ON THE SYMMETRIC ALGEBRA OF QUOTIENTS OF A C*-ALGEBRA

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(Received 17 May, 1989; revised 3 October, 1989)

1. Introduction. Let R be a semiprime ring (possibly without 1). The symmetric ring of quotients of R is defined as the set of equivalence classes of essentially defined double centralizers (f, g) on R; see [**1**], [**8**]. So, by definition, f is a left R-module homomorphism from an essential ideal I of R into R, g is a right R-module homomorphism from an essential ideal J of R into R, and they satisfy the balanced condition $f(x)y = xg(y)$ for $x \in I$ and $y \in J$. This ring was used by Kharchenko in his investigations on the Galois theory of semiprime rings [**4**] and it is also a useful tool for the study of crossed products of prime rings [**7**]. We denote the symmetric ring of quotients of a semiprime ring R by $Q(R)$.

If A is a C*-algebra then we can consider the filter $\mathcal{F}$ of *closed* essential ideals of A, directed downwards by inclusion. We denote by $Q_b(A)$ the *algebraic* inductive limit of $(M(I))_{I\in\mathcal{F}}$, where $M(I)$ denotes the C*-algebra of multipliers of I, and we call it the *symmetric normed algebra of quotients of* A. Clearly $Q_b(A)$ is a pre-C*-algebra and its completion, i.e. the C*-algebra inductive limit of $(M(I))_{I\in\mathcal{F}}$ is Pedersen's algebra of essential multipliers of A [**3**], [**9**]. However, we shall not consider this completion here. We also note that a symmetric normed algebra of quotients has been introduced and studied recently by Mathieu [**5**] in the setting of ultraprime Banach algebras.

It is shown in [**1**] that $Q_b(A)$ is the bounded subring of $Q(A)$. The purpose of this note is to use some recent results of N. C. Phillips [**11**] to prove a stronger relation between $Q_b(A)$ and $Q(A)$ (see Theorem 2.1 below). We use this theorem to obtain a characterization of the C*-algebras A such that $Q_b(A) = Q(A)$. In particular, we see that prime C*-algebras satisfy this condition.

2. The results. Let A be a C*-algebra. We view A as a subalgebra of $Q(A)$ via the regular representation $a \mapsto [(R_a, L_a)]$, where R_a (resp. L_a) denotes right (resp. left) multiplication by a. The involution of A extends to a positive definite involution on $Q(A)$ by the formula $[(f, g)]^* = [(g^*, f^*)]$; see [**1**]. By [**1**, Theorem 1.3], we can identify $Q_b(A)$ with the *-subalgebra of $Q(A)$ consisting of the elements of $Q(A)$ which are bounded with respect to the partial order on $Q(A)$ obtained by taking as a positive cone the set

$$\left\{\sum_{i=1}^{n} y_i^* y_i \mid y_i \in Q(A)\right\}.$$

Let R be a ring with unity and let M be the set of elements in $Z(R)$ which are not zero-divisors in $Z(R)$. If each element in M is not a zero-divisor in R, then we can form the *central localization* of R, RM^{-1}. The elements of RM^{-1} are of the form ab^{-1} where $a \in R$ and $b \in M$.

Let A be a C*-algebra and let x be an element in $Z(Q_b(A))$ such that x is not a zero-divisor in $Z(Q_b(A))$. Since $A \subset Q_b(A)$, x belongs to the *extended centroid* of A,

† This work was partially supported by CICYT grant PB 86-0353-C02-01.

$C(A)$, which is, by definition, the centre of $Q(A)$ and coincides with the centralizer of A in $Q(A)$; (see [**1**], [**6**]). Now $C(A)$ is a von Neumann regular ring [**6**, Theorem 3.3, (2)] and, since $C(A)$ has a proper involution, it is a *-regular ring in the sense of [**2**, p. 229]. By [**2**, Proposition 51.3], there exists $y \in C(A)$ such that $x = x^2 y$ and $e := xy$ is a projection in $C(A)$. In particular e is bounded and consequently $e \in Z(Q_b(A))$. Since $(1-e)x = 0$ we obtain $e = 1$ and so x is invertible in $Q(A)$. In particular x is not a zero-divisor in $Q_b(A)$.

It follows that we can form the central localization of $Q_b(A)$ and it is a subalgebra of $Q(A)$.

THEOREM 2.1. *If A is a C^*-algebra then $Q(A)$ is the central localization of $Q_b(A)$.*

Proof. Let $q = [(f, g)]$ be an element of $Q(A)$, where (f, g) is an essentially defined double centralizer on A. Obviously we can assume that f and g are defined on the same essential ideal I of A. Let K_I be the Pedersen's ideal of $\bar{I}$, the norm closure of I in A. Then $K_I \subset I$ and, since $K_I = K_I^2$, we see that $f(K_I) \subset K_I$ and $g(K_I) \subset K_I$. So (f, g) induces an element of the algebra of multipliers of K_I, and clearly K_I is an essential ideal of A.

Let $\{J_\lambda\}_{\lambda \in \Lambda}$ be a maximal family of nonzero pairwise orthogonal ideals of A such that $J_\lambda \subset K_I$ and with the property that $f_{|J_\lambda}$ and $g_{|J_\lambda}$ are bounded. We claim that $J = \bigoplus_{\lambda \in \Lambda} J_\lambda$ is an essential ideal of A. If J is not an essential ideal of A then there exists a nonzero closed ideal L of A such that $LJ_\lambda = 0$ for all $\lambda \in \Lambda$. Now choose a nonzero element $a \in (L \cap K_I)_+$. Then by [**11**, Theorem 2 and Proposition 3] we obtain a unique $(T, S) \in M(\overline{AaA})$ such that $T_{|\overline{aA}} = f_{|\overline{aA}}$ and $S_{|\overline{Aa}} = g_{|\overline{Aa}}$. It follows that (T, S) coincides with (f, g) on $K_I \cap \overline{AaA}$ and so the restrictions of f and g to $K_I \cap \overline{AaA}$ are bounded, which contradicts the maximality of the family $\{J_\lambda\}_{\lambda \in \Lambda}$.

Now set $U_\lambda = \{t \in \operatorname{Prim} A \mid J_\lambda \not\subset t\} = \{t \in \operatorname{Prim} A \mid \bar{J_\lambda} \not\subset t\}$. Then U_λ are pairwise disjoint open subsets of $\operatorname{Prim} A$, the *primitive spectrum* of A and $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ corresponds to $J = \bigoplus_{\lambda \in \Lambda} J_\lambda$. We define a function $\varphi : U \to \mathbf{C}$ by

$$\varphi(t) = \min\left\{1, \frac{1}{\|f_{|J_\lambda}\|}\right\} = \min\left\{1, \frac{1}{\|g_{|J_\lambda}\|}\right\}$$

if $t \in U_\lambda$. Then φ is a continuous bounded function on U and so by the Dauns–Hofmann Theorem [**10**, 4.4.8] there exists $z \in Z(M(J))$ such that $za + t = \varphi(t)a + t$ for all $a \in J$ and $t \in U$. It follows that $q_0 := zq$ is bounded on J. Since J is essential in A we obtain that $q_0 \in Q_b(A)$. Clearly, z is not a zero-divisor in $Z(Q_b(A))$ and so $q = z^{-1} q_0$, which shows that $Q(A)$ is the central localization of $Q_b(A)$. ■

PROPOSITION 2.2. *Let A be a C^*-algebra. The following conditions are equivalent:*

(i) $Q(A) = Q_b(A)$,
(ii) $Z(Q(A)) = Z(Q_b(A))$,
(iii) *any family of pairwise disjoint open subsets of* $\operatorname{Prim} A$ *is finite,*
(iv) $Z(Q(A))$ *is finite-dimensional,*
(v) *any double centralizer defined on an ideal of A is bounded.*

Proof. Obviously (i) $\Rightarrow$ (ii) and, by Theorem 2.1, (ii) $\Rightarrow$ (i).

By [**1**], $Z(Q(A)) \cong \varinjlim_{U \in \mathcal{D}} C(U)$ and $Z(Q_b(A)) \cong \varinjlim_{U \in \mathcal{D}} C_b(U)$, where $\mathcal{D}$ is the family of dense open subsets of $\operatorname{Prim} A$ and $C(U)$ (resp. $C_b(U)$) denotes the algebra of continuous (resp. bounded continuous) complex-valued functions on U. From this the implications (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) follow easily.

It is obvious that (v) implies (i).

Assume now that (i) holds and let (f, g) be a double centralizer defined on an ideal I of A. Let L be the left annihilator of I in A. Then L coincides with the right annihilator of I, L is an ideal of A and $I \oplus L$ is an essential ideal of A. By using $f(I)L = 0$ and $Lg(I) = 0$, we see that we can extend (f, g) to a double centralizer $(\bar{f}, \bar{g})$ on $I \oplus L$ by putting $\bar{f}(x+y) = f(x)$ and $\bar{g}(x+y) = g(x)$ for $x \in I$, $y \in L$. Therefore we can assume without loss of generality that I is an essential idea of A.

By (i) there exists an essential ideal J of A such that $J \subset I$ with $f_{|J}$ and $g_{|J}$ bounded. If f is not bounded then there exists a sequence $\{x_n\} \subset I$ such that $\|x_n\| \le 1$ and $\|f(x_n)\| \to \infty$. Now since $\bar{J}$ is essential in A we have $\|f(x_n)\| = \|R_{n|\bar{J}}\| = \|L_{n|\bar{J}}\|$ where R_n (resp. L_n) denotes right (resp. left) multiplication by $f(x_n)$. It follows that there exist $z_n \in J$ with $\|z_n\| \le 1$ such that $\|z_n f(x_n)\| \to \infty$. Since $z_n f(x_n) = f(z_n x_n)$ this leads to a contradiction. So any double centralizer defined on an ideal of A is bounded and consequently (v) holds. ■

Finally we state two immediate consequences of Proposition 2.2.

COROLLARY 2.3. (i) *If A is a prime C^*-algebra then every double centralizer defined on an ideal of A is automatically continuous.*

(ii) *If* Prim A *is Hausdorff then $Q(A) = Q_b(A)$ if and only if* Prim A *is finite.*

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