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INVARIANTS FOR AUTOMORPHISMS OF CERTAIN ITERATED SKEW POLYNOMIAL RINGS

by DAVID A. JORDAN and IMOGEN E. WELLS*

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Rings of invariants are identified for some automorphisms θ of certain iterated skew polynomial rings R, including the enveloping algebra of $sl_2(k)$, the Weyl algebra A_1 and their quantizations. We investigate how finite-dimensional simple R-modules split over the ring of invariants R^θ and how finite-dimensional simple R^θ-modules extend to R.

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1. Introduction

In a sequence of papers [3, 4, 5, 7] the first author has studied a class of iterated skew polynomial rings R in two indeterminates y and x over a finitely generated commutative k-algebra A, where k is an algebraically closed field. The principal example is the enveloping algebra of the Lie algebra $sl_2(k)$ [3]. Other examples include the quantized enveloping algebra of $sl_2(k)$ [3], the quantized Weyl algebra in two variables [5, 7], the coordinate rings for various quantum groups [3, 5, 7], and the enveloping algebra of the dispin Lie superalgebra [5, 7].

Given a positive integer n not divisible by char k, these algebras all admit an automorphism θ of order n acting as the identity on A and with $y \mapsto \omega y$ and $x \mapsto \omega^{-1}x$, where $\omega \in k$ is a primitive nth root of unity. The purpose of this paper is to study the ring of invariants for such automorphisms for a slightly more general iterated skew polynomial ring R in two variables, including, for example, the ordinary Weyl algebra A_1 as well as the quantized Weyl algebra. The ring of invariants turns out to be a factor of a ring constructed in the same way from the polynomial ring $A[w]$ as R is from A. As a consequence, the results in [7] determining the finite-dimensional simple modules over R may be applied to determine the finite-dimensional simple modules over the ring of invariants R^θ. Indeed it is possible to see how each finite-dimensional simple R-module splits over R^θ. We shall see that for a certain class of finite-dimensional simple R-module X, which often yields all the finite-dimensional simple modules, X is the direct sum of r simple R^θ-modules of dimension $q+1$ and, provided $q>0$, $n-r$ simple R^θ-modules of dimension q, where $\dim_k X = qn + r$, $0 \le r < n$. From this it follows, for example, that

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if $R = U(sl_2(k))$ then R^θ has n^2 simple modules of each positive dimension. This result for $U(sl_2(k))$ has been obtained independently by Kraft and Small [8].

2. Basic details of R and R^θ

2.1 The ring R. Let A be a finitely generated commutative algebra over an algebraically closed field k, let α be a k-automorphism of A, let $v \in A$ and let $\rho \in k\backslash\{0\}$. Let S be the skew polynomial ring $A[y; \alpha]$ and extend α to S by setting $\alpha(y) = \rho^{-1}y$. There is an α^{-1}-derivation δ of S such that $\delta(A) = 0$ and $\delta(y) = v$. This is a special case of a construction of skew derivations described in [2, 2.8]. The ring $R = R(A, \alpha, v, \rho)$ is the skew polynomial ring $S[x; \alpha^{-1}, \delta]$. Thus $xy - \rho yx = v$ and, for all $a \in A$, $xa = \alpha^{-1}(a)x$ and $ya = \alpha(a)y$.

This notation will be fixed throughout the paper.

2.2 Casimir element. If $v = u - \rho\alpha(u)$ for some $u \in A$ then the element $z = xy - u = \rho(yx - \alpha(u))$ is a normal element of R inducing a k-automorphism β of R such that $\beta(a) = a$ for all $a \in A$, $\beta(y) = \rho y$ and $\beta(x) = \rho^{-1}x$. We shall say that the 4-tuple (A, α, v, ρ) is *conformal* when v has this form $u - \rho\alpha(u)$. Examples of rings which arise in the conformal case are listed in [3], where $\rho = 1$, and [7]. If $\rho = 1$ then z is central and if ρ is an nth root of unity then z^n is central. In the general case, we set $w = xy = \rho yx + v$. Then $aw = wa$ for all $a \in A$ and when v is of the form $u - \rho\alpha(u)$, $w = z + u$ and $A[z] = A[w]$.

We extend α to a k-automorphism, also denoted α, of the ring $A[w]$ by setting $\alpha(w) = \rho^{-1}(w - v)$ and $\alpha^{-1}(w) = \rho w + \alpha^{-1}(v)$. In the conformal case, $\alpha(z) = \rho^{-1}z$ and, in general, $yw = \alpha(w)y$ and $xw = \alpha^{-1}(w)x$.

In the conformal case, the normal element z generates a non-zero proper ideal of R which consequently is not simple. The most obvious example of a ring of the form R in which v is not of the form $u - \rho\alpha(u)$ is the Weyl algebra A_1 where $A = k$, $\alpha = \mathrm{id}$, $\rho = 1$ and $v = 1$ whereas $u - \rho\alpha(u) = 0$ for all $u \in A$. The ring A_1 is well known to be simple if and only if $\operatorname{char} k = 0$. Conditions for R to be simple will be discussed in [11].

2.3 Identities. Set $v_0 = 0$ and, for $m \geq 1$, $v_m = \sum_{j=0}^{m-1} \rho^j\alpha^j(v)$. It is easy to see that, for all m, $m' \in \mathbb{N}$, $v_m + \rho^m\alpha^m(v_{m'}) = v_{m+m'}$ and that, in the conformal case, $v_m = u - \rho^m\alpha^m(u)$. The following identities, which hold for $m \geq 1$, can be checked inductively. In (3)–(6), α is extended to $A[w]$ as in 2.2.

(1) $xy^m - \rho^m y^m x = v_m y^{m-1}$.

(2) $x^m y - \rho^m yx^m = \alpha^{1-m}(v_m)x^{m-1}$.

(3) $x^m y^m = \prod_{j=0}^{m-1} \alpha^{-j}(w)$.

(4) $y^m x^m = \prod_{j=1}^{m} \alpha^j(w)$.

(5) $\alpha^m(w) = \rho^{-m}(w - v_m)$.

(6) $\alpha^{-m}(w) = \rho^m w + \alpha^{-m}(v_m)$.

Note that (1) is equivalent to saying that $\alpha^{-1}(y^m) = \rho^m y$ and $\delta(y^m) = v_m y^{m-1}$.

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2.4 Grading. Every element of R is a unique A-linear combination of the elements $y^i x^j, i \geq 0, j \geq 0$. For $m \in \mathbb{Z}$, let R_m be the set of all A-linear combinations of the elements $y^i x^j$ with $i - j = m$. Because of the relations used to define R as a ring extension of A, $R = \bigoplus_{m \in \mathbb{Z}} R_m$ is a $\mathbb{Z}$-graded ring. Note that $w \in R_0$. By 2.3 (3) and (4), $R_0 = A[w]$, while if $m > 0$ then $R_m = y^m A[w] = A[w] y^m$ and $R_{-m} = x^m A[w] = A[w] x^m$. Note that R_m can also be described as the set of all A-linear combinations of the elements $x^i y^j$ with $j - i = m$. In the conformal case, $R_m = A[z] y^m$ or $A[z] x^{-m}$ as appropriate.

2.5 The factor ring T. Suppose that (A, α, v, ρ) is *conformal*. We shall denote the factor ring R/zR, which was studied in [4] and [6], by T or, more explicitly, $T(A, \alpha, u)$. It is the ring extension of A generated by X and Y, the images of x and y respectively, subject to the relations

$$XY = u, \quad YX = \alpha(u)$$

and, for all $a \in A$,

$$Ya = \alpha(a)Y, \quad Xa = \alpha^{-1}(a)X.$$

Note that T is independent of the parameter ρ. As the normal element z is homogeneous of degree 0, there is an induced $\mathbb{Z}$-grading on T, $T = \bigoplus_{m \in \mathbb{Z}} T_m$, where $T_0 = A$, and, for $m > 0$, $T_m = Y^m A = AY^m$ and $T_{-m} = X^m A = AX^m$. Each element has a canonical form

$$a_n X^n + \ldots + a_1 X + a_0 + b_1 Y + \ldots + b_m Y^m, \quad \text{where } a_i, b_j \in A.$$

2.6 The automorphism θ of R and its invariants. Let n be a positive integer and let $\omega \in k$ be a primitive nth root of unity. Note that n must be invertible in k. There is a k-automorphism θ of R acting as the identity on A and with $\theta(y) = \omega y$ and $\theta(x) = \omega^{-1} x$. Clearly θ has order n and, for the grading described in 2.4, θ is a graded automorphism. It is easy to identify the ring R^θ of invariants in terms of this grading. For each m, θ acts on R_m by multiplication by ω^m. Thus

$$R^\theta = \bigoplus_{j \in \mathbb{Z}} R_{nj},$$

that is,

$$R^\theta = \left(\bigoplus_{j \geq 1} A[w] y^{nj} \right) \bigoplus A[w] \bigoplus \left(\bigoplus_{j \geq 1} A[w] x^{nj} \right).$$

Writing B for the polynomial ring $A[w]$,

$$R^\theta = \left(\bigoplus_{j \geq 1} BY^j \right) \bigoplus B \bigoplus \left(\bigoplus_{j \geq 1} BX^j \right).$$

By 2.2(3) and (4), $XY = U_n$ and $YX = \alpha^n(U_n)$, where $U_n = \prod_{j=0}^{n-1} \alpha^{-j}(w)$ so that $\alpha^n(U_n) = \prod_{j=1}^{n} \alpha^j(w)$.

It follows that there is a surjective graded homomorphism $\chi : T(B, \alpha^n, U_n) \to R^\theta$ given by $\chi(X) = x^n$ and $\chi(Y) = y^n$. Given that χ is graded, it is easily seen to be an isomorphism. Thus we have the following result.

Theorem. *The fixed ring R^θ of R for θ is isomorphic to $T(B, \alpha^n, U_n)$.*

In the remainder of the paper we shall use the isomorphism χ to identity R^θ with $T(B, \alpha^n, U_n)$. This includes the case $n = 1$ which gives a different way of viewing the ring R.

Corollary. $R(A, \alpha, v, \rho) = T(A[w], \alpha, w)$, *where* α *is extended to* $A[w]$ *by setting* $\alpha(w) = \rho^{-1}(w - v)$.

This corollary gives a way of generalizing known results from the conformal case with $\rho = 1$ to the general case. In 3.7 we shall do this for a result from [3] on extensions of simple modules.

2.7 The automorphism θ of T and its invariants. Consider a k-algebra of the form $T = T(A, \alpha, u) = R/zR$ where $R = R(A, \alpha, u - \alpha(u), 1)$. The automorphism θ of R introduced in 2.6 fixes z and induces a k-automorphism of order n, also to be denoted θ, of T acting as the identity on A and with $\theta(Y) = \omega Y$ and $\theta(X) = \omega^{-1}X$. It is easy to see that the fixed ring is the ring extension of A generated by Y^n and X^n subject to the relations

$$X^n Y^n = u_n, \quad Y^n X^n = \alpha^n(u_n),$$

where $u_n = \prod_{j=0}^{n-1} \alpha^{-j}(u)$. Hence T^θ can be identified with $T(A, \alpha^n, u_n)$, where the indeterminates are written as Y^n and X^n rather than Y and X.

When R is identified with $T(A[w], \alpha, w)$ using Corollary 2.6, the automorphism θ of R becomes the automorphism θ of $T(A[w], \alpha, w)$ as above, that is θ acts as the identity on $A[w]$, $\theta(Y) = \omega Y$ and $\theta(X) = \omega^{-1}X$. The descriptions in Theorem 2.6 and above of the invariants for R and rings of the form T both give the ring of invariants of $R = T(A[w], \alpha, w)$ to be $T(A[w], \alpha^n, U_n)$. As it is easier to work with rings of the form T than those of the form R, we shall often work with a general ring T and then use Corollary 2.6 to apply results to the general ring $R(A, \alpha, v, \rho)$.

2.8 Examples. One case for which the fixed ring already appears in the literature is when $A = k$, $\rho = 1 = v$ and $\alpha = \text{id}$ so that R is the Weyl algebra A_1. In the notation of this paper, the fixed ring is isomorphic to $T(k[w], \alpha^n, \prod_{j=0}^{n-1}(w + j))$ where $\alpha(w) = w - 1$. The algebras similar to the enveloping algebra of $sl_2(k)$ which are the subject of **[10]** are of the form $R = R(A, \alpha, v, \rho)$ with $A = k[w]$ and in **[10]** the fixed ring of the automorphism θ of A_1 is identified as a factor of such an algebra.

For another example, take $A = k[t]$, $\alpha(t) = t+2$, $u = -\frac{1}{4}(t-1)^2$ and $\rho = 1$ so that $v = u - \rho\alpha(u) = t$. In this case, R is the enveloping algebra of the Lie algebra $sl_2(k)$. (In the more standard notation for this algebra, x, y and t are written as e, f and h.) Here α extends to $A[w] = k[t, w]$ with $\alpha(w) = w - t$ and $\alpha^{-1}(w) = w + t - 2$. The element U_n in 2.6 is $\prod_{j=0}^{n-1}(w + jt - j^2 - j)$ and the fixed ring of θ is isomorphic to $T(k[t, w], \alpha^n . U_n)$.

Finally, suppose that $\operatorname{char} k \neq 2$ and take $A = k[t]$, $\alpha(t) = t+1$, $\rho = -1$ and $u = (2t-1)/4$ so that $v = u + \alpha(u) = t$. When $k = \mathbb{C}$, the ring R is the universal enveloping algebra of the dispin Lie superalgebra $B[0, 1]$, see [7, Example 1.3]. Here α extends to $A[w] = k[t, w]$ with $\alpha(w) = t - w$ and $\alpha^{-1}(w) = t - w - 1$. The fixed ring of θ is isomorphic to $T(k[t, w], \alpha^n, U_n)$, where, in the formula, $U_n = \prod_{j=0}^{n-1} \alpha^{-j}(w)$,

$$\alpha^{-j}(w) = \begin{cases} w - \frac{j}{2} & \text{if } j \text{ is even} \\ -w + t - \frac{j+1}{2} & \text{if } j \text{ is odd.} \end{cases}$$

3. Finite-dimensional simple modules

3.1 Finite-dimensional simple R-modules. If $\alpha^s(I) = I$ for an ideal I of A or $A[w]$ and some positive integer s then we say that I is *periodic* and call the least such s the *order* of I.

Theorem. (a) *Let $R = R(A, \alpha, v, \rho)$. Every finite-dimensional simple R-module is isomorphic to one of the following:*

(i) *the d-dimensional module $L(M) = R/(MR + xR + y^dR)$ for each maximal ideal M of A containing v_d for some (minimal) $d > 0$;*

(ii) *the s-dimensional modules $R/(NR + (y^s - \xi)R)$ and $R/(NR + (x^s - \xi)R)$ for each periodic maximal ideal N of $A[w]$ of order s and each $0 \neq \xi \in k$.*

(b) *Let $T = T(A, \alpha, u)$. Every finite-dimensional simple R-module is isomorphic to one of the following:*

(i) *the d-dimensional module $\mathcal{L}(M) = T/(MT + XT + Y^dT)$ fr each maximal ideal M of A containing u and $\alpha^d(u)$ for some minimal $d > 0$;*

(ii) *the s-dimensional modules $T/(MT + (Y^s - \xi)T)$ and $T/(MT + (X^s - \xi)T)$ for each periodic maximal ideal M of A of order s and each $0 \neq \xi \in k$.*

Proof. (a) is proved in [7] for the conformal case. We shall deduce (b) from the conformal case of (a) and the general case of (a) from (b).

Recall that $T = R/zR$, where $R = R(A, \alpha, u - \alpha(u), 1)$ and $z = xy - u$. The simple R-modules are then as given in (a). A simple R-module of type (i) is annihilated by z if and only if $u \in M$. Here $v_d = u - \alpha^d(u)$ and so those simple R-modules $L(M)$ which are also simple T-modules are those with $u \in M$ and $\alpha^d(u) \in M$. The periodic maximal ideals of $A[w] = A[z]$ for which the simple R-module of type (ii) gives rise to a simple T-module are of the form $MA[z] + zA[z]$ where M is a periodic maximal ideal of A. Thus (b) follows from the conformal case of (a).

For the general case of (a), recall from Corollary 2.6 that $R(A, \alpha, v, \rho) =$

$T(A[w], \alpha, w)$. For each $d \geq 1$, there is a bijection between the maximal ideals of A containing v_d and the maximal ideals of $A[w]$ containing w and $\alpha^d(w)$ given by $M \mapsto MA[w] + wA[w]$. Also, if $T' = T(A[w], \alpha, w)$ then, because $w = XY$, $(MA[w] + wA[w])T' + XT' + Y^dT' = MT' + XT' + Y^dT'$. The general case of (a) then follows from (b).

Note that if A has no periodic maximal ideals, as in the cases of the enveloping algebras of the Lie algebra $sl_2(k)$ and the dispin Lie superalgebra when char $k = 0$, only type (i) occurs.

3.2 Finite-dimensional simple T^θ-modules. Let $T = T(A, \alpha, u)$, let n be a positive integer and let θ be the automorphism of T introduced in 2.7. Thus $T^\theta = T(A, \alpha^n, u_n)$, where $u_n = \prod_{j=0}^{n-1} \alpha^{-j}(u)$ and the indeterminates are Y^n and X^n rather than Y and X.

Theorem 3.1(b) gives the following classification of the finite-dimensional simple T^θ-modules. Every finite-dimensional simple T^θ-module is isomorphic to one of the following:

(i) the d-dimensional module $\mathcal{L}_\theta(N) = T^\theta/(NT^\theta + X^nT^\theta + Y^{nd}T^\theta)$ for each maximal ideal N of A containing u_n and $\alpha^{nd}(u_n)$ for some (minimal) $d > 0$;

(ii) the s-dimensional modules $T^\theta/(NT^\theta + (Y^{ns} - \xi)T^\theta)$ and $T^\theta/(NT^\theta + (X^{ns} - \xi)T^\theta)$ for each periodic maximal ideal N of A of order s under α^n and each $0 \neq \xi \in k$.

A similar classification for the finite-dimensional simple R^θ-modules can be derived using Theorem 2.6.

We next analyse how the finite-dimensional simple T-modules split over T^θ.

Theorem. *Let M be a maximal ideal of A giving rise to a d-dimensional simple T-module $\mathcal{L}(M) = T/(MT + XT + Y^dT)$ as in 3.1(a)(i). Write $d = qn + r$ where $0 \leq r < n$. Then $\mathcal{L}(M)$ is isomorphic to the direct sum of r simple T^θ-modules $\mathcal{L}_\theta(\alpha^{-i}(M))$, $0 \leq i < r$, of dimension $q + 1$ and, if $q > 0$, $n - r$ simple T^θ-modules $\mathcal{L}_\theta(\alpha^{-i}(M))$, $r \leq i < n$, of dimension q. Furthermore every finite-dimensional T^θ-module of the form $\mathcal{L}_\theta(N)$ occurs as a T^θ-summand of a simple T-module $\mathcal{L}(\alpha^i(N))$ for some i.*

Proof. Recall that $d > 0$ is minimal with $u \in M$ and $\alpha^d(u) \in M$. For $0 \leq i \leq n-1$, set $N_i = \alpha^{-i}(M)$. Thus $u_n \in N_i$ for each i. Note that $\alpha^{nq}(u_n) = \prod_{j=0}^{n-1} \alpha^{nq-j}(u)$ and $\alpha^{n(q+1)}(u_n) = \prod_{j=1}^{n} \alpha^{nq+j}(u)$. Let $0 \leq i < r$. Then $\alpha^{nq+(r-i)}(u) = \alpha^{d-i}(u) \in N_i$ and so $\alpha^{n(q+1)}(u_n) \in N_i$. Moreover, $q + 1$ is the least positive integer j such that $\alpha^{nj}(u_n) \in N_i$. Thus there is a $q + 1$-dimensional simple T^θ-module $\mathcal{L}_\theta(N_i)$. Now suppose that $r \leq i < n$. Notice that this implies that $q > 0$ and $d \geq n$. Then $\alpha^{nq-(i-r)}(u) = \alpha^{d-i}(u) \in N_i$ and so $\alpha^{nq}(u_n) \in N_i$. Furthermore, q is the least positive integer j such that $\alpha^{nj}(u_n) \in N_i$. Hence there is a q-dimensional simple T^θ-module $\mathcal{L}_\theta(N_i)$. Thus, unless $q = 0$, in addition to the above r simple T^θ-modules of dimension $q + 1$, M gives rise to $n - r$ simple T^θ-modules of dimension q.

We next show that, over T^θ, the simple T-module $\mathcal{L}(M)$ splits as the direct sum of the simple modules $\mathcal{L}_\theta(N_i)$, $1 \leq i \leq m$, where m is the minimum of n and d. For $0 \leq j \leq d-1$, let $b_j = Y^j + (MT + XT + Y^dT)$. Then $\{b_j\}_{0 \leq j \leq d-1}$ is a basis for $\mathcal{L}(M)$ and each $Ab_j = b_jA$ is a one-dimensional A-submodule of $\mathcal{L}(M)$ with $\text{ann}_A b_j = \alpha^{-j}(M)$. The action of X and Y is given by the rules $b_0X = 0 = b_{d-1}Y$, $b_jY = b_{j+1}$ for $0 \le j \le d-2$ and, because $YX = \alpha(u)$, $b_jX = b_{j-1}\alpha(u)$ for $1 \le j \le d-1$.

Fix $0 \le i < m$ and let $e_i = \dim \mathcal{L}_0(N_i)$. Thus $e_i = q+1$ if $0 \le i < r$ and $e_i = q$ if $q > 0$ and $r \le i < n$. Then $\mathcal{L}_0(N_i)$ has a basis $\{c_{ij}\}_{0\le j\le e_i-1}$ analogous to the basis $\{b_j\}$ of $\mathcal{L}(M)$. Thus $c_{ij} = Y^{nj} + (MT^0 + X^nT^0 + Y^{ne_i}T^0)$, each $Ac_{ij} = c_{ij}A$ is a one-dimensional A-submodule of $\mathcal{L}_0(N_i)$, $\operatorname{ann}_A c_{ij} = \alpha^{-jn}(N_i)$, $c_{i0}X^n = 0 = c_{i,d-1}Y^n$, $c_{ij}Y^n = c_{i,j+1}$ for $0 \le j \le e_i - 2$, and $c_{ij}X^n = b_{j-1}\alpha^n(u_n)$ for $1 \le j \le e_i - 1$. Now let L_i be the subspace $\bigoplus_{j=0}^{e_i-1} b_{i+jn}A = b_iA + b_{i+n}A + \dots b_{i+(e_i-1)n}A$ of $\mathcal{L}(M)$. As T^0 is generated, as a ring extension of A, by Y^n and X^n, each L_i is an e_i-dimensional T^0-submodule of $L(M)$. Clearly $L(M) = \bigoplus_{i=0}^{m-1} L_i$.

Each L_i has basis $\{b_{i+jn}\}_{0\le j\le e_i-1}$ and $\operatorname{ann}_A b_{i+jn} = \alpha^{-(i+jn)}(M) = \alpha^{-nj}(N_i) = \operatorname{ann}_A c_{ij}$. Also $b_iX^n = 0 = b_{i+(e_i-1)n}Y^n$ and $b_{j+jn}Y^n = b_{i+(j+1)n}$ for $0 \le j \le e_i - 2$. If $1 \le j \le e_i - 1$ then $b_{i+jn}X^n = b_{i+(j-1)n}\alpha(u)\alpha^2(u)\dots\alpha^n(u) = b_{i+(j-1)n}\alpha^n(u_n)$. Hence there is a T^0-isomorphism $f : L_i \to \mathcal{L}_0(N_i)$ given by $f : b_{i+jn} \mapsto c_{ij}$. Identifying each L_i with $f(L_i)$, $\mathcal{L}(M)$ splits over T^0 as claimed.

Now let N be a maximal ideal of A for which there is a finite-dimensional simple T^0-module $\mathcal{L}_0(N)$. Then $u_n = \prod_{j=0}^{n-1} \alpha^{-j}(u) \in N$ and $\dim \mathcal{L}_0(N)$ is the minimal positive integer e such that $\alpha^{ne}(u_n) \in N$. As $u_n \in N$ there exists a minimal integer j_1 such that $0 \le j_1 \le n-1$ and $u \in \alpha^{j_1}(N)$. Let $M = \alpha^{j_1}(N)$. Also there exists a maximal integer j_2 with $0 \le j_2 \le n-1$ and $u \in \alpha^{j_2-ne}(N)$. If $d = ne - j_2 + j_1$ then $\alpha^d(u) \in \alpha^{j_1}(N)$ and, by the choice of e, j_1 and j_2, d is the least positive integer with $\alpha^d(u) \in M$. Thus there is a d-dimensional simple T-module $\mathcal{L}(M)$ and, in the above notation, $N = N_{j_1}$ is a T^0-summand of $\mathcal{L}(M)$.

3.3 Finite-dimensional simple *R*-modules and R^0-modules. Let θ be the automorphism of R introduced in 2.6. Thus $R^0 = T(A[w], \alpha^n, U_n)$. Theorem 3.2 can be applied to the finite-dimensional simple R-modules and R^0-modules.

Corollary. *Let M be a maximal ideal of A giving rise to a d-dimensional simple R-module $L(M) = R/(MR + xR + y^dR)$ as in 3.1(a)(i). Write $d = qn + r$, where $0 \le r < n$. Then $L(M)$ is isomorphic to the direct sum of r simple R^0-modules $\mathcal{L}(\alpha^{-i}(MA[w] + wA[w]))$, $0 \le i < r$, of dimension $q+1$ and, if $q > 0$, $n - r$ simple R^0-modules $\mathcal{L}(\alpha^{-i}(MA[w] + wA[w]))$, $r \le i < n$, of dimension q. Furthermore every finite-dimensional R^0-module of the form $\mathcal{L}(N)$ occurs as an R^0-summand of a simple R-module $L(\alpha^i(N) \cap A)$ for some i.*

Proof. When R is identified with $T(A[w], \alpha, w)$ as in 2.6, $L(M)$ becomes $\mathcal{L}(MA[w] + wA[w])$, see 3.1. The result follows on applying Theorem 3.2 to $T(A[w], \alpha, w)$.

3.4 Examples. With n and θ as in 2.6, we discuss three examples in which A has no periodic maximal ideals so that, by 3.1 and 3.3, every finite-dimensional R-module has the form $L(M)$ and splits as the direct sum of r simple R^0-modules of dimension $q+1$ and, if $q > 0$, $n - r$ simple modules of dimension q, where $\dim L(M) = qn + r$ and $0 \le r < n$. All finite-dimensional simple R^0-modules occur in this way.

(i) As is well-known, or can be seen from the classification in 3.1 (see [3, 3.17] for the details), if char $k = 0$ then $R = U(sl_2(k))$ has a unique d-dimensional simple module L_d for each positive integer d. Fix a positive integer j. The values of d for which L_d has j-dimensional simple R^0-summands are $d = n(j-1)+s$, $1 \le s \le n$, in which case there are s such summands, and $d = nj + s$, $1 \le s \le n$, in which case there are $n - s$. Thus the number of j-dimensional simple modules is $\sum_{s=1}^{n}(s+n-s) = n^2$. This result has been obtained independently by Kraft and Small, [8, Example 3].

(ii) The quantized enveloping algebra $U_q(sl_2(k))$ is a ring of the form R and, provided q is not a root of unity, has four d-dimensional simple modules for each positive integer d. See [3, 3.19] for details. A similar calculation to the above shows that R^0 has $4n^2$ simple modules of dimension j for each positive integer j.

(iii) Here we consider the case where R is the enveloping algebra of the dispin Lie superalgebra and char $k = 0$. Thus $A = k[t]$, $\alpha(t) = t+1$ and $\rho = -1$. Then R has a unique d-dimensional simple module L_d for each odd positive integer d and no simple modules of even degree. See [7, 4.2] for details.

Suppose that $n = 2m - 1$ is odd. Fix an odd positive integer j. The values of d for which L_d has j-dimensional simple R^0-summands are $d = n(j-1)+s$, $1 \le s \le n$, s odd, in which case there are s such summands, and $d = nj + s$, $1 \le s \le n$, s even, in which case there are $n - s$. This gives a total of $2m^2 - 2m + 1$ simple modules of each odd dimension j. A similar calculation shows that there are $2m^2 - 2m$ simple modules of each even dimension j.

Now suppose that $n = 2m$ is even and fix a positive integer j. The values of d for which L_d has j-dimensional simple R^0-summands are $d = n(j-1)+s$, $1 \le s \le n$, s odd, in which case there are s such commands, and $d = nj + s$, $1 \le s \le n$, s odd, in which case there are $n - s$. This gives a total of $2m^2$ simple modules of each dimension j, whether j is odd or even.

3.5 Extending simple T^0-modules to T. Let N be a maximal ideal of A for which there is a finite-dimensional simple T^0-module $\mathcal{L}_0(N)$. As shown in 3.2, there exists a maximal ideal M of A such that $\mathcal{L}_0(N)$ is a direct summand, over T^0, of the simple T-module $L(M)$. It is reasonable to ask whether $\mathcal{L}_0(N) \otimes_{T^0} T$ must be isomorphic to $\mathcal{L}(M)$. If, with e, j_1 and j_2 as in the proof of Theorem 3.2, j_1 is the unique integer such that $0 \le j_1 \le n-1$ and $u \in \alpha^{j_1}(N)$ and j_2 is the unique integer with $0 \le j_2 \le n-1$ and $u \in \alpha^{j_2 - ne}(N)$ then the answer is positive. The following example shows that it is not positive in general.

Let char $k = 0$, let A be the polynomial ring $k[t, w]$, and let α be the k-automorphism of A such that $\alpha(t) = t+1$ and $\alpha(w) = w + t(t-1)(4t+1)$. Let $n = 2$ and let M be the maximal ideal $tA + wA$. Form the ring $T = T(A, \alpha, w)$. Then $w \in M$, $\alpha(w) \in M$ and $a^2(w) \in M$ but $\alpha^{-1}(w) = w - (t-1)(t-2)(-4t+3) \equiv -6 \bmod M$. As $w \in M$ and $\alpha(w) \in M$, there is a one-dimensional T-module $\mathcal{L}(M) = T/(XT + YT + MT)$. As a T^0-module, this is $\mathcal{L}_0(M)$, its annihilator in T^0 is the ideal $X^2T^0 + Y^2T^0 + MT^0$ and $\mathcal{L}_0(M) \otimes_{T^0} T \simeq T/(X^2T + Y^2T + MT)$. Let $J = X^2T + Y^2T + MT$. Then $X^2Y = Xw = \alpha^{-1}(w)X \equiv -6X \bmod MT$. Hence $X \in J$ and so $J = XT + Y^2T + MT$. Also $Y^2X = Y\alpha(w) = \alpha^2(w)Y \in MT$ and therefore $J = XT + Y^2S + MT$, where $S = A[Y; \alpha]$. From this it follows easily that $Y \notin J$. Hence $\mathcal{L}_0(M) \otimes_{T^0} T \simeq T/J$ is not annihilated by Y, is two-dimensional with basis $\{1, \bar{Y}\}$ and is not isomorphic to $\mathcal{L}(M)$.

Note that, by Corollary 2.6, $T = T(k[t, w], \alpha, w) = R(k[t], \alpha, u - \alpha(u), 1)$ where $u = t^2(t-1)(t-2)$ so this example answers the corresponding question for rings of the form R. The simpler example with $u = t(t-1)(t-2)$ works equally here but the specified example has another role later. If the above example is amended so that $\alpha^2(u) \notin M$, for example, by taking $u = t^2(t-1)$ and so $\alpha(w) = w + t(3t+1)$, then $Y \in J$ and $\mathcal{L}_0 \otimes_{T^0} T \simeq \mathcal{L}(M)$. Calculations of this sort are used to establish that $\mathcal{L}_0(N) \otimes_{T^0} T \simeq \mathcal{L}(M)$ in the case claimed above.

3.6 Other finite-dimensional simple T-modules and T^0-modules. If there are periodic maximal ideals in A then there are finite-dimensional simple T-modules S not of the form $L(M)$. The way in which these modules split over T^0 is different to that for those of the form $\mathcal{L}(M)$. In particular, the summands all have the same dimension. Such a module S has one of the forms $T/(MT + (Y^s - \xi)T)$ or $T/(MT + (X^s - \xi)T)$ for some periodic maximal ideal M of A of order s and some $0 \neq \xi \in k$. Let m be the highest common factor of n and s and note that N has order s/m under α^n. Then it can be checked that, as a T^0-module, S is a direct sum of m simple T^0-modules, each of dimension s/m and of the form given in 3.2(ii). Also, for each of these T^0-modules S', $S' \otimes_{R^0} R \simeq S$.

3.7 Semisimplicity of finite-dimensional R-modules. Suppose that A has no periodic maximal ideals and let $R = R(A, \alpha, v, \rho)$. In [**3**, Section 5] it is shown that, in the conformal case with $\rho = 1$, all finite-dimensional R-modules are semisimple if and only if, for all maximal ideals M of A and all positive integers $d < e$,

$$u - \alpha^d(u) \in M \Rightarrow (u - \alpha^e(u) \notin M \quad \text{and} \quad M^2 + (u - \alpha^d(u))A = M).$$

It follows from this result and the action of the Casimir element z on the non-split extensions which can occur, that, for $T = T(A, \alpha, u) = R/zR$, all finite-dimensional T-modules are semisimple if and only if for all maximal ideals M of A and all positive integers $d < e$,

$$(u \in M \quad \text{and} \quad \alpha^d(u) \in M) \Rightarrow (\alpha^e(u) \notin M \quad \text{and} \quad M^2 + uA + \alpha^d(u)A = M).$$

Applying this to $R = T(A[w], \alpha, w)$, we obtain the following generalization of [**3**, 5.6].

Theorem. *Suppose that A has no periodic maximal ideals. All finite-dimensional R-modules are semisimple if and only if, for all maximal ideals M of A and positive integers $d < e$,*

$$v_d \in M \Rightarrow (v_e \notin M \quad \textit{and} \quad M^2 + v_d A = M).$$

Proof. By the above, all finite-dimensional R-modules are semisimple if and only if, for all maximal ideals N of $A[w]$ and positive integers $d < e$,

$$(w \in N \quad \text{and} \quad \alpha^d(w) \in N) \Rightarrow (\alpha^e(w) \notin N \quad \text{and} \quad N^2 + wA[w] + \alpha^d(w)A[w] = N).$$

There is a bijection between the set of maximal ideals N of $A[w]$ containing w and the set of maximal ideals M of A given by $N = MA[w] + wA[w] \leftrightarrow M = N \cap A$. As $\alpha^d(w) = \rho^{-d}(w - v_d)$ by 2.3(5), it is clear that $v_d \in M \Leftrightarrow \alpha^d(w) \in N$. Also $N^2 + wA[w] + \alpha^d(w)A[w] = N \Leftrightarrow N^2 + wA[w] + v_dA[w] = N \Leftrightarrow M^2 + v_dA = M$. The result follows.

3.8 Example. Suppose that A is the Laurent polynomial ring $k[t, t^{-1}]$ with $\alpha(t) = q^2t$ where $0 \neq q \in k$ is not a root of unity. Thus A is α-simple and is a principal ideal domain. Let $v = at + b$ for some $a, b \in k$ with $a \neq 0$ and consider the ring $R = R(A, \alpha.v, \rho)$ where $\rho = q^{-1}$. For $d \geq 1$, $v_d = (1 + q + \ldots q^{d-1})at + (1 + q^{-1} + \ldots q^{-(d-1)})b$ which generates the maximal ideal $M_d = (t + q^{d-1}\frac{b}{a})A$. As these maximal ideals are distinct, Theorem 3.7 applies to show that all finite-dimensional R-modules are semisimple. A particular case of interest is [7, Example 1.4(ii)] where $q = v^2$ and $v = v^{-1}(t + \frac{v^2}{v^2-1})$. This algebra R is the localization at the powers of t of the algebra, first considered by Woronowicz [12], obtained as above but with $A = k[t]$ rather than $k[t, t^{-1}]$. Alternative proofs of the semisimplicity of the finite-dimensional modules for the localization are given in [12] and [1].

3.9 Semisimplicity of finite-dimensional R^θ-modules. Applying the method of 3.7 to the fixed ring $R^\theta = T(A[w], \alpha^n, U_n)$ gives that all finite-dimensional R^θ-modules are semisimple if and only if, for all maximal ideals N of $A[w]$ and positive integers $d < e$,

$$(U_n \in N \quad \text{and} \quad \alpha^{nd}(U_n) \in N) \Rightarrow (\alpha^{ne}(U_n) \notin N \quad \text{and} \quad N^2 + U_nA[w] + \alpha^{nd}(U_n)A[w] = N).$$

It can be checked that this criterion is equivalent to the corresponding criterion for the case $n = 1$ in the proof of 3.7. Thus all finite-dimensional R-modules are semisimple if and only if the same is true for R^θ. The "only if" part of this is true in general for the ring of invariants $S = R^G$ of a finite group G of automorphisms of a right Noetherian algebra R provided $|G|$ is invertible in R. One proof involves using the trace map, see [9, p. 242], to show that for each right ideal I of S, $IR \cap S = I$. From this it follows that any finite-dimensional S-module S/I embeds in the R-module R/IR. As R is finitely generated as an R^θ-module by [9, 26.13(ii)], R/IR is finite-dimensional and hence semisimple as an R^θ-module. By [9, 26.13(iv)], R/IR is semisimple as an S-module and therefore S/I is semisimple. Alternatively, see [8, proof of Proposition 1]. The criterion in 3.7 can fail on either of two counts, $v_e \in M$ or $M^2 + v_dA \neq M$. The two give rise to different types of non-split extensions. The first gives rise to non-split extensions of $L(M)$ by $L(N)$ and of $L(N)$ by $L(M)$, where $N = \alpha^{-d}(M)$, and the second gives $\mathrm{Ext}^1_R(L(M), L(M))$ to be non-zero. See [3, Section 5] for details. Although there is a similar dichotomy for R^{θ}-modules, it is possible, as the next example shows, for R to have the property that $\mathrm{Ext}^1_R(X, X) = 0$ for all finite-dimensional simple R-modules X but for R^{θ} to fail to inherit this property.

3.10 Example. Consider the example of 3.5, that is $R = R(k[t], \alpha, u - \alpha(u), 1)$ where $\alpha(t) = t + 1$ and $u = t^2(t-1)(t-2)$ or, equivalently, $T = T(k[t, w], \alpha, w)$ with $\alpha(t) = t + 1$ and $\alpha(w) = w + t(t-1)(4t+1)$. As $k[t]$ is α-simple it follows that each finite-dimensional simple R-module has the form $L(M)$ for some maximal ideal M of A containing v_d for some positive integer d. The two-dimensional module $R/J(= T/J)$ in 3.5 is not semisimple.

Suppose that $\mathrm{Ext}^1_R(L(M), L(M)) \neq 0$ for some maximal ideal M of A. Then for some positive integer d, $v_d \in M$ but $M^2 + v_d A \neq M$. As M/M^2 is one-dimensional, it follows that $v_d \in M^2$. But $v_d - u - \alpha^d(u) = 4t^3 + (6d-9)t^2 + (4d^2 - 9d + 4)t + (d^3 - 3d^2 + 2d)$ so this cubic and its derivative share a common root which must be

$$\frac{4d^3 - 6d^2 - 11d + 12}{2(11 - 4d^2)}$$

From this it follows that d is a root of the polynomial

$$64d^6 - 528d^4 + 1452d^2 - 1088 = (4d^2 - 11)^3 + 243.$$

This polynomial has no integer roots and so $\mathrm{Ext}^1_R(L(M), L(M)) = 0$.

On the other hand, consider the fixed ring R^{θ} in the case $n = 2$. Let N be the maximal ideal $wA[w] + tA[w]$ of $A[w]$. Then $U_2 = w\alpha^{-1}(w) \in N$ and $\alpha^2(U_2) = (w - v_2)(w - v_1) = (w - u + \alpha^2(u))(w - u + \alpha(u)) \in N^2$ and so $N^2 + U_2A[w] + \alpha^2(U_2)A[w] \subseteq N^2 + wA[w] \subset N$. It follows that there is a one-dimensional simple R^{θ}-module $\mathcal{L}(N)$ with $\mathrm{Ext}^1_{R^{\theta}}(\mathcal{L}(N), \mathcal{L}(N)) \neq 0$.

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School of Mathematics and Statistics
University of Sheffield
The Hicks Building
Sheffield S3 7RH, UK