Annals of Actuarial Science (2025), 1–29
doi:10.1017/S1748499525000028

ORIGINAL RESEARCH PAPER

Optimal decision-making for consumption, investment, housing, and life insurance purchase in a couple with dependent mortality

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(Received 07 June 2024; revised 31 October 2024; accepted 06 February 2025)

Abstract
In this article, we study an optimization problem for a couple including two breadwinners with uncertain life times. Both breadwinners need to choose the optimal strategies for consumption, investment, housing, and life insurance purchasing to maximize the utility. In this article, the prices of housing assets and investment risky assets are assumed to be correlated. These two breadwinners are considered to have dependent mortality rates to include the breaking heart effect. The method of copula functions is used to construct the joint survival functions of two breadwinners. The analytical solutions of optimal strategies can be achieved, and numerical results are demonstrated.

Keywords: Consumption; copula functions; dependent mortality; housing; investment; life insurance; optimal strategies

1. Introduction

Life expectancy is experiencing a rapid increase over the last few decades, due to reduced mortality rate and improved population health (World Health Organization, WHO (2025)). The prevalent demographic trend can be observed all over the world which requires people to be more alert about their allocations of assets when they are approaching to the retirement. Households, who are at retirement, need to carefully make consumption and investment decisions to maximize their utility. Their behaviors are subject to previous savings that are accumulated throughout earlier life stage. The optimal retirement strategies can be determined via the life-cycle modeling.

Optimal life-cycle models have been widely examined and studied in the existing literatures. Optimal strategies of asset allocation and consumption model were originally developed for investors with a constant relative risk-aversion utility function in Merton (1969, 1971). Following Merton's work, a great number of optimal life cycle models have been developed with certain feature to resemble real world. For example, Richard (1975) further extended Merton's model to include the demand for life insurance and annuity, where the investors maximized their utility by allocating their wealth between assets, consumption, and insurance products. Pliska and Ye (2007) extended the model in Richard (1975) by assuming that the lifetime of the wage earner was random and unbounded. Kraft & Steffensen (2008) studied a life cycle model with mortality–disability–unemployment risk. More recently, Wang et al. (2021) examined the effects of model uncertainty and unknown income growth on the household decision makings.

There is some scientific evidence suggesting that the mortality rate of coupled lives might be correlated. The death of a spouse is associated with an increased risk of mortality for the surviving partner, which is known as breaking heart effect. In this sense, the optimal retirement strategy should be determined not only limited to an individual but also extended to a household of two people with time-dependent mortality being considered in the model. The breaking heart effect has been widely studied in existing literatures. For instance, Parkes et al. (1969) found the increasing morality rate for widowers during first half year of bereavement by using the empirical data. Stroebe (1994) stated that the vulnerability of the bereaved person can be explained by the social integration. Lack of social contact and supports during the bereavement can cause higher mortality. The results of Elwert & Christakis (2008) verified that the breaking heart effect varies significantly according to the causes of death of the precedent one. Spreeuw & Owadally (2013) used an augmented Markov model to demonstrate the short-term dependence of the couple's life times after the death of a partner. Lu (2017) used a mixed proportional hazards model to reflect the mortality dependence of the couple that is only due to the breaking heart effect by disengaging the breaking heart effect from other observed and unobserved heterogeneities. In the context of life cycle planning model, Wei et al. (2020) considered an optimization problem of consumption, investment, and life insurance purchasing for a couple, where the correlation of couple's life expectancy is modeled by using copula and common shock models.

Yates & Bradbury (2010) stated that Australian people are shifting to home ownership-based strategy to accumulate wealth and preventing poverty after retirement. There exist studies that have factored in a housing component when studying life cycle decision makings. For instance, Cocco et al. (2005) incorporated the housing factor by assuming housing prices are perfectly correlated with labor income, and that house renting is not allowed in the model. Kraft & Munk (2011) studied an optimal life cycle model with housing components, where real estate price, rental income, labor income, and investments were defined as correlated. They also assumed that there existed functional real estate investment trusts which ensures that investors could continuously adjust their real estate investment. Kung & Yang (2020) extended the model in Kraft & Munk (2011) by including insurance products.

To finance the accelerating aging-related cost, more and more people have the motivation to access their housing wealth through housing equity withdrawal. Ong et al. (2015) and Hanewald et al. (2016) investigated retired individuals' decision-making process when their primary source of wealth was home equity and they faced various risks. They used a discrete-time model to analyze consumption, investment, insurance, and annuity decisions, considering the option to access equity through a reverse mortgage or a home reversion plan. Reverse mortgage loans and home reversion plans give homeowners the opportunity to access their home equity by taking the lump-sum cash or annuity payments while still maintaining ownership of their properties Alai et al. (2014). Specifically, the provider lends the customer cash and, in return, takes a share or a mortgage charge on the customers' properties. The termination of reverse mortgage loans or home reversion plans can be trigged by the death or permanent move-out of the customers. Subsequently, when the property is sold, and a portion of the proceeds will be taken to settle the outstanding loan. To safeguard the interests of the provider, reverse mortgages typically include a no-negative-equity guarantee. This ensures that borrowers cannot owe more than the current value of their property Lee & Shi (2022).

In this article, we investigate the optimal household decision makings in investment, consumption, housing, and life insurance purchasing. More specifically, we consider stochastic housing price and rent and incorporate housing investment and housing consumption strategies. Also, the breaking heart effect is included to examine the dependence between the lifetimes of two wage earners in a household. Dynamic programming principle coupled with Hamilton–Jacobi–Bellman (HJB) equation has been adopted to solve the life cycle planning problem. Our article contributes to the literature in two aspects. First, we consider financial risks, uncertainties in housing price, and breaking heart effect in mortality risk simultaneously, and the interactions between a variety

https://doi.org/10.1017/S1748499525000028 Published online by Cambridge University Press

of risks have been examined. The spousal mortality dependence is captured by Gumbel–Hougaard copula model, and the parameters in the mortality model are calibrated by using the joint last survivor insurance policies data from a large Canadian insurance company. In this sense, our work extends the models in Kung & Yang (2020) and Wei et al. (2020). Second, we develop closed-form representations for optimal portfolio choice, life insurance demand, housing consumption, and housing investment for postretirement (i.e., no labor income) case. Also, we study how the optimal strategies vary w.r.t. time in the numerical illustrations. This analysis provides rich financial interpretations especially for the case when the housing investment strategy is negative because it implies the possibility of financing postretirement life for the retirees through reverse mortgage loans or home equity conversion.

The remainder of this article is organized as follows. Section 2 illustrates the life cycle model and formulates the household's stochastic optimization problem. Section 3 derives the analytical expressions for optimal strategies and value function using the copula model. We conduct some numerical studies in Section 4 and Section 5 concludes the article.

2. Model formulation

Let τ_i be the death time of the breadwinner i, for $i = 1, 2$. We assume that the marginal probability distribution functions of τ_i is given by

$$F_i(t) = P(\tau_i \le t) = 1 - e^{-\int_0^t \lambda_i(s)ds},$$

where λ_i is the force of mortality. It is assumed that the random variables τ_1 and τ_2 follow a joint probability distribution $F(\cdot, \cdot)$ with a density function of $f(\cdot, \cdot)$. We use T_1 to denote the time of the first death of the couple, that is, $T_1 = \tau_1 \wedge \tau_2$. We also use $F_{T_1}(\cdot)$ and $f_{T_1}(\cdot)$ to denote the probability distribution function and density function of T_1, respectively.

In our model, the dynamics of the risk-free asset B_t and risky asset S_t are assumed to be

$$\frac{dB(t)}{B(t)} = r(t)dt,$$
$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma_S(t)dZ_S(t),$$

where $r(t)$ is the risk-free interest rate, $\mu(t)$ is the appreciation rate, $\sigma_S(t)$ is the volatility, and $Z_S(t)$ is the Brownian motion with respect to risky asset price.

Based on Kung & Yang (2020) and Kraft & Munk (2011), we assume that household can invest in real estate asset at a unit price (e.g., unit can be defined as the price per square meter). The dynamics of the unit house price are

$$\frac{dH(t)}{H(t)} = (r(t) + \lambda_H\sigma_H(t) - \zeta)dt + \sigma_H(t)(\rho_{HS}dZ_S(t) + \rho_H dZ_H(t)),$$

where $\sigma_H(t)$ is the house price volatility, λ_H is the Sharpe ratio of the unit house price, $\rho_H = \sqrt{1 - \rho_{HS}^2}$, ρ_{HS} is the constant correlation between house and stock, and ζ is the imputed rent or the cost of holding house unit, and $Z_H(t)$ is the Brownian motion with respect to house price. By using this setting, the household has the option to take a short position in housing assets, allowing them to access housing wealth through housing equity withdrawal using reverse mortgages or home reversion plans.

The rent of a housing unit is assumed to have a constant relationship υ with the house price, where $\upsilon > 0$. Hence, the return of the household investing and renting out a unit house we have

$$\frac{dH(t) + \upsilon H(t)dt}{H(t)} = \left[r(t) + \hat{\lambda}_H\sigma_H(t)\right]dt + \sigma_H(t)\left[\rho_{HS}dZ_S(t) + \rho_H dZ_H(t)\right],$$

https://doi.org/10.1017/S1748499525000028 Published online by Cambridge University Press

where $\hat{\lambda}_H = \lambda_H + \frac{\upsilon - \zeta}{\sigma_H(t)}$.

We use $\pi(t)$ to denote the proportions of the wealth invested in risky assets. $\phi_1(t)$ is the units of housing units owned. $\phi_2(t)$ is the units of housing units rented. $\phi_3(t)$ is the units invested in REIT at time t. Housing consumption is defined as $\phi_4(t) = \phi_1(t) + \phi_2(t)$ and housing investment $\phi_5(t) = \phi_1(t) + \phi_3(t)$.

For each breadwinner, denoted as i where i can be either 1 or 2, $c_i(t)$ and $k_i(t)$ are the consumption amount and life insurance premium. Let $Y_i(t)$ be the deterministic income flow for breadwinner i during period $[0, T]$ $(i = 1, 2)$, where T represents the time when the last survivor of the couple passes away. Hence, the wealth dynamics is

$$\begin{aligned} dX(t) = &\{[r(t) + (\mu(t) - r(t))\pi(t)]X(t) - \phi_4(t)\upsilon H(t)\}dt + [\pi(t)X(t)\sigma_S(t) \\ &+ \phi_5(t)H(t)\rho_{HS}\sigma_H(t)]dZ_S(t) + \phi_5(t)H(t)\rho_H\sigma_H(t)dZ_H(t) \\ &- [\mathbf{1}_{\{t<\tau_1\}}\left(c_1(t) + k_1(t) - Y_1(t)\right) - \mathbf{1}_{\{t<\tau_2\}}\left(c_2(t) + k_2(t) - Y_2(t)\right)]dt. \end{aligned}$$

We assume a time-additive Cobb–Douglas style utility for the consumption, $U_1(c(,\phi_4) = \frac{(c^\beta \phi_4^{1-\beta})^\gamma}{\gamma}$ and the power utility for the bequest $U_2(x) = \frac{x^\gamma}{\gamma}$ for the terminal wealth, where β is the relative weighting between housing, $0 < \beta < 1$, and consumption, and γ is the relative risk aversion, $0 < \gamma < 1$.

We use δ to denote the discount factor. $\mathcal{A}_t$ is the set of all admissible strategies $U \triangleq (c_1, c_2, k_1, k_2, \pi, \phi_4, \phi_5)$. For an arbitrary admissible control $u \in \mathcal{A}_t$, the value function of this optimal problem is as follows:

$$\begin{aligned} V(t, x, h) = \max_{u \in \mathcal{A}_t} E\Bigg\{ &\int_t^{\tau_1 \wedge T} w_1 e^{-\delta s} U_1(c_1(s), \phi_4(s))ds + \int_t^{\tau_2 \wedge T} w_2 e^{-\delta s} U_1(c_2(s), \phi_4(s))ds \\ &+ w_3 \mathbf{1}_{\{\tau_1 \vee \tau_2 \le T\}} e^{-\delta(\tau_1 \vee \tau_2)} U_2\left(X(\tau_1 \vee \tau_2) + \sum_{i=1}^{2} \frac{k_i(\tau_i, X(\tau_i))}{\theta_i(\tau_i)} \mathbf{1}_{\{\tau_i = \tau_1 \vee \tau_2\}}\right) \\ &+ w_4 \mathbf{1}_{\{\tau_1 \vee \tau_2 > T\}} e^{-\delta T} U_2(X(T)) \Bigg\}, \end{aligned}$$

where $w_i \ge 0$, $i = 1, 2, 3, 4$ satisfies the condition of $\sum_{i=1}^4 w_i = 1$ which ranks the relative importance of the utility type, $X(t) = x$ and $H(t) = h$. For $i = 1, 2$, the insurance benefit $\frac{k_i(\tau_i, X(\tau_i))}{\theta_i(\tau_i)}$ is assumed to be paid to the beneficiary upon the death of the insured at time τ_i. Following Kung & Yang (2020), we define l as the loading factor and $\theta_i = (1 + l)\lambda_i$. In this context, $k_i(\tau_i, X(\tau_i))$ represents the life insurance premium. When $k_i(\tau_i, X(\tau_i))$ is positive, it indicates behavior consistent with purchasing life insurance. Conversely, when $k_i(\tau_i, X(\tau_i))$ is negative, it reflects behavior associated with purchasing a variable life annuity. The utility derived from the bequest and terminal wealth will only be realized upon the death of the last surviving individual. To ensure the utility function remains well-defined, we follow the approach in Kung & Yang (2020) by introducing a loading factor that may result in a zero-insurance condition.

3. Optimal results

Initially, we assume the presence of a breaking heart effect, where the time of death, denoted as τ_i, for the bereaved individual is influenced by the passing of their partner. This phenomenon highlights how deeply connected our emotions and mental well-being are to our physical health, especially in close relationships. When someone we love passes away, it can set off a chain reaction in our bodies and minds that affects how healthy we are overall. In our analysis, we first address the optimization problem in the scenario where one of the couple passes away, recognizing that the

https://doi.org/10.1017/S1748499525000028 Published online by Cambridge University Press

optimization strategy in the case where both breadwinners are alive may be informed by insights gained from the former scenario.

3.1 The optimization problem after the first death

We firstly consider the optimization problem after one of the couple dies, that is, $T_1 = \tau_1 < \tau_2$ or $T_1 = \tau_2 < \tau_1$. We simplify the notation $V_i(t, x, h)$ for the value function to V_i.

After T_1, the alive bread winner i has the value function of this optimal problem is as follows:

$$\begin{aligned} V_i = \max E\Big\{ \int_t^{\tau_i \wedge T} & w_i e^{-\delta(s-t)} U_1(c_i(s), \phi_4(s))ds \\ & + w_3 \mathbf{1}_{\{\tau_i \le T\}} e^{-\delta(\tau_i - t)} U_2\left(X_i(\tau_i) + \frac{k_i(\tau_i, X_i(\tau_i))}{\theta_i(\tau_i)}\right) \\ & + w_4 \mathbf{1}_{\{\tau_i > T\}} e^{-\delta(T-t)} U_2(X_i(T)) \Big\}, \end{aligned}$$

where $X_i(\cdot)$ is given by

$$\begin{aligned} dX_i(t) = & \{[r(t) + (\mu(t) - r(t))\pi_i(t)]X_i(t) - \phi_{4_i}(t)\upsilon H(t)\}dt + [\pi_i(t)X_i(t)\sigma_S(t) \\ & + \phi_{5_i}(t)H(t)\rho_{HS}\sigma_H(t)]dZ_S(t) + \phi_{5_i}(t)H(t)\rho_H\sigma_H(t)dZ_H(t) - \big(c_i(t) + k_i(t) - Y_i(t)\big)\, dt. \end{aligned}$$

The proof of the following proposition can be found in Appendix A.

Proposition 3.1. *The value function is given by*

$$V_i = \frac{1}{\gamma} g_i(t, h)^{1-\gamma} (x + b_i(t))^\gamma,$$

where $b_i(t) = \int_t^\omega e^{-\int_t^s (r(u)+\lambda_i(u))du} Y_i(s)ds$ *represents the human capital for breadwinner i. The optimal strategies are given by*

$$\begin{aligned} c_i^*(t) &= \frac{\beta}{(1-\beta)g_i(t,h)} \upsilon \eta_i h^k (x + b_i(t)), \\ k_i^*(t, x) &= \theta_i(t) \left[\left(\frac{\gamma}{w_3}\right)^{\frac{1}{\gamma-1}} g_i(t, h)^{-1} (x + b_i(t)) - x \right], \\ \pi_i^*(t) &= \frac{-(\mu(t) - r(t))}{(1-\rho_{HS}^2)\sigma_S^2(t)x(\gamma-1)} (x + b_i(t)), \\ \phi_{4_i}^*(t) &= \frac{\eta_i h^{q-1}}{g_i(t,h)} (x + b_i(t)), \\ \phi_{5_i}^*(t) &= \left[\frac{\rho_{HS}(\mu(t) - r(t))}{(1-\rho_{HS}^2)\sigma_S(t)\sigma_H(t)h(\gamma-1)} + \frac{g_{ih}(t,h)}{g_i(t,h)} \right] (x + b_i(t)). \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} g_i(t, h) &= \alpha_1 h^k \int_t^T e^{\alpha_2(s)} ds + \alpha_3 h^{\gamma(\beta+k-1)} \int_t^T e^{\alpha_4(s)} ds + \alpha_5, \\ \alpha_1 &= \frac{\eta_i \upsilon}{(1-\beta)(\gamma-1)}, \end{aligned}$$

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$$\alpha_2(s) = \frac{d_1(s-T)}{(1-\gamma)\alpha_1},$$

$$\alpha_3 = -\frac{w_1}{1-\gamma}\left(\frac{\beta}{1-\beta}\right)^{\beta\gamma} \eta_i^{\gamma} \upsilon^{\beta\gamma},$$

$$\alpha_4(s) = \frac{d_2(s-T)}{(1-\gamma)\alpha_3},$$

$$\alpha_5 = w_4^{\frac{1}{1-\gamma}} + \left(\frac{\gamma-1}{d_3} w_3^{\frac{1}{1-\gamma}} + w_4^{\frac{1}{1-\gamma}}\right)(e^{\frac{d_1(T-t)}{\gamma-1}} - 1),$$

$$d_1 = \alpha_1 d_3 + \alpha_1(\gamma-1)(\gamma-\zeta+\lambda_H\sigma_H(t))k - \frac{1}{2}\alpha_1(\gamma-1)\sigma_H^2(t)k(k-1),$$

$$d_2 = \alpha_3 d_3 + \alpha_3(\gamma-1)(\gamma-\zeta+\lambda_H\sigma_H(t))\gamma(\beta+k-1)$$
$$-\frac{1}{2}\alpha_3(\gamma-1)\sigma_H^2(t)\gamma(\beta+k-1)[\gamma(\beta+k-1)-1]$$

$$d_3 = \delta - (\gamma+1)\theta_i(t) - \gamma r - \frac{1}{2}\frac{\gamma(\mu-r)^2}{\sigma_S^2(\rho_{HS}^2-1)(\gamma-1)}$$

$$q = \frac{-\gamma+\beta\gamma}{1-\gamma},$$

$$\eta_i = (w_i\beta)^{\frac{1}{1-\gamma}}\left(\frac{\beta\upsilon}{1-\beta}\right)^{k-1}.$$

To ensure the non-negativity of $X_i(\tau_i) + \frac{k_i(\tau_i, X_i(\tau_i))}{\theta_i(\tau_i)}$, we need to verify that the expression for $g_i(t,h)$ is non-negative. Given that $w_i > 0$, $\upsilon > 0$, and $0 < \beta < 1$, it follows that $\eta_i > 0$. Given $\gamma < 1$, we can ensure that $\alpha_1 > 0$ and $\alpha_3 > 0$ and $X_i(\tau_i) + \frac{k_i(\tau_i, X_i(\tau_i))}{\theta_i(\tau_i)}$ can be non-negative.

For this case, we build upon the optimization framework proposed by Wei et al. (2020). Our approach initiates by considering the optimization problem after the first death and subsequently addresses the optimization problem before the first death. Compared to their work, our model further includes housing consumption and investment, thereby exploring the area of housing assets. By incorporating housing consumption and insurance demand, our model addresses two scenarios: after the first death, where the surviving breadwinner cannot purchase life insurance, a variable annuity, or real estate assets, which is considered an extension of the traditional Richard's model (Richard, 1975). Our extended model integrates housing elements such as consumptiona and investment component, distinguishing it from other extensions of Richard's model (e.g., Pliska & Ye, 2007; Zhang et al., 2021 and Chen et al., 2024), which initially did not consider these aspects. This incorporation broadens the scope of the model and allows for a more comprehensive analysis of the interplay between housing decisions and life insurance.

3.2 The optimization problem before the first death

We now consider the case that both breadwinners are alive. For each breadwinner, the optimal strategy has been discussed in Section 3.1 when $t > T_1$. The optimal strategies are written as $\bar{c}_i(t, x_t)$, $\bar{k}_i(t, x_t)$, $\bar{\pi}(t, x_t)$, $\bar{\phi}_{4_i}(t, x_t)$ and $\bar{\phi}_{5_i}(t, x_t)$ when both breadwinners are alive.

We simplify the notation $V_i(t, x, h)$ for the value function to V_i. The proof of the following dynamic equation is stated in Appendix B.

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$$
\begin{aligned}
V = \frac{1}{1-F_{T_1}(t)} \max_{u\in\mathscr{A}_t} E \Bigg\{ & \int_t^T \left[\int_z^\infty f(s,z)ds\right] V_1\left(z, X(z) + \frac{\bar{k}_2(z, X(z))}{\theta_2(z)}, H(z)\right) dz \\
& + \int_t^T \left[\int_s^\infty f(s,z)dz\right] V_2\left(s, X(s) + \frac{\bar{k}_1(s, X(s))}{\theta_1(s)}, H(z)\right) ds \\
& + \int_t^T \left[1 - F_{T_1}(s)\right] e^{-\delta(s-t)} \left[w_1 U_1(\bar{c}_1(s, X_1(s))) + w_2 U_1(\bar{c}_2(s, X_2(s)))\right] ds \\
& + w_4 e^{-\delta(T-t)} U_2(X(T)) \int_T^\infty \int_T^\infty f(s,z) ds dz \Bigg\}, \qquad (3.2)
\end{aligned}
$$

where

$$
\begin{aligned}
dX(t) = & [r(t) + (\mu(t) - r(t))\bar{\pi}(t)]X(t)dt + \sum_{i=1}^{2} [-\bar{\phi}_{4_i}(t)\upsilon H(t) - \left(\bar{c}_i(t) + \bar{k}_i(t) - Y_i(t)\right)]dt \\
& + \bar{\pi}(t)X(t)\sigma_S(t)dZ_S(t) + \sum_{i=1}^{2} [\bar{\phi}_{5_i}(t)H(t)\rho_{HS}\sigma_H(t)]dZ_S(t) \\
& + \sum_{i=1}^{2} \bar{\phi}_{5_i}(t)H(t)\rho_H\sigma_H(t)dZ_H(t).
\end{aligned}
$$

The proof of the following proposition can be found in Appendix C.

Proposition 3.2. *We write $\tilde{V} = (1 - F_{T_1}(t))V$. Rewriting this, We assume the value function follows the ansatz:*

$$
V = \frac{g(t,h)^{1-\gamma}}{\gamma[1 - F_{T_1(t)}]}(x + b(t))^\gamma
$$

and

$$
\tilde{V} = \frac{g(t,h)^{1-\gamma}}{\gamma}(x + b(t))^\gamma.
$$

The optimal strategies are given by

$$
\begin{aligned}
\bar{c}_i^*(t) &= \frac{\beta}{(1-\beta)g(t,h)}\upsilon\eta_i h^k (x + b(t)), \\
\bar{\pi}^*(t) &= \frac{-(\mu(t) - r(t))}{(1-\rho_{HS}^2)\sigma_S^2(t)x(\gamma - 1)}(x + b(t)), \\
\bar{\phi}_{4_i}^*(t) &= \frac{\eta_i h^{k-1}}{g(t,h)}(x + b(t)), \\
\bar{\phi}_5^*(t) &= \left[\frac{\rho_{HS}(\mu(t) - r(t))}{(1-\rho_{HS}^2)\sigma_S(t)\sigma_H(t)h(\gamma - 1)} + \frac{g_h(t,h)}{g(t,h)}\right](x + b(t)). \qquad (3.3)
\end{aligned}
$$

where

$$
\begin{aligned}
g(t,h) &= \left(w_4^{\frac{1}{1-\gamma}} + \frac{\tilde{d}_2}{\frac{\gamma}{1-\gamma}\tilde{d}_1}\right) e^{\frac{\gamma}{1-\gamma}\tilde{d}_1(T-t)} - \frac{\tilde{d}_2}{\frac{\gamma}{1-\gamma}\tilde{d}_1}, \\
k &= \frac{-\gamma + \beta\gamma}{1-\gamma},
\end{aligned}
$$

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$$\eta_i = (w_i\beta)^{\frac{1}{1-\gamma}}\left(\frac{\beta\upsilon}{1-\beta}\right)^{k-1},$$

$$b(t) = \sum_{i=1}^{2}\int_t^{\omega} e^{-\int_t^s (r(u)+\theta_i(u))du}Y_i(s),$$

$$\tilde{d}_1 = r(t) + \frac{(\mu(t)-r(t))^2}{(1-\rho_{HS}^2)\sigma_S^2(t)(\gamma-1)} + \frac{1}{2}\frac{(\mu(t)-r(t))^2}{(1-\rho_{HS}^2)^2\sigma_S^2(t)(\gamma-1)} + \frac{\tilde{P}_1}{\gamma} + \frac{\tilde{P}_3}{x},$$

$$\tilde{d}_2 = (1-F_{T_1}(t))^{-\frac{1}{\gamma-1}}\left[w_1\frac{\eta_1^{\gamma}}{\gamma} + w_2\frac{\eta_2^{\gamma}}{\gamma}\right]\left(\frac{\beta}{1-\beta}\right)^{\beta\gamma}\upsilon^{\beta\gamma}h^{\gamma(k+\beta-1)},$$

$$-(1-F_{T_1}(t))^{-\frac{1}{\gamma-1}}\frac{(\eta_1+\eta_2)\upsilon h^k}{1-\beta} + \tilde{P}_2\gamma^{-\frac{\gamma}{\gamma-1}}x^{\frac{1}{\gamma-1}},$$

$$\tilde{P}_1 = -\left(\delta + \int_t^{\infty} f(s,t)ds + \int_t^{\infty} f(t,z)dz\right),$$

$$\tilde{P}_2 = \theta_1(x+b_2) + \theta_2(x+b_1),$$

$$\tilde{P}_3 = \left(1-\frac{1}{\gamma}\right)\left(\frac{\theta_1}{\int_t^{\infty} f(t,z)dz}\right)^{\frac{\gamma}{\gamma-1}} g_2(t,h)\int_t^{\infty} f(t,z)dz$$

$$+\left(1-\frac{1}{\gamma}\right)\left(\frac{\theta_2}{\int_t^{\infty} f(s,t)ds}\right)^{\frac{\gamma}{\gamma-1}} g_1(t,h)\int_t^{\infty} f(s,t)ds.$$

Diverging from the framework presented by Kung & Yang (2020), we introduce a novel insurance component that extends Richard's foundational model (Richard, 1975). While Kung & Yang (2020) focus on integrating housing and life insurance decisions within a continuous time setting, our approach enhances this integration by specifically addressing the optimal consumption and investment strategies for households with two breadwinners following the first death. It is important to note that although $\bar{k}_i^*(t,x)$ can be negative, we must ensure that the expression $\frac{\bar{k}_i^*(t,x)}{\theta_i(t)} + x$ remains positive. Following the approach of Kung & Yang (2020), we apply a loading factor to reduce insurance demand. As demonstrated in Section 4, by choosing specific value of $l = 0.1$, we can ensure $\frac{k_i^*(t,x)}{\theta_i(t)} + x$ is non-negative.

The optimal solution after the first death represents a significant advancement over Richard's original model, as it now accounts for the housing components crucial for realistic financial planning and decision-making. By structuring the problem to capture the dynamic decision-making process of both individuals before the first death, our model accurately captures the dynamics of life insurance, housing, and financial decision-making over the life cycle of couples with correlated lifetimes, providing a more realistic and practical framework for optimizing consumption and investment strategies. Compared to Wei et al. (2020), our model includes housing elements for consumption and investment, broadening the study of optimal strategies. By incorporating this additional insurance component and integrating housing elements, our model provides a more robust framework for understanding and optimizing the consumption and investment behaviors of individuals facing the dual challenges of housing and life insurance decisions in a continuous time context.

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Table 1. Calibrated parameters in the mortality model

m_M	σ_M	m_F	σ_F	α_0	α_5	α_{10}
85.82	11.13	89.02	8.89	2.02	1.67	1.33

4. Numerical results and discussion

In our numerical demonstration, two cases are studied. For the optimisation problem after the first death (case 1), there is only one alive breadwinner. This person can purchase life insurance, a variable annuity, and real estate assets. For the second case (the optimisation problem before the first death), both breadwinners are alive. The couple can purchase life insurance, a variable annuity, and real estate assets. Calibration parameters in the mortality model are shown in Table 1.

The marginal probability distribution functions for male and female are assumed to be expressed by

$$F_1(t) = 1 - \exp\{\exp(-m_1/\sigma_1)[1 - \exp(t/\sigma_1)]\},$$
$$F_2(t) = 1 - \exp\{\exp(-m_2/\sigma_2)[1 - \exp(t/\sigma_2)]\}.$$

These two-parameter Gompertz distributions imply the following density functions of τ_1 and τ_2:

$$f_1(t) = \frac{1}{\sigma_1} \exp\left\{\exp(-m_1/\sigma_1)[1 - \exp(t/\sigma_1)] - \frac{m_1}{\sigma_1} + \frac{t}{\sigma_1}\right\},$$
$$f_2(t) = \frac{1}{\sigma_2} \exp\left\{\exp(-m_2/\sigma_2)[1 - \exp(t/\sigma_2)] - \frac{m_2}{\sigma_2} + \frac{t}{\sigma_2}\right\}.$$

In this case, the mortality rate of each wage earner is given by

$$\lambda_1(t) = \frac{1}{\sigma_1} e^{\frac{t-m_1}{\sigma_1}},$$
$$\lambda_2(t) = \frac{1}{\sigma_2} e^{\frac{t-m_2}{\sigma_2}}.$$

We use the following Gumbel–Hougaard copula to capture the dependence structures between the mortalities of a couple:

$$C(s, t) = e^{-[(-\ln s)^\alpha + (-\ln t)^\alpha]^{-1/\alpha}}, \quad \alpha \geq 1.$$

Under this model, the joint probability distribution of (τ_1, τ_2) is given by

$$F(t, t) = e^{-[(-\ln F_M(t))^\alpha + (-\ln F_F(t))^\alpha]^{\frac{1}{\alpha}}}.$$

The distribution function of T_1 is

$$F_{T_1}(t) = F_1(t) + F_2(t) - F(t, t).$$

Also, we have the following results:

$$f_1(t) = \int_t^\infty f(t, z)dz + \frac{f_1(t)F(t, t)}{F_1(t)} \left[1 + \left(\frac{\ln F_2(t)}{\ln F_1(t)}\right)^\alpha\right]^{\frac{1-\alpha}{\alpha}},$$
$$f_2(t) = \int_t^\infty f(s, t)ds + \frac{f_2(t)F(t, t)}{F_2(t)} \left[1 + \left(\frac{\ln F_1(t)}{\ln F_2(t)}\right)^\alpha\right]^{\frac{1-\alpha}{\alpha}}.$$

We use α_0, α_5 and α_{10} to denote the value of α in the copula model when the age difference between male and female is 0, 5 and 10, respectively.

In the numerical examples, we assume that the age of male is 75, and the age of female is 70. The age difference is 5, which makes $\alpha = 1.67$. Also, it is supposed that $T_1 = \tau_2$, which implies that the

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Table 2. Values of parameters in the numerical experiments

β	δ	r	μ	σ_S	γ	v	ω_1	ω_2	ω_3
0.2	0.03	0.05	0.0107	0.2669	0.3	0.025	0.4	0.45	0.1
ω_4	t	T	λ_H	σ_H	h	ρ_{HS}	ζ	x	α
0.05	0	5	0.5104	0.0601	2150	0.2503	0.015	200000	1.67

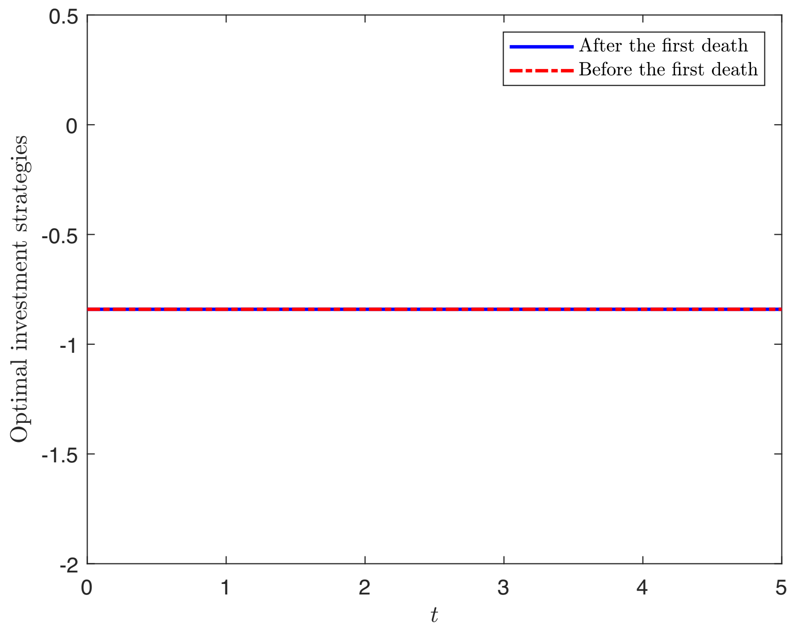

Figure 1. Effect of t on the optimal investment strategies.

wife dies before the husband, and so for the case of after the first death we focus on the strategies of the male. Most of the values of the model parameters shown in Table 2 are borrowed from Kung & Yang (2020). In Fig. 1, we show the effects of time t on the optimal investment strategies. As we can see, the household will not adjust the optimal investment decisions as time varies. The optimal investment strategy is a fixed negative constant which is independent of wealth level x and time t. This is consistent with the expressions for π_i^* in (3.5) and $\bar{\pi}^*$ in (3.14). This is because we assume that $\mu(t)$, $r(t)$ and $\sigma_S(t)$ are constants for simplicity, and also we do not consider income of the household in the numerical demonstration.

Fig. 2 displays the effects of time on the optimal consumption strategies. It can be observed that before the first death, the consumption strategies of male and female are both decreasing functions of time t. Comparatively speaking, the consumption policy for the case of after the first death is more sensitive to the change of time. Usually, it is a fact that the consumption strategy increases w.r.t. time. This is due to the fact that consumption strategy is an increasing function of labor income and usually the labor income increases as time goes by before the retirement time (see, for example, Kung & Yang, 2020 and Wei et al., 2020). However, this kind of result is not obtained in our article because we assume the household has no labor income in this section to illustrate the theoretical results. The household will reduce the consumption rates to keep the wealth for future use in our case. Similar arguments can be used to analyze the effects of time t on the housing consumption strategy which has been shown in Fig. 3.

In Fig. 4, we examine the effects of t on the optimal insurance strategy. The household usually spends more on life insurance as their ages increase to hedge against its mortality risk and

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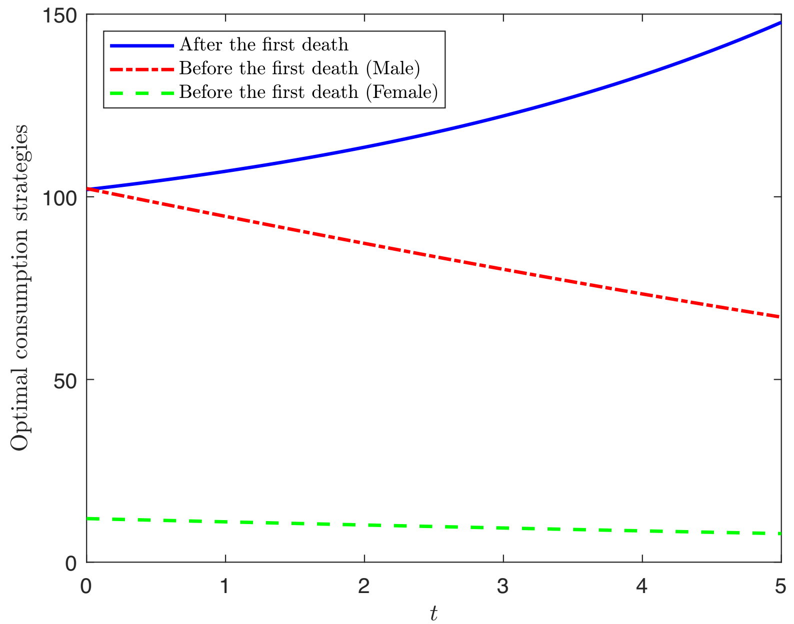

Figure 2. Effect of t on the optimal consumption strategies.

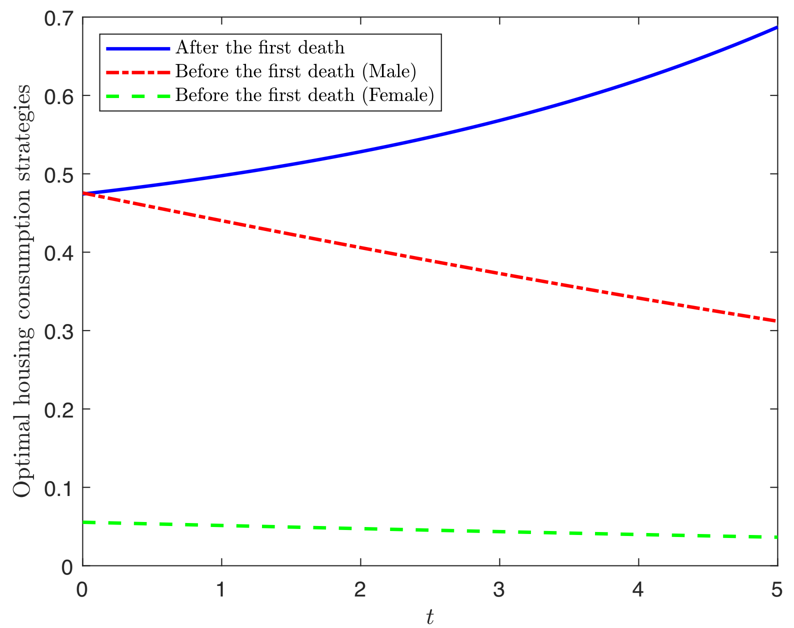

Figure 3. Effect of t on the optimal housing consumption strategies.

labor income risk. But in our case, the life insurance strategy declines as the human capital is not considered and purchasing life insurance makes little sense. Also, a negative life insurance strategy after the first death means the household receives payment from the insurance company. To ensure that $\frac{k_i^*(t,x)}{\theta_i(t)} + x$ remains non-negative, we set the loading factor to $l = 0.1$. This adjustment

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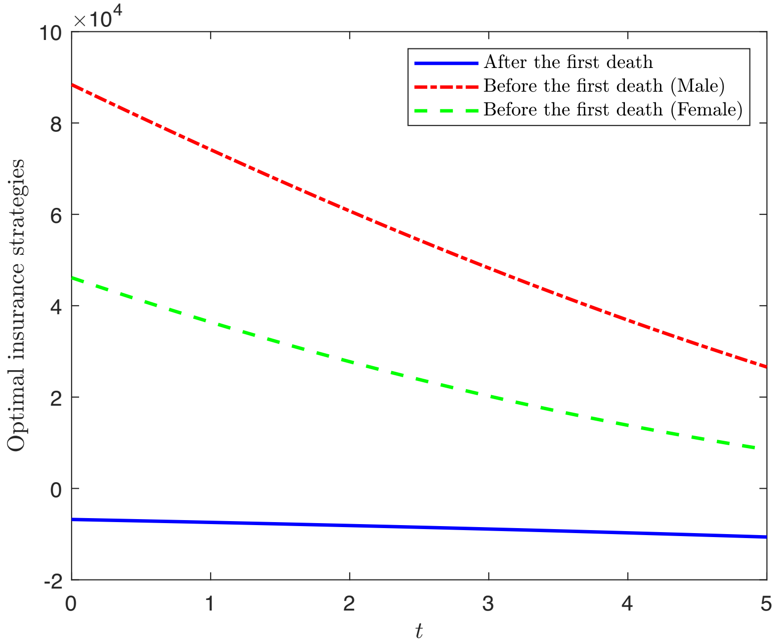

Figure 4. Effect of t on the optimal insurance strategies.

helps maintain positivity in the expression by moderating the impact of $k_i^*(t, x)$ when it is negative, effectively reducing the insurance demand. By following this approach, we ensure that the utility function $U_2(\cdot)$ remains well-defined and applicable within the value function.

Fig. 5 demonstrates how time t impacts optimal housing investment strategies for different scenarios. Considering that the pandemic has introduced significant uncertainty and volatility into the financial markets, which increases the demand for safer investments and leads to a decrease in stock prices and potentially lower expected returns, we assume a smaller value for the expected return of risky asset in this section. This is why the the values of the optimal investment strategy for risky asset in Fig. 1 are negative. On the other hand, housing and risk-free asset may appear relatively more attractive compared to stocks, which explains why the optimal housing investment strategies in Fig. 5 are positive. Finally, Fig. 6 shows the effects of wealth on the value functions for two cases. Not surprisingly, we find that the value functions increase w.r.t. the level of wealth.

In what follows, we show the effects of risk aversion parameter γ on the optimal control policies. In each of the following figures, we vary the value of γ from 0.01 to 0.5. Fig. 7 shows the impact of γ on the optimal investment decisions in risky asset. As we can observe, if the wage earner is more risk averse, they tend to sell more risky assets to mitigate financial risk, which makes intuitive sense.

In Figs. 8 and 9, we illustrate how the optimal consumption strategies and optimal housing consumption strategies vary w.r.t. γ, respectively. It can be seen that a more risk-averse wage earner consumes less. This stems from the tendency of risk-averse individuals to prefer more certain outcomes, often sacrificing potential higher utility from consumption to avoid potential financial instability or uncertainty. In this sense, our results are consistent with those in Chen et al. (2024).

From Fig. 10, we can see that optimal housing investment strategies increase as γ increases and the difference between before and after the first death is insignificant. In both cases, housing is considered as a relatively stable and less volatile asset compared to stocks, and hence more risk-averse breadwinner tends to invest more in housing.

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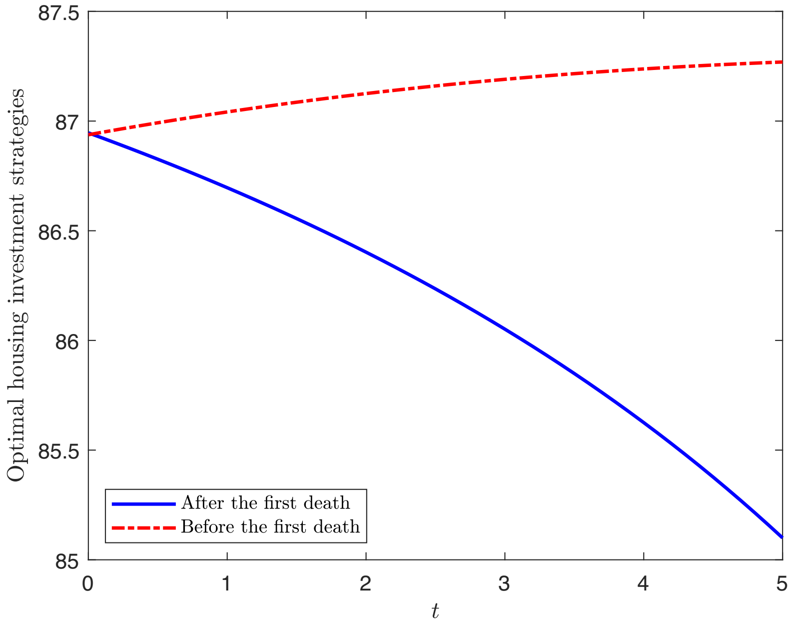

Figure 5. Effect of *t* on the optimal housing investment strategies.

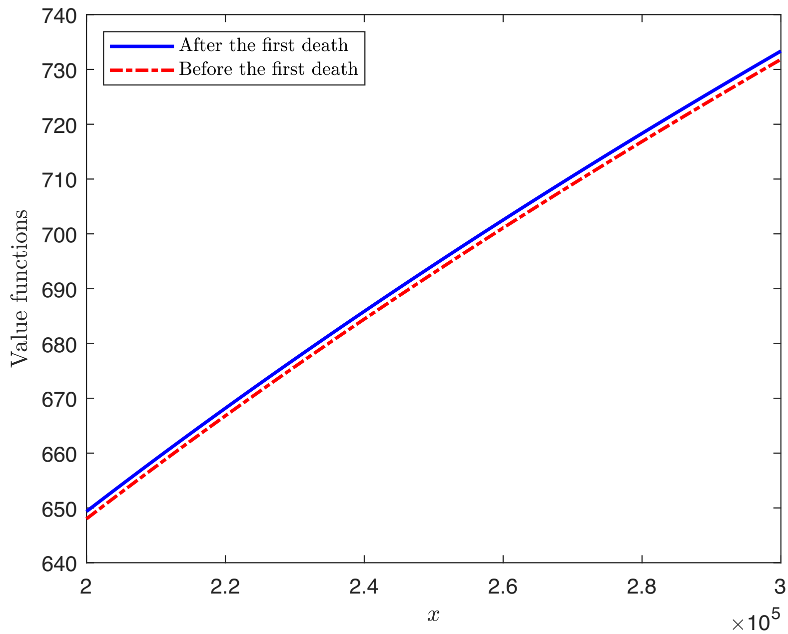

Figure 6. Effect of *x* on the value functions.

Fig. 11 displays how γ influences the optimal life insurance purchasing strategies. In the literature, different patterns of life insurance demand w.r.t. γ have been reveled, see, for example, Kwak & Lim (2014), Han & Hung (2017) and Chen et al. (2024). In our model setup, the results show the following economic implications. First, the impact of γ for the case of after the first death is not significant. Second, for the other two cases, the optimal insurance strategies increase and

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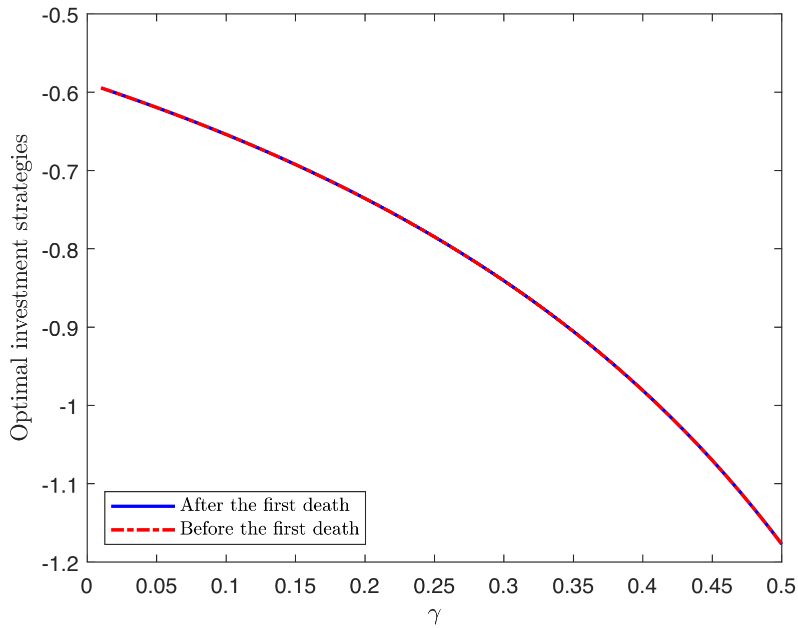

Figure 7. Effects of γ on the optimal investment strategies.

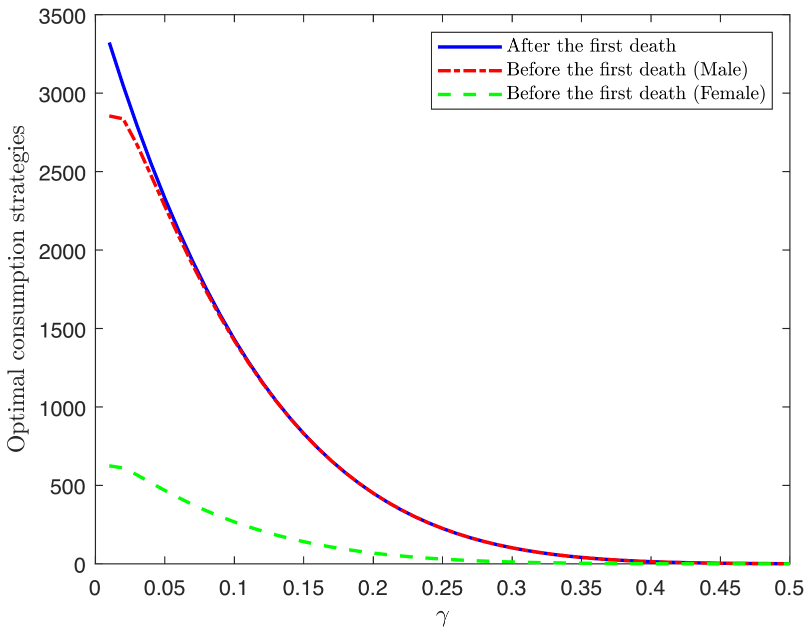

Figure 8. Effects of γ on the optimal consumption strategies.

then decrease w.r.t. γ. When γ is small, the wage earner who is more risk averse tends to purchase more life insurance to hedge against the potential losses from mortality risk and protect their family. However, if γ is relatively larger, housing investment may be considered as a more attractive and effective tool than life insurance for ensuring an adequate legacy and future consumption for dependents after the death of breadwinner, which results in a reduction in life insurance demand.

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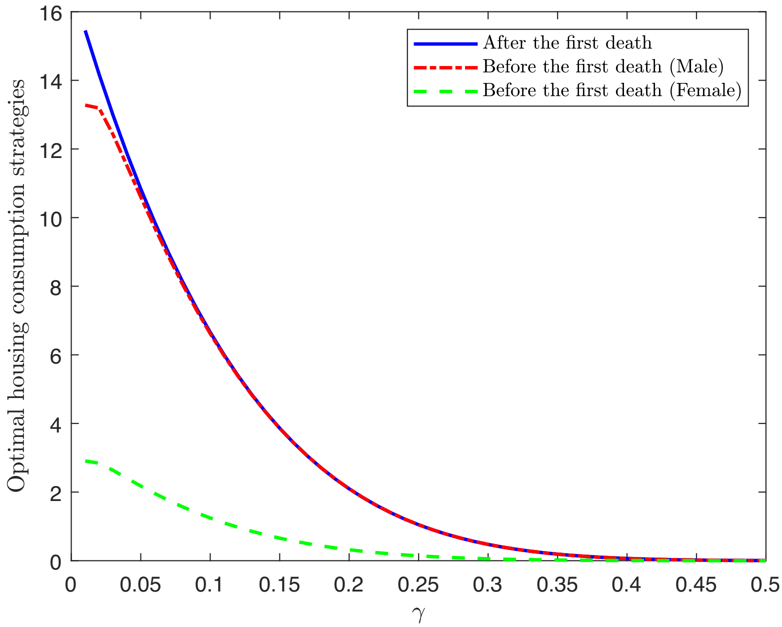

Figure 9. Effects of γ on the optimal housing consumption strategies.

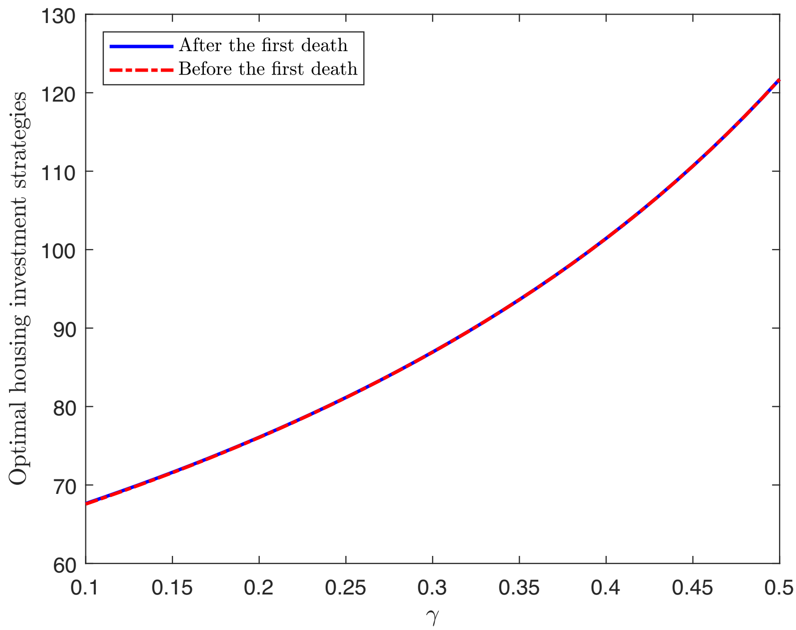

Figure 10. Effects of γ on the optimal housing investment strategies.

The numerical examples consider a couple with a 5-year age difference. Mortality dependency significantly influences consumption, investment, and insurance decisions. Before the first death, mortality dependency is reflected in a decreasing consumption pattern. Conversely, after the first death (without mortality dependency), consumption displays an increasing trend over time. The effect of mortality dependency is further evident in insurance purchase and housing investment. Mortality dependency prompts a demand for insurance, leading to a gradual decrease over time.

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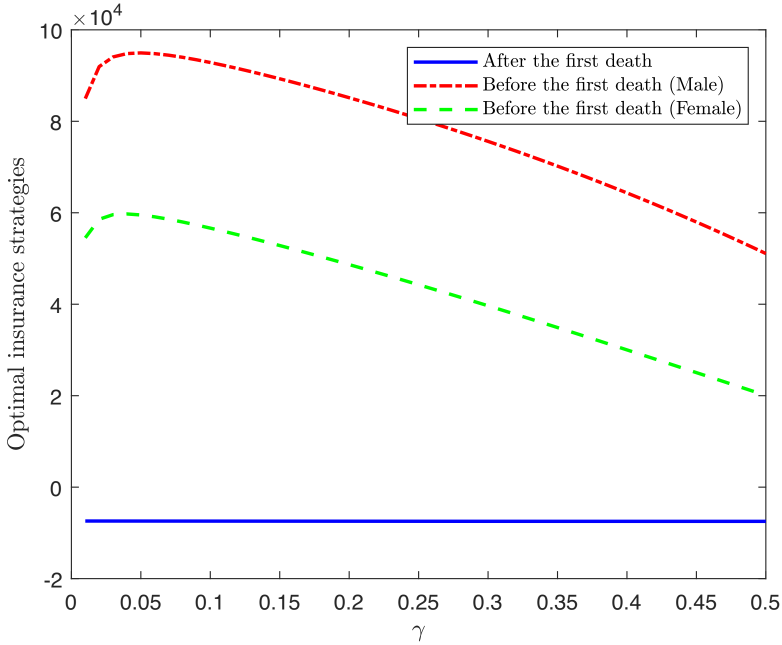

Figure 11. Effects of γ on the optimal insurance strategies.

In scenarios without mortality dependency (after the first death), the theoretical demand shifts towards annuities, as illustrated in our example. Moreover, mortality dependency influences housing investment, resulting in a rising trend in housing consumption. These findings highlight the intricate interplay between mortality dependency and diverse financial decisions within the specified scenarios.

5. Conclusion

In summary, this article investigates an optimal strategy problem within the context of a couple of two breadwinners with uncertain lifetimes. Optimal strategies for consumption, investment, housing, and life insurance have been determined to maximize utility. This study considers the correlated prices for housing assets and investment risky assets. Moreover, the model employs copula functions to account for correlated mortality rates and capture the breaking heart effect.

The analytical solutions for optimal strategies provide valuable insights, and the numerical results in this article enhance our understanding of complex dynamics. This research addresses knowledge gaps in life insurance, consumption, investment, and housing asset strategies for couples with uncertain lifetimes and mortality dependence. The findings offer a clear roadmap for decision-making in households dealing with financial uncertainties.

Regarding optimal consumption strategies, it can be shown that before the first death in a household, both male and female consumption decreased over time. This observed decrease contrasts with conventional expectations of increasing consumption over time due to rising labor income, which we excluded in the numerical demonstration to emphasize theoretical results.

After the first death, the consumption trend became less sensitive to time changes, indicating a stabilization in consumption rates. This deviation from traditional models, which predict a steady increase in consumption, reflects the household's strategy to preserve wealth for future use. By decreasing consumption gradually, households aim to accumulate savings as a financial buffer for

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retirement or unexpected expenses, maintaining a stable standard of living even after the loss of a partner.

The analysis also demonstrates the impact of time on housing consumption and life insurance strategies. Specifically, it shows that the decline in life insurance strategy over time, contrary to conventional expectations, can be attributed to the exclusion of human capital considerations in this particular case. The observed declining trend in housing assets suggests that retirees may increasingly consider using reverse mortgages to access home equity as a financial resource.

This article has the potential to influence policies related to housing, spending, investing, and insurance for retirees. By providing a deeper understanding of how housing assets, reverse mortgages, and retiree decisions interact, policymakers can develop more effective strategies for housing finance, retirement planning, and overall financial well-being. The findings may lead to the creation of more targeted policies that address the unique needs of retired couples, enhancing solutions for accessing housing equity and ensuring financial security during retirement. This improved policy framework could significantly benefit retirees by offering better options for managing their financial resources and maintaining their quality of life.

Data availability statement. Data sharing is not applicable. No new data is generated.

Funding statement. The research of Jiaqin Wei was supported by the National Natural Science Foundation of China (12071146, 12471447).

Competing interests. The authors declare none.

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Appendix A.

We can firstly write the continuous time Hamilton–Jacobi–Bellman (HJB) equation. Here, we simplify the notation $V_i(t, x, h)$ for the value function to V_i.

$$\begin{aligned}(\delta + \theta_i(t))V_i = \sup_{c_i,\pi_i,\phi_{5_i},\phi_{4_i},k_i} \Bigg\{ & w_i U_1(c_i(t), \phi_{4_i}(t)) + w_3\theta_i(t)U_2\left(x + \frac{k_i(t,x)}{\theta_i(t)}\right) + V_{it} \\ & + V_{iX}\{[r(t) + (\mu(t) - r(t))\pi_i(t)]x - \phi_{4_i}(t)\upsilon h - (c_i(t) + k_i(t) - Y_i(t))\} \\ & + V_{iH}[H(t)(r(t) + \lambda_H\sigma_H(t) - \zeta)] + \frac{1}{2}V_{iHH}[H^2(t)\sigma_H^2(t)] \\ & + \frac{1}{2}V_{iXX}[\pi_i^2(t)x^2\sigma_S^2(t) + \phi_{I_i}^2(t)h^2\sigma_H^2(t) \\ & + 2\rho_{HS}\pi_i(t)\phi_{5_i}(t)xh\sigma_S(t)\sigma_H(t)] \\ & + V_{iXH}\left[xh\sigma_S(t)\sigma_H(t)\rho_{HS}\pi(t) + \phi_{5_i}(t)h^2\sigma_H^2(t)\right]\Bigg\}, t \in [0, T) \end{aligned} \tag{A1}$$

and V_i at time T is equal to $w_4U_2(x)$.

We assume that the value function takes the form of

$$V_1 = \frac{1}{\gamma}g_1(t,h)^{1-\gamma}(x + b_1(t))^\gamma$$
$$V_2 = \frac{1}{\gamma}g_2(t,h)^{1-\gamma}(x + b_2(t))^\gamma,$$

where the $b_i(t) = \int_t^\omega e^{-\int_t^s (r(u)+\theta_i(u))du}Y(s)ds$ represents the human capital for breadwinner i. We can write $b_i(t) = Y\int_t^\omega e^{-r\times(s-t)+\theta_i(s)}ds = a_i(r, \omega - t)Y$ if we assume r and Y are constant.

The derivatives of V_i are stated as follows:

$$\begin{aligned}
V_{it} &= \gamma V_i \left[\frac{1-\gamma}{\gamma} \frac{g_{it}(t,h)}{g_i(t,h)} + \frac{b_{it}(t)}{x+b_i(t)} \right], \\
V_{ix} &= \frac{\gamma V_i}{x+b_i(t)}, \\
V_{ixx} &= \frac{\gamma(\gamma-1)V_i}{(x+b_i(t))^2}, \\
V_{ih} &= \frac{(1-\gamma)V_i g_{ih}(t,h)}{g_i(t,h)}, \\
V_{ihh} &= \gamma(1-\gamma)V_i \left[\frac{1}{\gamma} \frac{g_{ihh}(t,h)}{g_i(t,h)} - \left(\frac{g_{ih}(t,h)}{g_i(t,h)} \right)^2 \right], \\
V_{ixh} &= (1-\gamma)\gamma V_i \frac{g_{ih}(t,h)}{g_i(t,h)(x+b_i(t))}.
\end{aligned} \tag{A2}$$

According to the first-order conditions for optimal consumption, $c_i^*(t)$ and optimal housing consumption, $\phi_{4_i}^*(t)$, we have

$$\begin{aligned}
w_i\beta(\phi_{4_i}^*(t))^{1-\beta}(c_i^*(t))^{\beta-1}\left[(c_i^*(t))^{\beta}(\phi_{4_i}^*(t))^{1-\beta}\right]^{\gamma-1} &= V_{ix}, \\
w_i(1-\beta)(\phi_{4_i}^*(t))^{-\beta}(c_i^*(t))^{\beta}\left[(c_i^*(t))^{\beta}(\phi_{4_i}^*(t))^{1-\beta}\right]^{\gamma-1} &= \upsilon h V_{ix}
\end{aligned}$$

and further we find

$$\begin{aligned}
c_i^*(t) &= \frac{\beta}{1-\beta} \upsilon \eta_i h^k V_{ix}^{\frac{1}{\gamma-1}}, \\
\phi_{4_i}^*(t) &= \eta_i h^{q-1} V_{ix}^{\frac{1}{\gamma-1}}, \\
k_i^*(t,x) &= \theta_i(t) \left[\left(\frac{1}{w_3} V_{ix} \right)^{\frac{1}{\gamma-1}} - x \right],
\end{aligned} \tag{A3}$$

where $q = \frac{-\gamma+\beta\gamma}{1-\gamma}$ and $\eta_i = (w_i\beta)^{\frac{1}{1-\gamma}} \left(\frac{\beta\upsilon}{1-\beta} \right)^{k-1}$.

We can also find that

$$\begin{aligned}
U_1(c_i^*(t), \phi_{4_i}^*(t)) &= \frac{[(c_i^*(t))^{\beta}(\phi_{4_i}^*(t))^{1-\beta}]^{\gamma}}{\gamma} \\
&= \frac{\upsilon^{\beta\gamma} \eta_i^{\gamma} h^{(k+\beta-1)\gamma} \left(\frac{\beta}{1-\beta} \right)^{\gamma\beta}}{\gamma} V_{ix}^{\frac{\gamma}{\gamma-1}}, \\
U_2\left(x + \frac{k_i^*(t,x)}{\theta_i(t)} \right) &= \frac{w_3^{\frac{-\gamma}{\gamma-1}}}{\gamma} V_{ix}^{\frac{\gamma}{\gamma-1}}.
\end{aligned} \tag{A4}$$

According to the first-order conditions for the optimal proportions of the wealth invested in risky assets, $\pi_i^*(t)$, and optimal housing investment units, $\phi_{5_i}^*(t)$, we have

$$\begin{aligned}
0 &= (\mu(t)+r(t))xV_{ix} + \pi_i^*(t)x^2\sigma_S^2(t)V_{ixx} + \rho_{HS}\phi_{5_i}^*(t)xh\sigma_S\sigma_H V_{ixx} + \rho_{HS}xh\sigma_S(t)\sigma_H(t)V_{ixh}, \\
0 &= \phi_{5_i}^*(t)h^2\sigma_H^2(t)V_{ixx} + \pi_i^*(t)\rho_{HS}xh\sigma_S(t)\sigma_H(t)V_{ixx} + h^2\sigma_H^2 V_{ixh}
\end{aligned}$$

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and further we find

$$
\begin{aligned}
\pi_i^*(t) &= \frac{-(\mu(t)-r(t))V_{ix}}{(1-\rho_{HS}^2)\sigma_S^2(t)xV_{ixx}},\\
\phi_{5_i}^*(t) &= \frac{\rho_{HS}(\mu(t)-r(t))V_{ix}}{(1-\rho_{HS}^2)\sigma_S(t)\sigma_H(t)hV_{ixx}} - \frac{V_{ixh}}{V_{ixx}}\\
&= \frac{\rho_{HS}(\mu(t)-r(t))V_{ix} - (1-\rho_{HS}^2)\sigma_S(t)\sigma_H(t)hV_{ixh}}{(1-\rho_{HS}^2)\sigma_S(t)\sigma_H(t)hV_{ixx}}. \qquad \text{(A5)}
\end{aligned}
$$

Here, to simplify the equations, we denote c_i as $c(t)$, θ_i as $\theta(t)$, μ as $\mu(t)$, r as $r(t)$, π_i as $\pi_i(t)$, S as $S(t)$, H as $H(t)$ σ_S as $\sigma_S(t)$, σ_H as $\sigma_H(t)$, k_i as $k_i(t,x)$, Y_i as $Y_i(t)$, b_i as $b_i(t)$, ϕ_{4_i} as $\phi_{4_i}(t)$, and ϕ_{5_i} as $\phi_{5_i}(t)$. Then, we can have the continuous time HJB equation

$$
\begin{aligned}
(\delta+\theta_i)V_i = \sup_{c_i,\pi_i,\phi_{5_i},\phi_{4_i},k_i} \Bigg\{ & w_iU_1(c_i,\phi_{4_i}) + w_3\theta_iU_2\left(x+\frac{k_i}{\theta_i}\right)\\
& V_{it} + V_{iX}\{[r+(\mu-r)\pi_i]x - \phi_{4_i}\upsilon h - (c_i+k_i-Y_i)\}\\
& + V_{iH}[h(r+\lambda_H\sigma_H-\zeta)] + \frac{1}{2}V_{iHH}[h^2\sigma_H^2]\\
& + \frac{1}{2}V_{iXX}[\pi_i^2x^2\sigma_S^2 + \phi_{5_i}^2h^2\sigma_H^2\\
& + 2\rho_{HS}\pi_i\phi_{5_i}xh\sigma_S\sigma_H]\\
& + V_{iXH}\left[xh\sigma_S\sigma_H\rho_{HS}\pi_i + \phi_{5_i}h^2\sigma_H^2\right]\Bigg\} \text{ for } t\in[0,T). \qquad \text{(A6)}
\end{aligned}
$$

By substituting (A2)–(A5) into (A6), we can have

$$
\begin{aligned}
0 = {} & V_i(\delta-\theta_i) - (1-\gamma)w_3^{-\frac{1}{\gamma-1}}\gamma^{-\frac{1}{\gamma-1}}(x+b_i)^{-\frac{\gamma}{\gamma-1}}V_i^{\frac{\gamma}{\gamma-1}}\\
& - w_i\eta_i^\gamma h^{\gamma(\beta+k-1)}\upsilon^{\beta\gamma}\left(\frac{\beta}{1-\beta}\right)^{\beta\gamma}\gamma^{-\frac{1}{\gamma-1}}(x+b_i)^{-\frac{\gamma}{\gamma-1}}V_i^{\frac{\gamma}{\gamma-1}}\\
& + V_i\left[\frac{g_{it}(t,h)(\gamma-1)}{g_i(t,h)}\right] - V_i\gamma\theta_i - \gamma^{-\frac{1}{\gamma-1}}(x+b_i)^{-\frac{\gamma}{\gamma-1}}V_i^{\frac{\gamma}{\gamma-1}}\frac{\eta_ih^k\upsilon}{\beta-1}\\
& - V_i\gamma\left[r+\frac{(\mu-r^2)}{\sigma_S^2(\rho_{HS}^2-1)(\gamma-1)}\right] + V_i(\gamma-1)h(\gamma-\zeta+\lambda_H\sigma_H)\frac{g_{ih}(t,h)}{g_i(t,h)}\\
& - \frac{1}{2}V_i(\gamma-1)h^2\sigma_H^2\frac{g_{ihh}(t,h)}{g_i(t,h)} + \frac{1}{2}V_i\gamma\frac{\mu-r}{\sigma_S^2(\rho_{HS}^2-1)(\gamma-1)}. \qquad \text{(A7)}
\end{aligned}
$$

Then, we divide (A7) by $\frac{(x+b_i)^\gamma}{\gamma g_i^\gamma(t,h)}$ and find

$$
\begin{aligned}
0 = {} & g_i(t,h)d_3 + g_{it}(t,h)(\gamma-1) + g_{ih}(t,h)(\gamma-1)h(\gamma-\zeta+\lambda_H\sigma_H)\\
& - \frac{1}{2}(\gamma-1)h^2\sigma_H^2(t)g_{ihh}(t,h) + d_4, \qquad \text{(A8)}
\end{aligned}
$$

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where

$$d_3 = \delta - (\gamma + 1)\theta_i(t) - \gamma r - \frac{1}{2}\frac{\gamma(\mu - r)^2}{\sigma_S^2(\rho_{HS}^2 - 1)(\gamma - 1)},$$
$$d_4 = (\gamma - 1)w_3^{\frac{1}{1-\gamma}} - w_1\left(\frac{\beta}{1-\beta}\right)^{\beta\gamma}\eta_i^{\gamma}h^{\gamma(\beta+k-1)}\upsilon^{\beta\gamma} + \frac{1}{1-\beta}\eta_i h^k\upsilon.$$

According to terminal condition, $g_i(t, h)$ has the boundary condition, that is, $g_i(T, h) = w_4^{\frac{1}{1-\gamma}}$. Hence, closed form result of $g_i(t, h)$ can be found from (A8).

$$g_i(t, h) = \alpha_1 h^k \int_t^T e^{\alpha_2(s)}ds + \alpha_3 h^{\gamma(\beta+k-1)}\int_t^T e^{\alpha_4(s)}ds + \alpha_5,$$

where

$$\alpha_1 = \frac{\eta_i\upsilon}{(1-\beta)(\gamma-1)},$$
$$\alpha_2(s) = \frac{d_1(s-T)}{(1-\gamma)\alpha_1},$$
$$\alpha_3 = -\frac{w_1}{1-\gamma}\left(\frac{\beta}{1-\beta}\right)^{\beta\gamma}\eta_i^{\gamma}\upsilon^{\beta\gamma},$$
$$\alpha_4(s) = \frac{d_2(s-T)}{(1-\gamma)\alpha_3},$$
$$\alpha_5 = w_4^{\frac{1}{1-\gamma}} + \left(\frac{\gamma-1}{d_3}w_3^{\frac{1}{1-\gamma}} + w_4^{\frac{1}{1-\gamma}}\right)(e^{\frac{d_1(T-t)}{\gamma-1}} - 1),$$
$$d_1 = \alpha_1 d_3 + \alpha_1(\gamma-1)(\gamma - \zeta + \lambda_H\sigma_H)k - \frac{1}{2}\alpha_1(\gamma-1)\sigma_H^2 k(k-1),$$
$$d_2 = \alpha_3 d_3 + \alpha_3(\gamma-1)(\gamma - \zeta + \lambda_H\sigma_H)\gamma(\beta+k-1)$$
$$-\frac{1}{2}\alpha_3(\gamma-1)\sigma_H^2(t)\gamma(\beta+k-1)[\gamma(\beta+k-1)-1].$$

Appendix B.

We firstly define the following conditional probability distribution functions.

$$F_1(s;t) = P(\tau_1 \le s \mid \tau_1 > t),$$
$$F_2(s;t) = P(\tau_2 \le s \mid \tau_2 > t),$$
$$F_{T_1}(s;t) = P(T_1 \le s \mid T_1 > t),$$
$$F(s_1, s_2;t) = P(\tau_1 \le s_1, \tau_2 \le s_2 \mid T_1 > t),$$

The corresponding conditional density functions are defined as $f_1(s;t)$, $f_2(s;t)$, $f_{T_1}(s;t)$ and $f(s_1, s_2;t)$.

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Lemma B1.

$$f_i(x;t) = \frac{f_i(x)}{1 - F_i(t)}, \quad i = 1, 2,$$
$$f_{T_1}(x;t) = \frac{f_{T_1}(x)}{1 - F_{T_1}(t)}$$
$$f(x, y;t) = \frac{f(x, y)}{1 - F_{T_1}(t)}$$

where $T_1 = \tau_1 \wedge \tau_2$ *and* $F_{T_1}(t) = F_1(t) + F_2(t) - F(t, t)$.

Since the proof of the above lemma closely resembles that of Lemma 3.2 in Wei et al. (2020), the details are omitted here.

To obtain Equation (3.2), we firstly write the follows terms which are

$$\begin{aligned}
&\int_t^{\tau_1\wedge T} w_1 e^{-\delta(s-t)} U_1(c_1(s, X(s)))ds + \int_t^{\tau_2\wedge T} w_2 e^{-\delta(s-t)} U_1(c_2(s, X(s)))ds \\
&= \int_t^{T_1\wedge T} w_1 e^{-\delta(s-t)} U_1(c_1(s, X(s)))ds + \int_t^{T_1\wedge T} w_2 e^{-\delta(s-t)} U_1(c_2(s, X(s)))ds \\
&\quad + \int_{T_1\wedge T}^{\tau_1\wedge T} w_1 e^{-\delta(s-t)} U_1(c_1(s, X(s)))ds + \int_{T_1\wedge T}^{\tau_2\wedge T} w_2 e^{-\delta(s-t)} U_1(c_2(s, X(s)))ds \\
&= \int_t^{T_1\wedge T} w_1 e^{-\delta(s-t)} U_1(c_1(s, X(s)))ds + \int_t^{T_1\wedge T} w_2 e^{-\delta(s-t)} U_1(c_2(s, X(s)))ds \\
&\quad + \mathbf{1}_{\{T_1=\tau_2<\tau_1, T_1\le T\}} \int_{T_1}^{\tau_1\wedge T} w_1 e^{-\delta(s-t)} U_1(c_1^*(s, X(s)))ds \\
&\quad + \mathbf{1}_{\{T_1=\tau_1<\tau_2, T_1\le T\}} \int_{T_1}^{\tau_2\wedge T} w_2 e^{-\delta(s-t)} U_1(c_2^*(s, X(s)))ds,
\end{aligned} \tag{B1}$$

$$\begin{aligned}
&w_3 \mathbf{1}_{\{\tau_1\vee\tau_2\le T\}} e^{-\delta(\tau_1\vee\tau_2)} U_2\left(X(\tau_1\vee\tau_2) + \sum_{i=1}^{2} \frac{k_i(\tau_i, X(\tau_i))}{\theta_i(\tau_i)} \mathbf{1}_{\{\tau_i=\tau_1\vee\tau_2\}}\right) \\
&= w_3 \mathbf{1}_{\{T_1=\tau_2<\tau_1\le T\}} e^{-\delta\tau_1} U_2\left(X(\tau_1) + \frac{k_1^*(\tau_1, X(\tau_1))}{\theta_1(\tau_1)}\right) \\
&\quad + w_3 \mathbf{1}_{\{T_1=\tau_1<\tau_2\le T\}} e^{-\delta\tau_2} U_2\left(X(\tau_2) + \frac{k_1^*(\tau_2, X(\tau_2))}{\theta_2(\tau_2)}\right),
\end{aligned} \tag{B2}$$

and

$$\begin{aligned}
&w_4 \mathbf{1}_{\{\tau_i>T\}} e^{-\delta(T-t)} U_2(X(T)) \\
&= w_4 \left(\mathbf{1}_{\{T_1=\tau_2\le T<\tau_1\}} + \mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}\right) e^{-\delta(T-t)} U_2(X(T)) \\
&\quad + w_4 \left(\mathbf{1}_{\{T_1=\tau_1\le T<\tau_2\}} + \mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}\right) e^{-\delta(T-t)} U_2(X(T)).
\end{aligned} \tag{B3}$$

Following Equations (B1)–(B3), we can find

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$$\begin{aligned}
V = &\max_{u\in\mathcal{A}_t} E\bigg\{ \int_t^{\tau_i\wedge T} w_i e^{-\delta(s-t)} U_1(c_i(s), \phi_4(s))ds \\
&+ w_3 \mathbf{1}_{\{\tau_i\le T\}} e^{-\delta(\tau_i-t)} U_2\left(X(\tau_i) + \frac{k_i(\tau_i, X(\tau_i))}{\theta_i(\tau_i)}\right) \\
&+ w_4 \mathbf{1}_{\{\tau_i> T\}} e^{-\delta(T-t)} U_2(X(T))\bigg\} \\
= &\max_{u\in\mathcal{A}_t} E\bigg\{ \int_t^{T_1\wedge T} w_1 e^{-\delta(s-t)} U_1(\bar{c}_1(s, X(s)))ds + \int_t^{T_1\wedge T} w_2 e^{-\delta(s-t)} U_1(\bar{c}_2(s, X(s)))ds \\
&+ \mathbf{1}_{\{T_1=\tau_2<\tau_1, T_1\le T\}}\bigg[\int_{T_1}^{\tau_2\wedge T} w_2 e^{-\delta(s-t)} U_1(c_2^*(s, X(s)))ds \\
&+ w_3 \mathbf{1}_{\{\tau_2\le T\}} e^{-\delta\tau_2} U_2\left(X(\tau_2) + \frac{k_2^*(\tau_2, X(\tau_2))}{\theta_2(\tau_2)}\right) + w_4 \mathbf{1}_{\tau_2>T\}} e^{-\delta(T-t)} U_2(X(T))\bigg] \\
&+ \mathbf{1}_{\{T_1=\tau_1<\tau_2, T_1\le T\}}\bigg[\int_{T_1}^{\tau_1\wedge T} w_1 e^{-\delta(s-t)} U_1(c_1^*(s, X(s)))ds \\
&+ w_3 \mathbf{1}_{\{\tau_1\le T\}} e^{-\delta\tau_1} U_2\left(X(\tau_1) + \frac{k_1^*(\tau_1, X(\tau_1))}{\theta_1(\tau_1)}\right) + w_4 \mathbf{1}_{\tau_1>T\}} e^{-\delta(T-t)} U_2(X(T))\bigg] \\
&+ w_4 \left(\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}} + \mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}\right) e^{-\delta(T-t)} U_2(X(T))\bigg\}.
\end{aligned} \tag{B4}$$

We can further simplify the terms in Equation (B4) as follows:

$$\begin{aligned}
&\max_{u\in\mathcal{A}_t} E\bigg\{ \int_t^{T_1\wedge T} e^{-\delta(s-t)} \left[w_1 U_1(\bar{c}_1(s, X(s))) + w_2 U_1(\bar{c}_2(s, X(s)))\right] ds\bigg\} \\
&= \max_{u\in\mathcal{A}_t} E\bigg\{ \mathbf{1}_{\{t<T_1<\le T\}} \int_t^{T_1} e^{-\delta(s-t)} \left[w_1 U_1(\bar{c}_1(s, X(s))) + w_2 U_1(\bar{c}_2(s, X(s)))\right] ds \\
&\quad + \mathbf{1}_{\{T_1>T\}} \int_t^{T} e^{-\delta(s-t)} \left[w_1 U_1(\bar{c}_1(s, X(s)))ds + w_2 U_1(\bar{c}_2(s, X(s)))\right] ds\bigg\} \\
&= \max_{u\in\mathcal{A}_t} E\bigg\{ \int_t^{T} \left[\frac{1-F_{T_1}(s)}{1-F_{T_1}(t)}\right] e^{-\delta(s-t)} \left[w_1 U_1(\bar{c}_1(s, X(s))) + w_2 U_1(\bar{c}_2(s, X(s)))\right] ds\bigg\},
\end{aligned} \tag{B5}$$

$$\begin{aligned}
&\max_{u\in\mathcal{A}_t} E\bigg[\mathbf{1}_{\{T_1=\tau_2<\tau_1, T_1\le T\}} V_1\left(T_1, X(T_1) + \frac{\bar{k}_2(T_1, X(T_1))}{\theta_2(T_1)}\right)\bigg] \\
&= \max_{u\in\mathcal{A}_t} E\bigg[(\mathbf{1}_{\{\tau_2\le T, \tau_1>T\}} + \mathbf{1}_{\{\tau_2\le\tau_1\le T\}}) V_1\left(T_1, X(T_1) + \frac{\bar{k}_2(T_1, X(T_1))}{\theta_2(T_1)}\right)\bigg] \\
&= \max_{u\in\mathcal{A}_t} E\bigg\{ \int_t^{T} \left[\frac{\int_z^\infty f(s,z)ds}{1-F_{T_1}(t)}\right] V_1\left(z, X(z) + \frac{\bar{k}_2(z, X(z))}{\theta_2(z)}\right) dz\bigg\},
\end{aligned} \tag{B6}$$

$$\begin{aligned}
&\max_{u\in\mathcal{A}_t} E\bigg[\mathbf{1}_{\{T_1=\tau_1<\tau_2, T_1\le T\}} V_2\left(T_1, X(T_1) + \frac{\bar{k}_1(T_1, X(T_1))}{\theta_1(T_1)}\right)\bigg] \\
&= \max_{u\in\mathcal{A}_t} E\bigg[(\mathbf{1}_{\{\tau_1\le T, \tau_2>T\}} + \mathbf{1}_{\{\tau_1\le\tau_2\le T\}}) V_2\left(T_1, X(T_1) + \frac{\bar{k}_1(T_1, X(T_1))}{\theta_1(T_1)}\right)\bigg] \\
&= \max_{u\in\mathcal{A}_t} E\bigg\{ \int_t^{T} \left[\frac{\int_s^\infty f(s,z)dz}{1-F_{T_1}(t)}\right] V_2\left(s, X(s) + \frac{\bar{k}_1(s, X(s))}{\theta_1(s)}\right) ds\bigg\},
\end{aligned} \tag{B7}$$

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and

$$\begin{aligned}
&\max_{u\in\mathcal{A}_t} E\left[w_4\left(\mathbf{1}_{\{T<T_1=\tau_2<\tau_1\}}+\mathbf{1}_{\{T<T_1=\tau_1<\tau_2\}}\right)e^{-\delta(T-t)}U_2(X(T))\right]\\
&=\max_{u\in\mathcal{A}_t} E\left\{\frac{w_4e^{-\delta(T-t)}U_2(X(T))}{1-F_{T_1}(t)}\left[\int_T^\infty\int_z^\infty f(s,z)dsdz+\int_T^\infty\int_s^\infty f(s,z)dzds\right]\right\}\\
&=\max_{u\in\mathcal{A}_t} E\left\{\frac{w_4e^{-\delta(T-t)}U_2(X(T))}{1-F_{T_1}(t)}\int_T^\infty\int_T^\infty f(s,z)dsdz\right\}. \qquad \text{(B8)}
\end{aligned}$$

Based on Equations (B5)–(B8), we can have

$$\begin{aligned}
V=&\frac{1}{1-F_{T_1}(t)}\max_{u\in\mathcal{A}_t} E\Big\{\int_t^T\left[1-F_{T_1}(s)\right]e^{-\delta(s-t)}\left[w_1U_1(\bar{c}_1(s,X(s)))+w_2U_1(\bar{c}_2(s,X(s)))\right]ds\\
&+\int_t^T\left[\int_z^\infty f(s,z)ds\right]V_1\left(z,X(z)+\frac{\bar{k}_2(z,X(z))}{\theta_2(z)}\right)dz\\
&+\int_t^T\left[\int_s^\infty f(s,z)dz\right]V_2\left(s,X(s)+\frac{\bar{k}_1(s,X(s))}{\theta_1(s)}\right)ds\\
&+w_4e^{-\delta(T-t)}U_2(X(T))\int_T^\infty\int_T^\infty f(s,z)dsdz\Big\}. \qquad \text{(B9)}
\end{aligned}$$

Appendix C.

Here, to simplify the equations, we denote $\bar{c}_i$ as $\bar{c}_i(t)$, θ_i as $\theta(t)$, μ as $\mu(t)$, r as $r(t)$, π_i as $\pi_i(t)$, S as $S(t)$, H as $H(t)$ σ_S as $\sigma_S(t)$, σ_H as $\sigma_H(t)$, $\bar{k}_i$ as $\bar{k}_i(t,x)$, Y_i as $Y_i(t)$, b_i as $b_i(t)$, $\bar{\phi}_{4_i}$ as $\bar{\phi}_{4_i}(t)$, and $\bar{\phi}_5$ as $\bar{\phi}_5(t)$. We write $\tilde{V}=(1-F_{T_1}(t))V$. According to It's formula, the dynamics of the value function $\tilde{V}$ is

$$\begin{aligned}
d\tilde{V}=&\tilde{V}_tdt+\tilde{V}_X\{[r+(\mu-r)\bar{\pi}]X-\sum_{i=1}^{2}\left[\bar{\phi}_{4_i}\upsilon H+(\bar{c}_i+\bar{k}_i-Y_i)\right]\}dt\\
&+\tilde{V}_H[H(r+\lambda_H\sigma_H-\zeta)]dt+\frac{1}{2}\tilde{V}_{HH}[H^2\sigma_H^2]dt\\
&+\frac{1}{2}\tilde{V}_{XX}[\bar{\pi}^2X^2\sigma_S^2+\bar{\phi}_5^2H^2\sigma_H^2+2\rho_{HS}\bar{\pi}\bar{\phi}_5XH\sigma_S\sigma_H]dt\\
&+\tilde{V}_{XH}\left[XH\sigma_S\sigma_H\rho_{HS}\bar{\pi}+\bar{\phi}_5H^2\sigma_H^2\right]dt\\
&+\{\tilde{V}_X[\bar{\pi}X\sigma_S+\bar{\phi}_5H\rho_{HS}\sigma_H]+\tilde{V}_H\rho_{HS}\sigma_H\}dZ_S\\
&+[\tilde{V}_X\bar{\phi}_5H\rho_H\sigma_H+\tilde{V}_XH\rho_H\sigma_H]dZ_H \text{ for } t\in[0,T].
\end{aligned}$$

We can then obtain the HJB equation.

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$$\left(\delta+\int_t^\infty f(s,t)ds+\int_t^\infty f(t,z)dz\right)\tilde{V}$$
$$=\sup_{\bar{c}_i,\bar{\pi},\bar{\phi}_5,\bar{\phi}_{4_i},\bar{k}_i}\left\{(1-F_{T_1})\sum_{i=1}^2 w_iU_1(\bar{c}_i,\bar{\phi}_{4_i})+\tilde{V}_t\right.$$
$$+V_1\left(x+\frac{\bar{k}_2(t,x)}{\theta_2},h\right)\int_t^\infty f(s,t)ds+V_2\left(x+\frac{\bar{k}_1(t,x)}{\theta_1},h\right)\int_t^\infty f(t,z)dz$$
$$+\tilde{V}_X\{[r+(\mu-r)\bar{\pi}]x-\sum_{i=1}^2[\bar{\phi}_{4_i}\upsilon h+(\bar{c}_i+\bar{k}_i-Y_i)]\}$$
$$+\tilde{V}_H[h(r+\lambda_H\sigma_H-\zeta)]+\frac{1}{2}\tilde{V}_{HH}[h^2\sigma_H^2]$$
$$+\frac{1}{2}\tilde{V}_{XX}[\bar{\pi}^2x^2\sigma_S^2+\bar{\phi}_5^2h^2\sigma_H^2+2\rho_{HS}\bar{\pi}\bar{\phi}_5xh\sigma_S\sigma_H]$$
$$\left.+\tilde{V}_{XH}\left[xh\sigma_S\sigma_H\rho_{HS}\bar{\pi}+\bar{\phi}_5h^2\sigma_H^2\right]\right\}\text{ for } t\in[0,T), \tag{C1}$$

$$\tilde{V}=w_4U_2(X(T))\text{ for } t>T.$$

We assume that the value function has the ansatz of

$$V=\frac{g(t,h)^{1-\gamma}}{\gamma[1-F_{T_1(t)}]}(x+B)^\gamma$$

and

$$\tilde{V}=\frac{g(t,h)^{1-\gamma}}{\gamma}(x+b)^\gamma,$$

where $B=\sum_{i=1}^2\int_t^\omega e^{-\int_t^s(r(u)+\theta_i(u))du}Y_ids$ represents the human capital for breadwinner.

The derivatives of V are stated as follows:

$$\tilde{V}_t=\gamma\tilde{V}\left[\frac{1-\gamma}{\gamma}\frac{g_t(t,h)}{g(t,h)}+\frac{B_t}{x+B}\right],$$
$$\tilde{V}_x=\frac{\gamma\tilde{V}}{x+B},$$
$$\tilde{V}_{xx}=\frac{\gamma(\gamma-1)\tilde{V}}{(x+B)^2},$$
$$\tilde{V}_h=\frac{(1-\gamma)\tilde{V}g_H(t,h)}{g(t,h)},$$
$$\tilde{V}_{hh}=\gamma(1-\gamma)\tilde{V}\left[\frac{1}{\gamma}\frac{g_{HH}(t,h)}{g(t,h)}-\left(\frac{g_H(t,h)}{g(t,h)}\right)^2\right],$$
$$\tilde{V}_{xh}=(1-\gamma)\gamma\tilde{V}\frac{g_H(t,h)}{g(t,h)(x+B)}. \tag{C2}$$

According to the first-order conditions for optimal consumption, $\bar{c}_i^*$ and optimal housing consumption, $\bar{\phi}_{4_i}^*$, we have

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$$(1-F_{T_1}(t))w_i\beta(\bar{\phi}_{4_i}^*)^{1-\beta}(\bar{c}_i^*)^{\beta-1}\left[(\bar{c}_i^*)^{\beta}(\bar{\phi}_{4_i}^*)^{1-\beta}\right]^{\gamma-1}=\tilde{V}_x,$$
$$(1-F_{T_1}(t))w_i(1-\beta)(\bar{\phi}_{4_i}^*)^{-\beta}(\bar{c}_i^*)^{\beta}\left[(\bar{c}_i^*)^{\beta}(\bar{\phi}_{4_i}^*)^{1-\beta}\right]^{\gamma-1}=\upsilon h\tilde{V}_x$$

and by first-order condition, we can find

$$\bar{c}_i^*=\frac{\beta}{1-\beta}\upsilon\eta_i h^k V_x^{\frac{1}{\gamma-1}},$$
$$\bar{\phi}_{4_i}^*=\eta_i h^{k-1}V_x^{\frac{1}{\gamma-1}} \tag{C3}$$

and

$$\begin{aligned}\bar{\pi}^*&=\frac{-(\mu-r)\tilde{V}_x}{(1-\rho_{HS}^2)\sigma_S^2 x\tilde{V}_{xx}}\\ \bar{\phi}_5^*&=\frac{\rho_{HS}(\mu-r)\tilde{V}_x}{(1-\rho_{HS}^2)\sigma_S\sigma_H h\tilde{V}_{xx}}-\frac{\tilde{V}_{xh}}{\tilde{V}_{xx}}\\ &=\frac{\rho_{HS}(\mu-r)\tilde{V}_x-(1-\rho_{HS}^2)\sigma_S\sigma_H h\tilde{V}_{xh}}{(1-\rho_{HS}^2)\sigma_S\sigma_H h\tilde{V}_{xx}}.\end{aligned} \tag{C4}$$

where $k=\frac{-\gamma+\beta\gamma}{1-\gamma}$ and $\eta_i=(w_i\beta)^{\frac{1}{1-\gamma}}\left(\frac{\beta\upsilon}{1-\beta}\right)^{k-1}$.

Now, we consider the following bivariate function

$$\begin{aligned}\psi(\bar{k}_1,\bar{k}_2)&=-(\bar{k}_1+\bar{k}_2)\tilde{V}_x+V_1\left(t,x+\frac{\bar{k}_2}{\theta_2(t)}\right)\int_t^\infty f(s,t)ds\\ &\quad+V_2\left(t,x+\frac{\bar{k}_1(t)}{\theta_1(t)}\right)\int_t^\infty f(t,z)dz\\ &\quad-\left(\delta+\int_t^\infty f(s,t)ds+\int_t^\infty f(t,z)dz\right)\tilde{V}\\ &=(\bar{k}_1+\bar{k}_2)\tilde{V}_x+\frac{g_1^{1-\gamma}(t,h)\left(x+\frac{\bar{k}_2^*}{\theta_2}+b_2\right)^{\gamma}}{\gamma}\int_t^\infty f(s,t)ds\\ &\quad+\frac{g_2^{1-\gamma}(t,h)\left(x+\frac{\bar{k}_1^*}{\theta_1}+b_1\right)^{\gamma}}{\gamma}\int_t^\infty f(t,z)dz\\ &\quad-\left(\delta+\int_t^\infty f(s,t)ds+\int_t^\infty f(t,z)dz\right)\tilde{V}.\end{aligned} \tag{C5}$$

We define $(\bar{k}_1,\bar{k}_2)$ as a critical point. Since $\psi_{\bar{k}_1}<0$, $\psi_{\bar{k}_2}<0$ and $\psi_{\bar{k}_1\bar{k}_1}\psi_{\bar{k}_2\bar{k}_2}-\psi_{\bar{k}_1\bar{k}_2}^2>0$, $\psi(\bar{k}_1,\bar{k}_2)$ has the relative maximum point at $(\bar{k}_1^*,\bar{k}_2^*)$. Hence, we have

$$\psi_{\bar{k}_1^*}(\bar{k}_1^*,\bar{k}_2^*)=-\tilde{V}_X+\frac{g_2^{1-\gamma}(t,h)}{\theta_1}\left(x+\frac{\bar{k}_1^*}{\theta_1}+b_2\right)^{\gamma-1}\int_t^\infty f(t,z)dz=0,$$
$$\psi_{\bar{k}_2^*}(\bar{k}_1^*,\bar{k}_2^*)=-\tilde{V}_X+\frac{g_1^{1-\gamma}(t,h)}{\theta_2}\left(x+\frac{\bar{k}_2^*}{\theta_2}+b_1\right)^{\gamma-1}\int_t^\infty f(s,t)ds=0. \tag{C6}$$

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Equation (C5) can be further simplified as

$$\psi_{\bar{k}_1^*}(\bar{k}_1^*,\bar{k}_2^*)=-\tilde{V}_X+\frac{g_2^{1-\gamma}(t,h)}{\theta_1}\left(x+\frac{\bar{k}_1^*}{\theta_1}+b_2\right)^{\gamma-1}\int_t^\infty f(t,z)dz=0,$$
$$\psi_{\bar{k}_2^*}(\bar{k}_1^*,\bar{k}_2^*)=-\tilde{V}_X+\frac{g_1^{1-\gamma}(t,h)}{\theta_2}\left(x+\frac{\bar{k}_2^*}{\theta_2}+b_1\right)^{\gamma-1}\int_t^\infty f(s,t)ds=0. \qquad \text{(C7)}$$

From Equation (C7), we can obtain

$$\left(x+\frac{\bar{k}_1^*}{\theta_1}+b_2\right)^{\gamma-1}=\frac{\tilde{V}_X(t,x)\theta_1}{g_2^{1-\gamma}(t,h)\int_t^\infty f(t,z)dz}$$
$$\left(x+\frac{\bar{k}_2^*}{\theta_2}+b_1\right)^{\gamma-1}=\frac{\tilde{V}_X(t,x)\theta_2}{g_1^{1-\gamma}(t,h)\int_t^\infty f(s,t)ds} \qquad \text{(C8)}$$

and

$$\bar{k}_1^*=\theta_1(-x-b_2)+\left(\frac{\tilde{V}_X\theta_1}{g_2^{1-\gamma}(t,h)\int_t^\infty f(t,z)dz}\right)^{\frac{1}{\gamma-1}}$$
$$\bar{k}_2^*=\theta_2(-x-b_1)+\left(\frac{\tilde{V}_X\theta_2}{g_1^{1-\gamma}(t,h)\int_t^\infty f(s,t)ds}\right)^{\frac{1}{\gamma-1}}. \qquad \text{(C9)}$$

Based on Equations (C8) and (C9), Equation (C5) can be written as

$$\begin{aligned}\psi(\bar{k}_1^*,\bar{k}_2^*)&=-(\bar{k}_1^*+\bar{k}_2^*)\tilde{V}_X+g_1^{1-\gamma}(t,h)\frac{\left(\frac{\tilde{V}_X\theta_2}{g_1^{1-\gamma}(t,h)\int_t^\infty f(s,t)ds}\right)^{\frac{\gamma}{\gamma-1}}}{\gamma}\int_t^\infty f(s,t)ds\\
&\quad+g_2^{1-\gamma}(t,h)\frac{\left(\frac{\tilde{V}_X(t,x)\theta_1}{g_2^{1-\gamma}(t,h)\int_t^\infty f(t,z)dz}\right)^{\frac{\gamma}{\gamma-1}}}{\gamma}\int_t^\infty f(t,z)dz\\
&\quad-\left(\delta+\int_t^\infty f(s,t)ds+\int_t^\infty f(t,z)dz\right)\tilde{V}\\
&=(\theta_1(x+b_2)+\theta_2(x+b_1))\tilde{V}_x\\
&\quad+\left(1-\frac{1}{\gamma}\right)\left(\frac{\theta_1}{\int_t^\infty f(t,z)dz}\right)^{\frac{\gamma}{\gamma-1}}g_2(t,h)\int_t^\infty f(t,z)dz\tilde{V}_x^{\frac{\gamma}{\gamma-1}}\\
&\quad+\left(1-\frac{1}{\gamma}\right)\left(\frac{\theta_2}{\int_t^\infty f(s,t)ds}\right)^{\frac{\gamma}{\gamma-1}}g_1(t,h)\int_t^\infty f(s,t)ds\tilde{V}_x^{\frac{\gamma}{\gamma-1}}\\
&\quad-\left(\delta+\int_t^\infty f(s,t)ds+\int_t^\infty f(t,z)dz\right)\tilde{V}.\end{aligned} \qquad \text{(C10)}$$

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Based on Equations (C3) and (C4), Equation (C1) can be written as

$$0=\sup_{\bar{c}_i,\bar{\pi},\bar{\phi}_5,\bar{\phi}_{4_i},\bar{k}_i}\Bigg\{(1-F_{T_1}(t))\left[w_1\frac{\eta_1^{\gamma}}{\gamma}+w_2\frac{\eta_2^{\gamma}}{\gamma}\right]\left(\frac{\beta}{1-\beta}\right)^{\beta\gamma}\upsilon^{\beta\gamma}h^{\gamma(k+\beta-1)}V_X^{\frac{\gamma}{\gamma-1}}+\tilde{V}_t$$
$$+\tilde{V}_X\left[rx+\frac{(\mu-r)^2\tilde{V}_X}{(1-\rho_{HS}^2)\sigma_S^2\tilde{V}_{XX}}+Y_1+Y_2\right]$$
$$+\tilde{V}_H[h(r+\lambda_H\sigma_H-\zeta)]-\tilde{V}_X\frac{(\eta_1+\eta_2)\upsilon h^k}{1-\beta}V_X^{\frac{1}{\gamma-1}}$$
$$+\frac{1}{2}\tilde{V}_{HH}[h^2\sigma_H^2]+\frac{1}{2}\tilde{V}_{XX}\Bigg[\frac{\tilde{V}_X^2(\mu-r)^2}{(1-\rho_{HS}^2)^2\sigma_S^2\tilde{V}_{XX}^2}$$
$$+\frac{\rho_{HS}^2(\mu-r)^2\tilde{V}_X^2}{(1-\rho_{HS}^2)^2\sigma_S^2\tilde{V}_{XX}^2}+\frac{\tilde{V}_{XH}^2h^2\sigma_H^2}{\tilde{V}_{XX}^2}+\frac{-2\rho_{HS}^2(\mu-r)^2\tilde{V}_X^2}{(1-\rho_{HS}^2)^2\sigma_S^2\tilde{V}_{XX}^2}\Bigg]$$
$$+\tilde{V}_{XH}\Bigg[\frac{-\rho_{HS}(\mu-r)\tilde{V}_Xh\sigma_H}{(1-\rho_{HS}^2)\sigma_S\tilde{V}_{XX}}$$
$$+\frac{\rho_{HS}(\mu-r)\tilde{V}_Xh\sigma_H+(1-\rho_{HS}^2)\tilde{V}_{XH}h^2\sigma_H^2\sigma_S}{(1-\rho_{HS}^2)\sigma_S\tilde{V}_{XX}}\Bigg]\Bigg\}+\psi(\bar{k}_1^*,\bar{k}_2^*). \tag{C11}$$

Substitute (C2) into (C11), we have

$$0=\sup_{\bar{c}_i,\bar{\pi},\bar{\phi}_5,\bar{\phi}_{4_i},\bar{k}_i}\Bigg\{\gamma\tilde{V}\left[\frac{1-\gamma}{\gamma}\frac{g_t(t,h)}{g(t,h)}+\frac{B_t}{x+B}\right]$$
$$+(1-F_{T_1}(t))^{-\frac{1}{\gamma-1}}\left[w_1\frac{\eta_1^{\gamma}}{\gamma}+w_2\frac{\eta_2^{\gamma}}{\gamma}\right]\left(\frac{\beta}{1-\beta}\right)^{\beta\gamma}\upsilon^{\beta\gamma}h^{\gamma(k+\beta-1)}\gamma^{\frac{\gamma}{\gamma-1}}\tilde{V}^{\frac{\gamma}{\gamma-1}}(x+B)^{-\frac{\gamma}{\gamma-1}}$$
$$+\frac{\gamma}{x+B(t)}\tilde{V}\left[r(t)x+\frac{(\mu(t)-r(t))^2(x+B)}{(1-\rho_{HS}^2)\sigma_S^2(\gamma-1)}+Y_1+Y_2\right]$$
$$+\frac{(1-\gamma)g_H(t,h)}{g(t,h)}\tilde{V}[h(r+\lambda_H\sigma_H-\zeta)]$$
$$+\frac{1}{2}\gamma(1-\gamma)\tilde{V}\left[\frac{1}{\gamma}\frac{g_{HH}(t,h)}{g(t,h)}-\left(\frac{g_H(t,h)}{g(t,h)}\right)^2\right](h^2\sigma_H^2)$$
$$-(1-F_{T_1}(t))^{-\frac{1}{\gamma-1}}\frac{(\eta_1+\eta_2)\upsilon h^k}{1-\beta}\gamma^{\frac{\gamma}{\gamma-1}}\tilde{V}^{\frac{\gamma}{\gamma-1}}(x+B)^{-\frac{\gamma}{\gamma-1}}$$
$$+\frac{1}{2}\frac{\gamma(\gamma-1)\tilde{V}}{(x+B)^2}\left[\frac{(\mu-r)^2(x+B)^2}{(1-\rho_{HS}^2)^2\sigma_S^2(\gamma-1)^2}+\frac{(x+B)^2h^2\sigma_H^2g_H^2(t,h)}{g^2(t,h)}\right]$$
$$+\gamma(1-\gamma)\frac{h^2\sigma_H^2g_H^2(t,h)\tilde{V}}{g^2(t,h)}+\psi(\bar{k}_1^*,\bar{k}_2^*). \tag{C12}$$

https://doi.org/10.1017/S1748499525000028 Published online by Cambridge University Press

Then, the continuous time (C12) becomes

$$0 = g(t,h)\tilde{d}_1 + g_t(t,h)\frac{1-\gamma}{\gamma} + g_h(t,h)\frac{1-\gamma}{\gamma}[h(r+\lambda_H\sigma_H-\zeta)] + g_{hh}(t,h)\frac{1}{2}\frac{1-\gamma}{\gamma}h^2\sigma_H^2 + \tilde{d}_2, \tag{C13}$$

where

$$\tilde{d}_1 = r + \frac{(\mu-r)^2}{(1-\rho_{HS}^2)\sigma_S^2(\gamma-1)} + \frac{1}{2}\frac{(\mu-r)^2}{(1-\rho_{HS}^2)^2\sigma_S^2(\gamma-1)} + \frac{\tilde{P}_1}{\gamma} + \frac{\tilde{P}_3}{x},$$

$$\tilde{d}_2 = (1-F_{T_1}(t))^{-\frac{1}{\gamma-1}}\left[w_1\frac{\eta_1^\gamma}{\gamma} + w_2\frac{\eta_2^\gamma}{\gamma}\right]\left(\frac{\beta}{1-\beta}\right)^{\beta\gamma}\upsilon^{\beta\gamma}h^{\gamma(k+\beta-1)},$$

$$-(1-F_{T_1}(t))^{-\frac{1}{\gamma-1}}\frac{(\eta_1+\eta_2)\upsilon h^k}{1-\beta} + \tilde{P}_2\gamma^{-\frac{\gamma}{\gamma-1}}x^{\frac{1}{\gamma-1}},$$

$$\tilde{P}_1 = -\left(\delta + \int_t^\infty f(s,t)ds + \int_t^\infty f(t,z)dz\right),$$

$$\tilde{P}_2 = \theta_1(x+b_2) + \theta_2(x+b_1),$$

$$\tilde{P}_3 = \left(1-\frac{1}{\gamma}\right)\left(\frac{\theta_1}{\int_t^\infty f(t,z)dz}\right)^{\frac{\gamma}{\gamma-1}} g_2(t,h)\int_t^\infty f(t,z)dz + \left(1-\frac{1}{\gamma}\right)\left(\frac{\theta_2}{\int_t^\infty f(s,t)ds}\right)^{\frac{\gamma}{\gamma-1}} g_1(t,h)\int_t^\infty f(s,t)ds.$$

From (C13), we can find

$$g(t,h) = \left(w_4^{\frac{1}{1-\gamma}} + \frac{\tilde{d}_2}{\frac{\gamma}{1-\gamma}\tilde{d}_1}\right)e^{\frac{\gamma}{1-\gamma}\tilde{d}_1(T-t)} - \frac{\tilde{d}_2}{\frac{\gamma}{1-\gamma}\tilde{d}_1}.$$

Cite this article: Zhang J, Wei J and Wang N (2025). Optimal decision-making for consumption, investment, housing, and life insurance purchase in a couple with dependent mortality, *Annals of Actuarial Science*, 1–29. https://doi.org/10.1017/S1748499525000028