Proceedings of the Edinburgh Mathematical Society (1989) 32, 19–29 ©

MULTIPARAMETER ROOT VECTORS

by PAUL BINDING*

(Received 27th February 1987, revised June 1987)

0. Preliminaries

The concept of "root vectors" is investigated for a class of multiparameter eigenvalue problems

$$W_m(\lambda)x_m = 0 \neq x_m, \qquad m = 1, \ldots, k \tag{$*$}$$

where $W_m(\lambda) = T_m - \sum_{n=1}^{k} \lambda_n V_{mn}$ operate in Hilbert spaces H_m and $\lambda \in \mathbb{C}^k$. Previous work on this "uniformly elliptic" class has demonstrated completeness of the decomposable tensors $x_1 \otimes \cdots \otimes x_k$ in a subspace G of finite codimension in $H = H_1 \otimes \cdots \otimes H_k$, but questions remain about extending this to a basis of H. In this work, bases of elements y_m, in general nondecomposable but satisfying recursive equations of the type $W_m(\lambda)y_m = \sum_{n=1}^{k} V_{mn} z_{mn}$, are constructed for the "root subspaces" corresponding to $\lambda \in \mathbb{R}^k$.

1. Introduction

Let T_m, V_{mn} be self-adjoint operators in Hilbert spaces H_m, T_m being bounded below with compact resolvent, and V_{mn} being bounded, for $1 \leqq m, n \leqq k$. We are interested in a spectral decomposition of the Hilbert Space tensor product $H = H_1 \otimes \cdots \otimes H_k$ by the eigenvalue problem $(*)$ of Section **0**.

Let us begin with the case $k = 1$, when $(*)$ becomes, with subscripts suppressed,

$$W(\lambda)x = 0, W(\lambda) = T - \lambda V.$$

Despite the self-adjointness assumptions, λ need not be real and the eigenvectors x need not be complete in H. Under a suitable nondegeneracy condition (e.g. if V is $1-1$), it can be shown [6] that the span G of the eigenvectors has a finite dimensional complement F which is in turn spanned by elements x^j satisfying equations of the form

$$W(\lambda)x^j = Vx^{j-1}, \qquad j = 0, \ldots, l-1$$

where $x^{-1} = 0$. Evidently this is equivalent to the Jordan chain condition

$$(\Gamma - \lambda)x^j = x^{j-1} \tag{1.1}$$

*Research supported by a Canadian NSERC Operating Grant.

where $\Gamma = V^{-1}T$ so $x^{j-1} \in N(\Gamma - \lambda I)^j$. The x^j are called *root vectors* and

$$N(\Gamma - \lambda I)^d \tag{1.2}$$

$d = \dim F$, is called the *root subspace* associated with λ.

For $k > 1$, there seems to be no analogue in the literature, although various authors have addressed the problem. Atkinson [**1**] raises the question of how to define root vectors for $k > 1$ and gives one answer as follows, at least in finite dimensions [**2**, Chapter 6]. With † denoting induced operators in H (e.g. $V_{11}^\dagger = V_{11} \otimes I_2 \otimes \cdots \otimes I_k$), we set

$$\Delta_0 = \det [V_{mn}^\dagger] \tag{1.3}$$

which is well defined since the elements of different rows commute. Then Δ_n is defined as the determinant in (1.3) but with column n replaced by $[T_1^\dagger, \ldots, T_k^\dagger]^T$. Under a suitable nondegeneracy condition (e.g. if Δ_0 is $1-1$) the operators $\Gamma_n = \Delta_0^{-1} \Delta_n$ commute for $n = 1, \ldots, k$ and thus H admits a decomposition into *joint* root subspaces of the form

$$J(\lambda) = \bigcap_{n=1}^{k} N((\Gamma_n - \lambda_n I)^\nu) \tag{1.4}$$

where $\nu \leqq \dim H$, cf. (1.2).

This leads to a rather complicated definition of root vectors, since an element of (1.4) will in general belong to different Jordan chains for each Γ_n, cf. (1.1), and moreover such chains are not defined directly in terms of the data in (∗). This is particularly important when $\dim H = \infty$, since the construction and commutativity of the Γ_n are then by no means obvious. In a more general situation, Isaev [**10**] has addressed the relation between elements of (1.4) and equations of the form

$$W_m(\lambda)^\dagger x = \sum_{n=1}^{k} V_{mn}^\dagger z_n \tag{1.5}$$

in H, but concludes that the topic "faces essential difficulties". Gadzhiev [**9**] has shown the relevance of tensors, formed from generalized chains satisfying equations of the form

$$W_m(\lambda) x_m^j = \sum_{n=1}^{k} V_{mn} x_m^{j-1} \tag{1.6}$$

in H_m, to systems of differential equations with multiple time scales. Our root vectors will be formed from a generalisation of (1.6) and will satisfy (1.5), for a class of problems obeying a "definiteness condition" defined below.

The simplest of many definiteness conditions in the literature on (∗) is uniform right definiteness (URD) where $\Delta_0 \gg 0$, i.e. has a positive definite bounded inverse, on H. It is known that URD holds if $(u, \Delta_0 u)$ has a positive lower bound for unit *decomposable* tensors, giving a condition expressible directly in terms of the data in (∗). Also URD implies that each $\lambda \in \mathbb{R}^k$ in (*), that each exponent ν in (1.4) may be taken as unity, and that $\bigcap_{n=1}^k N(\Gamma_n - \lambda_n I)$ is spanned by eigentensors, i.e. elements

$$x^{\otimes} = x_1 \otimes \cdots \otimes x_k, \tag{1.7}$$

where x_m satisfy (*). References for these facts are [**2**, **3**, **11**].

Another important definiteness condition, with application to various separation of variables problems, is uniform ellipticity (UE) where, instead of Δ_0, the *cofactors* of Δ_0, labelled Δ_{0mn}, $\gg 0$ on H. For various equivalent conditions, see [**3**] where UE is labelled LD_δ—again UE may be checked directly in terms of the data in (*). Under a suitable nondegeneracy condition, e.g. if Δ_0 is $1-1$, the span G of the eigentensors (1.7) has a finite dimensional complement F which is in turn spanned by joint root subspaces (1.4). This is an easy consequence of [**7**, Lemma 4.2] and will be demonstrated in Section 2. In the special case when each $T_m \gg 0$ on H_m, known as uniform left definiteness (ULD), each exponent ν may be taken as unity in (1.4), so the eigentensors (1.7) span H, as for URD, cf. [**4**, **13**]. Actually this holds under the weaker condition of UE and $\Delta_n \gg 0$ for some n. This will be seen in Section 4, but has already been observed for the case of $k=2$ Sturm–Liouville equations (*) in [**8**, Theorem 4.3].

This work of Faierman makes important contributions both to the completeness of eigentensors in G (cf. the discussion in [**7**, Section 1]) and to the nature of root vectors required to span F. In the case when $\Delta_2 \geqq 0$, [**8**, Theorem 5.5] gives a basis for F in terms of the data in (*), and we shall discuss this further in Section 4, noting here that in general λ has real components and $\nu=2$ suffices in (1.4), cf. [**7**, Theorem 5.4]. When Δ_2 is indefinite, [**8**, Theorem 9.2] gives a basis of $N(\Gamma_2 - \lambda_2 I)$ and in Section 3 we shall give an extension of this to general ν, k and $\lambda \in \mathbb{R}^k$, for our abstract formulation. While our methods also have a bearing on $\lambda \notin \mathbb{R}^k$, they do not cover all possibilities, and we hope to discuss the nonreal situation separately. In Section 2 we discuss the non-defective case ($\nu=1$) and we embed (*) in a parametric family which is almost always non-defective. In Section 3 we use analytic perturbation theory, cf. [**5**], to discuss the defective case by a limiting process, and we connect our work with (1.5) and (1.6). Section 4 is devoted to remarks on determination of the root vectors, on Jordan structure of the Γ_n and on the case where one of the $\Delta_n \geqq 0$. We conclude with a numerical example.

2. The nondefective case

We shall need certain constructions from [**7**]. Self adjoint operators T_m and V_{mn} are induced in H by T_m and V_{mn}, and Δ_0 is defined by (1.3), with Δ_{0mn} as the (m,n) cofactor of this determinant. We assume (i) UE, i.e. each $\Delta_{0mn} \gg 0$, and (ii) Δ_0 is $1-1$. Then each operator

$$\sum_{m=1}^{k} \Delta_{0mn} T_m^\dagger$$

has a self-adjoint closure in H, denoted by Δ_n. If, for fixed m, we replace V_{mn} by $\delta_{mn} I_m$ (I_m = identity on H_m) for $n=1,\ldots,k$ then Δ_l is replaced by a "cofactor" operator which

https://doi.org/10.1017/S0013091500006866 Published online by Cambridge University Press

we denote by Δ_{lmm}. As in [7, Theorem 2.5] we may assume (by translating the λ origin if necessary) that each Δ_n is bounded below with compact inverse, and we define

$$B_n = \Delta_n^{-1}\Delta_0, \Gamma_n = \Delta_0^{-1}\Delta_n, \qquad n = 1, \ldots, k.$$

Theorem 2.1. *H is the closure of $F \dot{+} G$ where F is a finite dimensional direct sum of joint root subspaces (1.4) and G is a linear span of eigentensors (1.7).*

Proof. In [7, Lemma 4.2] it is shown that $D(|\Delta_1|^{1/2})$ is the closure, in a norm stronger than that in H, of $F \dot{+} G$ say where $\dim F < \infty$ and the eigentensors span G. F is a direct sum of joint root subspaces for the B_n, and an easy computation shows that $N(B_n - \lambda_n^{-1} I)^\nu = N(\Gamma_n - \lambda_n I)^\nu$, so the result follows from density of $D(|\Delta_1|^{1/2})$ in H. □

From now on we shall concentrate on the subspace F. If $\nu = 1$ suffices in (1.4) for a fixed λ then we say that λ is nondefective. If each eigenvalue λ is nondefective then (*) is nondefective.

Corollary 2.2. *If (*) is nondefective then F, and hence H, is spanned by eigentensors.*

Proof. By [7, Theorem 3.2] the equations

$$(B_n - \lambda_n^{-1})x = 0, \qquad n = 1, \ldots, k$$

are equivalent to

$$W_m(\lambda)^\dagger x = 0, \qquad m = 1, \ldots, k,$$

and hence to

$$x \in \bigotimes_{m=1}^{k} N(W_m(\lambda)).$$

It suffices therefore to construct an eigentensor basis out of arbitrary basis elements $x_m \in N(W_m(\lambda))$, for each λ corresponding to a joint root subspace in F. □

The basis of our subsequent analysis in an embedding with T_k replaced by $T_k + \mu I_k$, $\mu \in \mathbb{R}$. Then Δ_0 remains unchanged but Δ_n is replaced by $\Delta_n + \mu\Delta_{0kn}$.

Theorem 2.3. *The set of μ values for which (*) is defective has no finite accumulation.*

Proof. Eliminating all but λ_n from (*) we obtain

$$(\Delta_n + \mu\Delta_{0kn} - \lambda_n \Delta_0)x^\otimes = 0$$

i.e.

$$(\tilde{\Delta}_n + \mu I - \lambda_n \tilde{\Delta}_0)x^\otimes = 0 \tag{2.1}$$

https://doi.org/10.1017/S0013091500006866 Published online by Cambridge University Press

where $\tilde{\Delta}_j = \Delta_{0kn}^{-1}\Delta_j$ $(j=0,n)$ are self-adjoint in H_{0kn}. Here H_{0kn} denotes H with inner product given by $(x,y)_{0kn}=(x,\Delta_{0kn}y)$. We shall prove that the set of μ values for which (2.1) is defective (as a problem in λ_n) has no finite accumulation for any fixed n, and hence for all n. For other values of μ, λ_n will be a nondefective eigenvalue of

$$\Gamma_n(\mu):=\tilde{\Delta}_0^{-1}(\tilde{\Delta}_n+\mu I) \tag{2.2}$$

and so $\nu=1$ will suffice in (1.4).

For large real μ, $\tilde{\Delta}_n+\mu I \gg 0$ and hence has a positive square root S. Thus $\Gamma_n(\mu)^{-1} = S^{-2}\tilde{\Delta}_0$ is compact symmetric in $D(S)$ with inner product given by $[x,y]=(Sx,Sy)$. It follows that all eigenvalues of $\Gamma_n(\mu)^{-1}$, and hence of $\Gamma_n(\mu)$, are nondefective. Moreover $\Gamma_n(\mu)$ is holomorphic in μ [**5**, Lemma 3.2] and we then conclude that the eigennilpotents for $\Gamma_n(\mu)$ vanish for large real μ, and hence for all μ [**13**, Theorem VII.1.8].

Suppose λ_j is a defective (i.e. nonsemisimple) eigenvalue for $\Gamma_n(\mu_j)$, with $\mu_j\to\mu_0$ as $j\to\infty$. Without loss of generality we may assume $\lambda_j\to\lambda_0$ by virtue of [**5**, Theorem 3.7]. Appealing to [**13**, Section VII.1.3] we may separate $\sigma(\Gamma_n(\mu_j))$ by means of a small contour in $\mathbb{C}$ encircling λ_0. This leads to a finite dimensional problem with a defective eigenvalue for each sufficiently large μ_j. From the previous paragraph, such μ_j are exceptional in the sense of [**13**, p. 64], and their accumulation at μ_0 is therefore a contradiction. □

In summary, we find that the eigentensors are complete in H for almost all μ. On the other hand $\mu=0$ may still yield a defective problem, and we turn next to this case.

3. The defective case

We fix our attention on a defective $\lambda^*\in\mathbb{R}^k$ corresponding to $\mu=0$. For notational ease, we shall consider first the *simple* case, when $\tilde{\Delta}_k-\lambda_k^*\tilde{\Delta}_0$, which is a self-adjoint operator on H_{0kk} in the notation of (2.1), has nullity one. By [**13**, Theorem VII.3.9] there exist real $\mu(\lambda_k)$, and $x(\lambda_k)$ of unit norm in H_{0kk}, holomorphic at λ_k^*, such that

$$N(\lambda_k):=N(\tilde{\Delta}_k+\mu(\lambda_k)I-\lambda_k\tilde{\Delta}_0)=N(\Gamma_k(\mu(\lambda_k))-\lambda_k I) \tag{3.1}$$

is spanned by $x(\lambda_k)$, in the notation of (2.2). Moreover the $\Gamma_n(\mu(\lambda_k))$ commute for each λ_k [**7**, Theorem 3.1], so they have eigenvalues $\lambda_n(\lambda_k)$ and a common eigenvector $x(\lambda_k)$. By [**7**, Theorem 3.2], $x(\lambda_k)$ is a decomposable tensor $x^{\otimes}(\lambda_k)$ say, where

$$\begin{aligned} W_m(\lambda(\lambda_k))x_m(\lambda_k)&=0, \qquad 1\leqq m<k \\ W_k(\lambda(\lambda_k))x_k(\lambda_k)&=-\mu(\lambda_k)x_k(\lambda_k). \end{aligned} \tag{3.2}$$

Eliminating all but λ_j and λ_k from the first $k-1$ equations (3.2), we obtain

$$(\Delta_{jkk}-\lambda_j(\lambda_k)\Delta_{0kk}+\lambda_k\Delta_{0kj})x^{\otimes}(\lambda_k)=0 \tag{3.3}$$

https://doi.org/10.1017/S0013091500006866 Published online by Cambridge University Press

in terms of the cofactor operators introduced in the first paragraph of Section 2. Operating by Δ_{0kk}^{-1}, we derive an equation analogous to (2.1), viz.

$$(\tilde{\Delta}_{jkk}-\lambda_j(\lambda_k)I+\lambda_k\tilde{\Delta}_{0kj})x^{\otimes}(\lambda_k)=0,$$

involving self-adjoint operators on H_{0kk}. It follows that $N(\lambda_k)$ is invariant for $\tilde{\Delta}_{jkk}+\lambda_k\tilde{\Delta}_{0kj}$. Applying [13, p. 386] to the (H_{0kk}) orthoprojector $P(\lambda_k)$ onto $N(\lambda_k)$, we construct an (H_{0kk}) unitary operator $U(\lambda_k)$, holomorphic at λ_k^*, such that

$$U(\lambda_k)^{-1}P(\lambda_k^*)U(\lambda_k)=P(\lambda_k).$$

Thus

$$A(\lambda_k):=U(\lambda_k)^{-1}(\tilde{\Delta}_{jkk}+\lambda_k\tilde{\Delta}_{0kj})U(\lambda_k)\,|\,N(\lambda_k^*) \tag{3.4}$$

is (H_{0kk}) self-adjoint on $N(\lambda_k^*)$ and is holomorphic at λ_k^*, and its eigenvalue $\lambda_j(\lambda_k)$ is therefore real and holomorphic at λ_k^*.

In summary, the $\lambda(\lambda_k)$ and $x_m(\lambda_k)$ of (3.2) can be taken holomorphic at λ_k^*, and, since $\mu(\lambda_k)$ is nonconstant [5, Corollary 2.4],

$$\mu(\lambda_k^*)=\mu'(\lambda_k^*)=\cdots=\mu^{(\nu-1)}(\lambda_k^*)=0\neq\mu^{(\nu)}(\lambda_k^*) \tag{3.5}$$

for some finite ν. We are now ready for the construction of root vectors.

Theorem 3.1. *In the simple case satisfying (3.2) and (3.5), the joint root subspace* $J(\boldsymbol{\lambda}^*):=\bigcap_{n=1}^{k}N(\Gamma_n-\lambda_n^*I)^d$, $d=\dim F$, *has a basis consisting of elements*

$$y_j=\sum_{i_1+\cdots+i_k=j}y_1^{i_1}\otimes\cdots\otimes y_k^{i_k},\qquad 0\leqq j<\nu$$

where

$$W_m(\boldsymbol{\lambda}^*)y_m^l=\sum_{n=1}^{k}V_{mn}\sum_{i=0}^{l-1}\gamma_n^{l-i}y_m^i,\qquad 1\leqq l<\nu \tag{3.6}$$

$\gamma_n^i=\lambda_n^{(i)}(\lambda_k^*)/i!$ *and* $y_m^0=x_m$ *as in* (*), $1\leqq m,n\leqq k$.

Proof. By simplicity and [5, Theorem 3.3], $J(\boldsymbol{\lambda}^*)$ is contained in $N(\Gamma_k-\lambda_k^*I)^\nu$ which has a basis $B=\{x^{\otimes}(\lambda_k^*),x^{\otimes\prime}(\lambda_k^*),\ldots,x^{\otimes(\nu-1)}(\lambda_k^*)\}$. Moreover

$$(\Gamma_n(\mu(\lambda_k))-\lambda_n(\lambda_k))x^{\otimes}(\lambda_k)=0,\qquad n=1,\ldots,k$$

and repeated differentiation, together with (3.5), gives

$$(\Gamma_n-\lambda_n^*I)x^{\otimes(l)}(\lambda_k^*)=\sum_{i=0}^{l-1}l!\gamma_n^{l-i}x^{\otimes(i)}(\lambda_k^*)/i!,\qquad 0\leqq l<\nu. \tag{3.7}$$

It follows inductively that

$$(\Gamma_n - \lambda_n^* I)^l x^{\otimes(l-1)}(\lambda_k^*) = 0$$

so $J(\boldsymbol{\lambda}^*)$ contains B, which is therefore a basis as required.

Thus it suffices to prove that

$$y_m^j = x_m^{(j)}(\lambda_k^*)/j!, \qquad 0 \leqq j < v,$$

satisfy (3.6). This is clear for $j=0$, so assume $v>1$. Since $x_m = x_m(\lambda_k)$ is holomorphic at λ_k^*, we have by repeated differentiation of $(*)$

$$W_m(\boldsymbol{\lambda}^*) x_m^{(l)}(\lambda_k^*) = \sum_{i=0}^{l-1} (l!) \gamma_n^{l-i} V_{mn} x_m^{(i)}(\lambda_k^*)/i! \tag{3.8}$$

for $1 \leqq m \leqq k$, and also for $m=k$ by virtue of (3.5).

Finally we compute

$$y_j = x^{\otimes(j)}(\lambda_k^*)/j! = \sum_{i_1 + \cdots + i_k = j} y_1^{i_1} \otimes \cdots \otimes y_k^{i_k}$$

and (3.6) is established. □

Remark 3.2. (1.6) is the special case of (3.6) obtained by setting $\lambda_1 = \lambda_2 = \cdots = \lambda_k$ and $y_m^j = x_m^j$.

Remark 3.3. Evidently $(*)$ yields

$$W_m(\boldsymbol{\lambda}^*)^\dagger x^\otimes = 0$$

and repeated differentiation leads to

$$W_m(\boldsymbol{\lambda}^*)^\dagger x^{\otimes(l)}(\lambda_k^*)/l! = \sum_{n=1}^{k} V_{mn}^\dagger \sum_{i=0}^{l-1} \gamma_n^{l-i} x^{\otimes(i)}(\lambda_k^*)/i!$$

$$= \sum_{n=1}^{k} V_{mn}^\dagger z_n,$$

say. Thus our basis elements automatically satisfy equations of the form (1.5).

We return now to the general case, when $\dim N(\Gamma_k - \lambda_k^* I)$ is an arbitrary finite number. Geometrically, (3.2) generates n_c curves parameterized by $\boldsymbol{\lambda}(\lambda_k)$ and touching each of the $n_k := \dim N(W_k(\boldsymbol{\lambda}^*))$ surfaces corresponding to the kth equation of $(*)$. Each of the $n_c n_k$ possible combinations leads to a different set of vectors satisfying (3.6), each with its own initial element y_0 and its own length v. These $n_c n_k$ sets form our basis of $J(\boldsymbol{\lambda}^*)$.

Theorem 3.4. *If $\lambda \in \mathbb{R}^k$ then $J(\lambda^*)$ is a direct sum of subspaces spanned by sets of root vectors y_j as in Theorem 3.1 where the various initial elements $y_0 = x^{\otimes}$ form a basis for $\bigcap_{n=1}^{k} N(\Gamma_n - \lambda_n^* I) = \bigotimes_{m=1}^{k} N(W_m(\lambda^*))$.*

Proof. $N(\lambda_k)$, defined as in (3.1), is now finite dimensional, so several branches $(\mu(\lambda_k), x(\lambda_k))$ may exist holomorphic at λ_k^*. By Theorem 2.3, the $\Gamma_n(\mu(\lambda_k))$ on each set of coincident branches continue to generate a common eigenvector basis of $N(\lambda_k)$, provided $\mu(\lambda_k)$ is small and nonzero. We may now repeat the analysis of the simple case, choosing basis elements $x(\lambda_k)$ to be decomposable and to satisfy (3.2) for some $\lambda(\lambda_k)$, which are again $\mathbb{R}^k$-valued and holomorphic at λ_k^* by (H_{0kk}) self-adjointness and holomorphy of the operators $A(\lambda_k)$ defined as in (3.3). Thus the $W_m(\lambda(\lambda_k))$ in (3.2) are H_m self-adjoint and holomorphic, and so we may choose the $x_m(\lambda_k)$ to be holomorphic at λ_k^*.

We now apply Theorem 3.1 to each branch in turn. An easy extension of [**5**, Theorem 3.3] shows that the $x^{\otimes(l)}(\lambda^*)$ form a basis of $N(\Gamma_k - \lambda_k^* I)^d$. Repeating the argument with k replaced by each n in turn, we automatically restrict the y_0 to $\bigcap_{n=1}^{k} N(\Gamma_n - \lambda_n^* I)$ and the y_j generate a basis of $J(\lambda^*)$ as required.

4. Remarks and special cases

4.1. Determination of λ_n^i. At first sight this seems to require the eigenvalues λ_n as *functions* of λ_k, but in fact much less information is needed. Let us illustrate for small v, using lower case letters for quadratic forms, e.g. $v_{mn}(x) = (x, V_{mn}x)$, $\delta_{0kk}(y) = (y, \Delta_{0kk}y)$.

From (3.8) with $l = 1$ we have, with $x_m = x_m(\lambda_k^*)$,

$$0 = (x_m, W_m(\lambda^*)x_m'(\lambda_k^*)) = \sum_{n=1}^{k} \lambda_n'(\lambda_k^*) v_{mn}(x_m), \qquad 1 \leqq m < k.$$

Since $\Delta_{0kk} \gg 0$, we thus have a uniquely soluble system of linear equations in the unknowns $\lambda_n'(\lambda_k^*)$, $1 \leqq n < k$. In fact

$$\lambda_n'(\lambda_k^*) = \delta_{0kn}(x^{\otimes}) / \delta_{0kk}(x^{\otimes})$$

i.e. a quotient of $(k-1) \times (k-1)$ determinants with entries of the form $v_{mn}(x_m)$, and no explicit differentiation is required to calculate γ_n^1.

We now use (3.8) to find $x_m'(\lambda_k^*)$, again without explicit differentiation and proceed to $l = 2$, giving

$$\begin{aligned} 0 &= (x_m, W_m(\lambda^*)x_m''(\lambda_k^*)) \\ &= 2\sum_{n=1}^{k} \lambda_n'(\lambda_k^*)(x_m, V_{mn}x_m(\lambda_k^*)) + \sum_{n=1}^{k} \lambda_n''(\lambda_k^*) v_{mn}(x_m) \end{aligned}$$

which may be solved uniquely for $\lambda_n''(\lambda_k^*)$, $1 \leqq n < k$. This yields γ_n^2, and so on.

4.2. Jordan structure of the Γ_n. In the simple case, (3.7) shows that the $x^{\otimes(i)}/i!$ form a Jordan basis for Γ_k, i.e. Γ_k has Jordan block structure relative to this basis. Similarly Γ_n

has Toeplitz structure. Since any set of matrices commuting with a Jordan block will be of this form, the Γ_n thus inherit no special properties (other than commutativity) from the multiparameter connection in the simple case. In the general case, however, the Γ_n are direct sums of blocks as above, and this is a considerable specialization from the arbitrary commuting case.

4.3. Nonnegative Δ_n. If at least one of the Δ_n is nonnegative definite, say $\Delta_k \geqq 0$, then λ_k must be real [7, Lemma 5.1]. Thus (3.3) gives

$$\lambda_j = \delta_{0kk}(x^{\otimes})^{-1}(\lambda_k \delta_{0kj}(x^{\otimes}) + \delta_{jkk}(x^{\otimes}))$$

in the quadratic form notation of 4.1, and so $\boldsymbol{\lambda} \in \mathbb{R}^k$.

If $\Delta_k \gg 0$ then λ_k is an eigenvalue of the compact self-adjoint operator $B_k = \Delta_k^{-1}\Delta_0$ on $D(\Delta_k^{1/2})$ with inner product given by $[x,y] = (\Delta_k^{1/2}x, \Delta_k^{1/2}y)$, and is thus a nondefective eigenvalue. The analysis of 4.2 thus shows that one may take $v=1$ in (1.4). This case occurs e.g. when $T_m \gg 0$, i.e. ULD.

If $\Delta_k \geqq 0$ but not $\gg 0$, i.e. $N(\Delta_k)$ is nontrivial, then Γ_k has Jordan chains of length at most two, and if the length is two then $\lambda_k = 0$ [7, Lemma 5.1]. Appealing again to 4.2, then, we see that F is spanned by Jordan chains of the form $\{x^{\otimes}\}$ or $\{x^{\otimes}, x^{\otimes\prime}\}$.

The analysis of 4.1 thus gives a complete description of F in terms of the original data in (*): $x^{\otimes} = x_1 \otimes \cdots \otimes x_k, x_m$ as in (*) and

$$x^{\otimes\prime} = \sum_{m=1}^{k} x_1 \otimes \cdots \otimes x_{m-1} \otimes x'_m \otimes x_{m+1} \otimes \cdots \otimes x_k$$

where

$$W_m(\boldsymbol{\lambda})x'_m = \sum_{n=1}^{k} \delta_{0kn}(x^{\otimes}) V_{mn} x_m / \delta_{0kk}(x^{\otimes}). \tag{4.1}$$

In the case of $k=2$ Sturm–Liouville equations, this result can be obtained from [9, Theorem 5.5] although it is stated differently. In the case where each $T_m \geqq 0$ (so *each* $\Delta_n \geqq 0$) the Jordan chain structure of F (and its dimension) were analysed in [7, Section 5] but without explicit formulae for $x^{\otimes\prime}$.

4.4 An example. Let $k=2$, $H_1 = H_2 = \mathbb{C}^2$,

$$T_1 = T_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_{11} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad V_{21} = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}, \quad V_{22} = -2V_{12} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then $0 = \det W_1(\boldsymbol{\lambda}) = \varepsilon^2 + \varepsilon - \lambda_1^2$, $\varepsilon = 2\lambda_1 - \lambda_2$ and $0 = \det W_2(\boldsymbol{\lambda}) = 4\varepsilon^2 - 2\varepsilon - \lambda_1^2$. The solutions are $\varepsilon = 1$, giving $\boldsymbol{\lambda} = (\pm\sqrt{2}, \pm 2\sqrt{2} - 1)$, and $\varepsilon = 0$, giving $\boldsymbol{\lambda} = \mathbf{0}$ (a double root). When $\varepsilon = 1$, we calculate eigenvectors

$$x_1 = \begin{bmatrix} \mp 1 \\ \sqrt{2} \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} \pm\sqrt{2} \\ 1 \end{bmatrix}.$$

https://doi.org/10.1017/S0013091500006866 Published online by Cambridge University Press

When

$$\varepsilon=0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_m = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{say } x_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, m=1,2.$$

The root vector $x^{\otimes\prime} = x_1' \otimes x_2 + x_1 \otimes x_2'$ may be calculated via (4.1). Evidently

$$\delta_{021}(x^{\otimes}) = -v_{12}(x_1) = 1 \quad \delta_{022}(x^{\otimes}) = v_{11}(x_1) = 2$$

so

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_1' = 1/2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \quad \text{say } x_1' = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_2' = 1/2 \begin{bmatrix} 1 \\ -4 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \quad \text{say } x_2' = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}.$$

Using the isomorphism $H^2 \cong \mathbb{C}^4$, we may write the two eigentensors corresponding to $\varepsilon=1$ as $(-\sqrt{2}, \mp 1, \pm 2, \sqrt{2})$, the one corresponding to $\varepsilon=0$ as $(0,0,0,1)$ and the root vector $x^{\otimes\prime}$ as $(0,1/2,0,0)+(0,0,1/2,0)=(0,1/2,1/2,0)$. It is readily verified that these four elements are indeed a basis of $\mathbb{C}^4$.

Acknowledgement. I thank the Universities of Dundee and Strathclyde for their hospitality during the preparation of this work. I also thank Patrick Browne for various helpful discussions and for criticizing the original draft.

REFERENCES

1. F. V. Atkinson, Multiparameter spectral theory, *Bull. Amer. Math. Soc.* **74** (1968), 1–27.

2. F. V. Atkinson, *Multiparameter Eigenvalue Problems Vol. 1* (Academic Press, 1972).

3. P. A. Binding, Multiparameter definiteness conditions, *Proc. Roy. Soc. Edinburgh* **89A** (1981), 319–332.

4. P. A. Binding, Left definite multiparameter eigenvalue problems, *Trans. Amer. Math. Soc.* **272** (1982), 476–486.

5. P. A. Binding and P. J. Browne, Applications of two parameter spectral theory to symmetric generalised eigenvalue problems, preprint.

6. P. A. Binding and K. Seddighi, On root vectors for self-adjoint pencils, *J. Funct. Anal.* **70** (1987), 117–125.

7. P. A. Binding and K. Seddighi, Elliptic multiparameter eigenvalue problems, *Proc. Edinburgh Math. Soc.* **30** (1987), 215–228.

8. M. Faierman, Expansions in eigenfunctions of two-parameter system of differential equations I–IV, preprints.

9. G. A. Gadzhiev, On a multitime equation and its reduction to a multiparameter spectral problem, *Soviet Math. Dokl.* **32** (1985), 710–713.

10. G. A. Isaev, On root elements of multiparameter spectral problems, *Soviet Math. Dokl.* **21** (1980), 127–130.

11. A. Kallstrom and B. D. Sleeman, Solvability of a linear operator system, *J. Math. Anal. Appl.* **55** (1976) 785–793.

12. A. Kallstrom and B. D. Sleeman, A left definite multiparameter eigenvalue problem in ordinary differential equations, *Proc. Roy. Soc. Edinburgh* **74A** (1976), 145–155.

13. T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, 1976).

Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada T2N IN4