$

Proof ($\Longrightarrow$): Let K be an element of $\mathcal{I}$. Then $\pi(K) = 0$. So for every $T \in \mathcal{A}$ such that $\pi(T)$ is invertible, $\pi(T + K) = \pi(T) + \pi(K)$ is invertible.

($\Longleftarrow$): Let $T \in \mathcal{A}$ be an element such that $\pi(T)$ is invertible. Then $\pi(T) + \pi(K) = \pi(T + K)$ is invertible. Hence $\pi(K) = 0$ by Lemma 2.1. This yields $K \in \mathcal{I}$. ■

Theorem 2.3 *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero and $\mathcal{I}$ be a closed 2-sided ideal of $\mathcal{A}$. Assume that $\phi\colon \mathcal{A} \to \mathcal{A}$ is a linear map that is surjective up to $\mathcal{I}$. Then the following conditions are equivalent:*

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(i) *ϕ preserves the set $\{A \in \mathcal{A} \mid \pi(A)$ is invertible$\}$ in both directions.*
(ii) *$\phi(\mathcal{J}) \subseteq \mathcal{J}$ and the induced map $\tilde{\phi}\colon \mathcal{A}/\mathcal{J} \to \mathcal{A}/\mathcal{J}$ is the composition of a Jordan automorphism and left multiplication by an invertible element of $\mathcal{A}/\mathcal{J}$.*

Proof (i) $\Rightarrow$ (ii): We first show that $\phi(\mathcal{J}) \subseteq \mathcal{J}$. Let $K \in \mathcal{J}$ and $T \in \mathcal{A}$ be an element such that $\pi(T)$ is invertible. Then there is $T' \in \mathcal{A}$ such that $T = \phi(T') + K'$ for some $K' \in \mathcal{J}$. As $\phi(T') = T - K'$, we have that $\pi(\phi(T'))$ is invertible. Then $\pi(T') = \pi(T' + K)$ is invertible. So, $\pi(\phi(T' + K))$ is invertible, and hence $\pi(T + \phi(K)) = \pi(\phi(T' + K) + K')$ is invertible by Lemma 2.2. This yields $\phi(K) \in \mathcal{J}$ by Lemma 2.2 again, so that $\phi(\mathcal{J}) \subseteq \mathcal{J}$. It is clear that $\tilde{\phi}$ is surjective, since ϕ is surjective up to $\mathcal{J}$. Since $\pi(\phi(I))$ is invertible in $\mathcal{A}/\mathcal{J}$, there exists $S \in \mathcal{A}$ such that $\pi(S)\pi(\phi(I)) = \pi(\phi(I))\pi(S) = \pi(I)$. Let $\psi = L_{\pi(S)} \circ \tilde{\phi}\colon \mathcal{A}/\mathcal{J} \to \mathcal{A}/\mathcal{J}$, where $L_{\pi(S)}$ is the left multiplication map of $\mathcal{A}/\mathcal{J}$ by $\pi(S)$. Noting that for $T \in \mathcal{A}$,

$$\psi(\pi(T)) = L_{\pi(S)}(\pi(\phi(T))) = \pi(S)\pi(\phi(T)) = \pi(S\phi(T)),$$

we have that ψ is unital. We also have that $\pi(T)$ is invertible if and only if $\pi(\phi(T))$ is invertible if and only if $\pi(S)\pi(\phi(T))$ is invertible. This shows that ψ preserves the invertible elements in both directions. Then $\sigma(\psi(\pi(T)) \subseteq \sigma(\pi(T))$, so that ψ is spectrally bounded, that is, the spectral radius of $\psi(\pi(T))$ is less than or equal to that of $\pi(T)$. Then ψ is continuous by [1, Theorem 5.5.2]. Let $\psi(\pi(T)) = 0$. Then for all invertible $\pi(A) \in \mathcal{A}/\mathcal{J}$, we have that

$$\psi(\pi(T + A)) = \psi(\pi(T) + \pi(A)) = \psi(\pi(A))$$

is invertible. Hence $\pi(T) + \pi(A) = \pi(T + A)$ is invertible. Then by Lemma 2.1, $\pi(T) = 0$. This shows that ψ is injective.

Summing up, we have that ψ is a unital continuous linear bijection preserving invertible elements in both directions. Then, by [2, Theorem 1.2], ψ transforms the set of mutually orthogonal idempotents into a set of mutually orthogonal idempotents. In particular, ψ maps idempotents into idempotents. Since the real rank of $\mathcal{A}$ is zero, the real rank of $\mathcal{A}/\mathcal{J}$ is also zero by [3, Theorem 3.14]. Then, by [7, Theorem 4.1] ψ is a Jordan automorphism. As $\psi = L_{\pi(S)} \circ \tilde{\phi}$, we get

$$\tilde{\phi} = L_{\pi(S)}^{-1} \circ \psi = L_{\pi(S)^{-1}} \circ \psi = L_{\pi(\phi(I))} \circ \psi.$$

(ii) $\Rightarrow$ (i): It is well known that Jordan automorphisms preserve invertibility. Thus we have

$$\pi(A) \text{ is invertible in } \mathcal{A}/\mathcal{J} \iff \tilde{\phi}(\pi(A)) = \pi(\phi(A)) \text{ is invertible in } \mathcal{A}/\mathcal{J}.$$

This completes the proof. ∎

As a corollary we can recapture a special case of [2, Theorem 1.3], which states that a surjective spectrum preserving linear map between von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ is a Jordan isomorphism. In the following we consider the case $\mathcal{A} = \mathcal{B}$.

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Corollary 2.4 *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero. Assume that $\phi\colon \mathcal{A} \to \mathcal{A}$ is a surjective linear map. Then ϕ preserves the set of invertible elements in both directions if and only if ϕ is the composition of a Jordan automorphism and left multiplication by an invertible element of $\mathcal{A}$.*

Proof It follows directly from Theorem 2.3 by taking $\mathcal{I} = \{0\}$. ∎

Now we consider the relation between the condition that "$\pi(A)$ is invertible if and only if $\pi(\phi(A))$ is invertible" and their spectra. For $A \in \mathcal{A}$ let $\sigma_{\mathcal{I}}(A) = \sigma(\pi(A))$.

Theorem 2.5 *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero and $\mathcal{I}$ be a closed 2-sided ideal of $\mathcal{A}$. Assume that $\phi\colon \mathcal{A} \to \mathcal{A}$ is a linear map that is surjective up to $\mathcal{I}$. Then the following conditions are equivalent:*

(i) $\sigma_I(A) = \sigma_I(\phi(A)) \quad (A \in \mathcal{A})$.
(ii) *$\pi(A)$ is invertible if and only if $\pi(\phi(A))$ is invertible for $A \in \mathcal{A}$ and $\phi(I) = I + K$ for some $K \in \mathcal{I}$.*

Proof (i) ⇒ (ii): Since $\pi(A)$ is invertible if and only if $\pi(\phi(A))$ is invertible, it suffices to show that $\phi(I) - I \in \mathcal{I}$. Let $\phi(I) - I = K$ and let $T \in \mathcal{A}$ be an arbitrary element. Then there exist $T' \in \mathcal{A}$ and $K' \in \mathcal{I}$ such that $T = \phi(T') + K'$. We compute

$$\begin{aligned}\sigma(\pi(T) + \pi(K)) &= \sigma(\pi(T) + \pi(\phi(I) - I)) \\ &= \sigma(\pi(\phi(T' + I))) - 1 \\ &= \sigma((\pi(T' + I)) - 1 \\ &= \sigma(\pi(T')) \\ &= \sigma(\pi(\phi(T')) \\ &= \sigma(\pi(T)).\end{aligned}$$

Then $\pi(K)$ is in the radical of $\mathcal{A}/\mathcal{I}$ by Zemanek's characterization of the radical [1, Theorem 5.3.1]. As the radical is $\{0\}$, it follows that $K \in \mathcal{I}$.

(ii) ⇒ (i): As $\pi(\phi(I)) = \pi(I)$, it follows that $\psi = \tilde{\phi}$ is the Jordan automorphism. Then

$$\sigma_I(\phi(T)) = \sigma(\pi(\phi(T)) = \sigma(\tilde{\phi}(\pi(T)) = \sigma(\psi(\pi(T))) = \sigma(\pi(T)) = \sigma_I(T).$$

This completes the proof. ∎

3 Linear Maps Preserving Operators of Zero Products in $\mathcal{A}/\mathcal{I}$

Similarly to Theorem 2.3, we consider linear maps on $\mathcal{A}$ that preserve zero products on the quotient algebra $\mathcal{A}/\mathcal{I}$. Note that in the following theorem, the map ϕ is not necessarily continuous.

Theorem 3.1 *Let $\mathcal{A}$ be a properly infinite von Neumann algebra and $\mathcal{I}$ be a proper ideal of $\mathcal{A}$. Assume that $\phi\colon \mathcal{A} \to \mathcal{A}$ is a linear map that is surjective up to $\mathcal{I}$. Then the following condition* (i) *implies condition* (ii).

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(i) $\pi(\phi(A))\pi(\phi(B)) = 0$ *if* $\pi(A)\pi(B) = 0$ *for* $A, B \in \mathcal{A}$.
(ii) $\phi(\mathcal{J}) \subseteq \mathcal{J}$ *and the induced map* $\tilde{\phi}\colon \mathcal{A}/\mathcal{J} \to \mathcal{A}/\mathcal{J}$ *has the form* $\tilde{\phi} = \tilde{\phi}(\pi(I))\Phi$, *where* $\Phi\colon \mathcal{A}/\mathcal{J} \to \mathcal{A}/\mathcal{J}$ *is a multiplicative surjection and* $\tilde{\phi}(\pi(I))$ *is a nonzero central operator.*

Proof Condition (i) can be rephrased as follows: for $A, B \in \mathcal{A}$, if $AB \in \mathcal{J}$, then $\phi(A)\phi(B) \in \mathcal{J}$. Let A be an element of $\mathcal{J}$. Then $\phi(A)\phi(B) \in \mathcal{J}$ for every $B \in \mathcal{A}$. Since ϕ is surjective up to $\mathcal{J}$, we have $I = \phi(B') + K'$ for some $B' \in \mathcal{A}$ and $K' \in \mathcal{J}$. As

$$\phi(A) = \phi(A)I = \phi(A)(\phi(B') + K') \in \mathcal{J},$$

we have $\phi(A) \in \mathcal{J}$. This yields that $\phi(\mathcal{J}) \subseteq \mathcal{J}$ and the induced map $\tilde{\phi}$ is defined. It is easy to show that $\tilde{\phi}$ is surjective as ϕ is surjective up to $\mathcal{J}$. It preserves the zero product, that is, if $\pi(A)\pi(B) = 0$, then $\tilde{\phi}(\pi(A))\tilde{\phi}(\pi(B)) = 0$. By [12, Theorem 4], every operator in a properly infinite von Neumann algebra is the finite sum of idempotents. Then by [4, Theorem 2.6], $\tilde{\phi}(\pi(I))$ is a nonzero central operator and $\tilde{\phi} = \tilde{\phi}(\pi(I))\Phi$, where $\Phi\colon \mathcal{A}/\mathcal{J} \to \mathcal{A}/\mathcal{J}$ is multiplicative. This completes the proof. ■

Corollary 3.2 *Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and let $\phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a linear map that is surjective up to $\mathcal{K}(\mathcal{H})$. Then the following conditions are equivalent:*

(i) $\pi(\phi(A))\pi(\phi(B)) = 0$ *if* $\pi(A)\pi(B) = 0$ *for* $A, B \in \mathcal{B}(\mathcal{H})$.
(ii) $\phi(\mathcal{K}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H})$, *and the induced map* $\tilde{\phi}\colon \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ *is a nonzero scalar multiple of an automorphism of* $\mathcal{C}(\mathcal{H})$.

Proof (i) $\Rightarrow$ (ii): By Theorem 3.1, $\tilde{\phi}(\pi(I))$ is a nonzero operator in the center of $\mathcal{C}(\mathcal{H})$. Hence it is the scalar operator c and $\tilde{\phi} = c\Phi$, where $\Phi\colon \mathcal{C}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ is multiplicative. As $\mathcal{C}(\mathcal{H})$ is simple, it follows that Φ is an automorphism.

(ii) $\Rightarrow$ (i): It is clear. ■

If $\mathcal{A}$ is a general C^*-algebra and ϕ is a continuous linear map, we have the following result.

Corollary 3.3 *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{J}$ be a proper closed 2-sided ideal of $\mathcal{A}$. Assume that $\phi\colon \mathcal{A} \to \mathcal{A}$ is a continuous linear map that is surjective up to $\mathcal{J}$. Then the following conditions are equivalent:*

(i) $\pi(\phi(A))\pi(\phi(B)) = 0$ *if* $\pi(A)\pi(B) = 0$ *for* $A, B \in \mathcal{A}$.
(ii) $\phi(\mathcal{J}) \subseteq \mathcal{J}$, *and the induced map* $\tilde{\phi}\colon \mathcal{A}/\mathcal{J} \to \mathcal{A}/\mathcal{J}$ *has the form* $\tilde{\phi} = \tilde{\phi}(\pi(I))\Phi$, *where* $\Phi\colon \mathcal{A}/\mathcal{J} \to \mathcal{A}/\mathcal{J}$ *is a multiplicative surjection and* $\tilde{\phi}(\pi(I))$ *is an invertible operator in the center of* $\mathcal{A}/\mathcal{J}$.

Proof (i) $\Rightarrow$ (ii): As in the proof of Theorem 3.1, $\tilde{\phi}$ is a surjective linear map preserving zero products. It is easy to show that $\tilde{\phi}$ is continuous. Then by [4, Theorem 4.11], $\tilde{\phi}(\pi(I))$ is a nonzero central operator and $\tilde{\phi} = \tilde{\phi}(\pi(I))\Phi$, where $\Phi\colon \mathcal{A}/\mathcal{J} \to \mathcal{A}/\mathcal{J}$ is multiplicative.

(ii) $\Rightarrow$ (i): It is clear. ■

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Remark 3.4 There is a comment that is related to Corollary 3.2. Phillips and Weaver [13] constructed an outer automorphism on $\mathcal{C}(\mathcal{H})$ using the Continuum Hypothesis, which answered negatively a long-standing problem regarding whether or not every automorphism of the Calkin algebra is inner. But in [6, Theorem 1] Farah showed that all automorphisms of the Calkin algebra are inner with some consistent set-theoretic axioms called OCA_∞ and MA (see [6] for their statements). If we assume the axioms OCA_∞ and MA, then Corollary 3.2(ii) is equivalent to the following condition.

(iii) $\phi(A) = cTAS + \alpha(A) \quad (A \in \mathcal{B}(\mathcal{H}))$, where c is a nonzero scalar, T and S are Fredholm operators such that $\pi(T)^{-1} = \pi(S)$, and $\alpha\colon \mathcal{B}(\mathcal{H}) \to \mathcal{K}(\mathcal{H})$ is a linear map.

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