Canad. Math. Bull. Vol. **45** (4), 2002 pp. 623–633

Fermionic and Bosonic Representations of the Extended Affine Lie Algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$

Dedicated to Professor Robert Moody on the occasion of his 60th birthday

Yun Gao

Abstract. We construct a class of fermions (or bosons) by using a Clifford (or Weyl) algebra to get two families of irreducible representations for the extended affine Lie algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$ of level $(1, 0)$ (or $(-1, 0)$).

0 Introduction

Spinor representations for the affine Kac-Moody Lie algebras were first developed by Frenkel [F1] and Kac-Peterson [KP] independently. As one interesting application, Frenkel [F1] obtained addition formulas for elliptic θ-functions. Their ideas were to use an Clifford algebra with infinitely many generators to construct certain quadratic elements. It turns out that these quadratic elements plus the identity element span an orthogonal affine Kac-Moody Lie algebra. Thereafter, Feingold-Frenkel [FF] constructed representations for all classical affine Kac-Moody Lie algebras by using Clifford or Weyl algebras with infinitely many generators. Following [FF] we call these corresponding representations fermionic or bosonic.

The Clifford (or Weyl) algebras have natural representations on the exterior (or symmetric) algebras of polynomials over half of generators. Those representations are also important in quantum and statistical mechanics where the generators are interpreted as operators which create or annihilate particles and satisfy Fermi (or Bose) statistics.

Motivated by [F1], [KP] and [FF], we start from the Clifford (or Weyl) algebras with generators $a_i(n)$, $a_i^*(n)$, $1 \le i \le N$, $n \in \mathbb{Z}$, subject to relations

$$\{a_i(n), a_j(m)\}_\rho = \{a_i^*(n), a_j^*(m)\}_\rho = 0,$$
$$\{a_i(n), a_j^*(m)\}_\rho = \rho\delta_{ij}\delta_{n+m,0},$$

where $\{a, b\}_\rho = ab + \rho ba$ is the Jordan (or Lie) bracket depending on $\rho = \pm 1$. We then go on to construct a class of fermions (or bosons) and obtain representations for the newly developed extended affine Lie algebra (or double affine Lie algebra)

Received by the editors February 1, 2002; revised April 25, 2002.

The author gratefully acknowledges the grant support from the Natural Sciences and Engineering Research Council of Canada. He also wishes to thank Professors B. Allison, S. Berman, E. Neher and A. Pianzola for their support during the preparation of this work.

AMS subject classification: 17B65, 17B67.

$\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$ by enlarging the underlying module of the Clifford (or Weyl) algebra, where $\mathbb{C}_q = \mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ is a quantum torus of two variables. More precisely, let c_x and c_y denote the two central elements corresponding to the variables x and y respectively, the pair (λ, μ) is called the level for $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$ if c_x and c_y act as the scalars λ and μ respectively. The modules we obtained are completely reducible and we thus got two families of irreducible representations for the extended affine Lie algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$ of level $(1, 0)$ and $(-1, 0)$ respectively. Extended affine Lie algebras are a high dimensional generalization of affine Kac-Moody Lie algebras which were introduced in [H-KT] and further studied in [BGK] and [AABGP]. A representation for the extended affine Lie algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$ of level $(1, 0)$ has been obtained in [G1] and [BS] by using the principal vertex operator construction and in [G2] by using the homogeneous vertex operator construction. These vertex operator representations for $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$ were constructed only when q is not a root of unity. Our fermionic and bosonic representations are carried out for any non-zero q. A notion of highest weight modules was also proposed in [G2]. Some unitary representations which are related to our algebra have been given by Jakobsen-Kac [JK] and Wakimoto [W].

Throughout this paper, we denote the field of complex numbers and the ring of integers by $\mathbb{C}$ and $\mathbb{Z}$ respectively.

1 Extended Affine Lie Algebras

In this section, we will recall some basics on construction of the extended affine Lie algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$, which is of type A_{N-1} with nullity 2. For more information on extended affine Lie algebras, see [AABGP], [BGK] and [H-KT].

Let q be a non-zero complex number. A quantum torus associated to q (see [M]) is the unital associative $\mathbb{C}$-algebra $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ (or, simply $\mathbb{C}_q$) with generators $x^{\pm 1}, y^{\pm 1}$ and relations

$$xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1 \quad \text{and} \quad yx = qxy. \tag{1.1}$$

Then

$$x^{m_1} y^{n_1} x^{m_2} y^{n_2} = q^{n_1 m_2} x^{m_1+m_2} y^{n_1+n_2} \tag{1.2}$$

and

$$\mathbb{C}_q = \sum_{m,n \in \mathbb{Z}} \oplus \mathbb{C} x^m y^n. \tag{1.3}$$

Set $\Lambda(q) = \{n \in \mathbb{Z} : q^n = 1\}$. q is said to be generic if $\Lambda(q) = \{0\}$.

Define $\kappa, \chi \colon \mathbb{C}_q \to \mathbb{C}$ to be $\mathbb{C}$-linear functions given by

$$\kappa(x^m y^n) = \begin{cases} 1, & \text{if } m = 0 \text{ and } n \in \Lambda(q), \\ 0, & \text{otherwise;} \end{cases} \tag{1.4}$$

$$\chi(x^m y^n) = \begin{cases} 1, & \text{if } m = 0 \text{ and } n = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

Let d_x, d_y be the degree operators on $\mathbb{C}_q$ defined by

$$d_x(x^m y^n) = mx^m y^n, \quad d_y(x^m y^n) = nx^m y^n \tag{1.6}$$

for $m, n \in \mathbb{Z}$. Moreover, for any $n \in \Lambda(q)$, we define the operator

$$d(n) = y^n d_x. \tag{1.7}$$

We shall identify d_x with $d(0)$. Then the operators d_y and $d(n)$, $n \in \Lambda(q)$, form a derivation subalgebra of $\mathbb{C}_q$. They satisfy

$$[d(n), d(n')] = 0, \quad [d_y, d(n)] = nd(n), \tag{1.8}$$

for $n, n' \in \Lambda(q)$. Note that $y^n d_x$ is not a derivation if $n \in \mathbb{Z} \setminus \Lambda(q)$.

For the associative algebra $\mathbb{C}_q$ over $\mathbb{C}$, we have the matrix algebra $M_N(\mathbb{C}_q)$ with entries from $\mathbb{C}_q$. We will write $A(x) \in M_N(\mathbb{C}_q)$ for $A \in M_N(\mathbb{C})$ and $x \in \mathbb{C}_q$. Let $\mathfrak{gl}_N(\mathbb{C}_q)$ be the Lie algebra $M_N(\mathbb{C}_q)^-$ as usual. The Lie algebra $\mathfrak{gl}_N(\mathbb{C}_q)$ has a nondegenerate invariant form given by

$$\big(A(a), B(b)\big) = \operatorname{tr}(AB)\chi(ab), \quad \text{for } A, B \in M_N(\mathbb{C}), a, b \in \mathbb{C}_q. \tag{1.9}$$

Note that $\chi(ab) = \chi(ba)$ for $a, b \in \mathbb{C}_q$.

Let $c(n)$ be symbols indexed by $n \in \Lambda(q)$. We sometimes denote $c(0)$ by c_x. Now we form a central extension of $\mathfrak{gl}_N(\mathbb{C}_q)$,

$$\widehat{\mathfrak{gl}_N(\mathbb{C}_q)} = \mathfrak{gl}_N(\mathbb{C}_q) \oplus \Big(\sum_{n \in \Lambda(q)} \oplus \mathbb{C}c(n) \Big) \oplus \mathbb{C}c_y \tag{1.10}$$

with Lie bracket

$$\begin{aligned}[A(x^{m_1} y^{n_1}), B(x^{m_2} y^{n_2})] &= A(x^{m_1} y^{n_1})B(x^{m_2} y^{n_2}) - B(x^{m_2} y^{n_2})A(x^{m_1} y^{n_1}) \\ &\quad + \operatorname{tr}(AB)\kappa\big((d_x x^{m_1} y^{n_1})x^{m_2} y^{n_2}\big)\, c(n_1 + n_2) \\ &\quad + \operatorname{tr}(AB)\chi\big((d_y x^{m_1} y^{n_1})x^{m_2} y^{n_2}\big)\, c_y\end{aligned} \tag{1.11}$$

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, $A, B \in M_N(\mathbb{C})$, where $c(n)$, for $n \in \Lambda(q)$ and c_y are central elements of $\widehat{\mathfrak{gl}_N(\mathbb{C}_q)}$. Note that

$$\kappa(x^{m_1+m_2} y^{n_1+n_2}) = 0 \quad \text{if} \quad n_1 + n_2 \in \mathbb{Z} \setminus \Lambda(q). \tag{1.12}$$

The derivations d_y and $d(n)$ for $n \in \Lambda(q)$ can be extended to derivations on $\mathfrak{gl}_N(\mathbb{C}_q)$. Now we can define the semi-direct product of the Lie algebra $\widehat{\mathfrak{gl}_N(\mathbb{C}_q)}$ and those derivations:

$$\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)} = \widehat{\mathfrak{gl}_N(\mathbb{C}_q)} \oplus \Big(\sum_{n \in \Lambda(q)} \oplus \mathbb{C}d(n) \Big) \oplus \mathbb{C}d_y, \tag{1.13}$$

https://doi.org/10.4153/CMB-2002-057-3 Published online by Cambridge University Press

with Lie bracket

$$
\begin{aligned}
(1.14)\qquad &[d(n'), A(x^m y^n)] = mA(x^m y^{n+n'}), \quad [d_y, A(x^m y^n)] = nA(x^m y^n),\\
&[d(n'), c(n'')] = [d(n'), c_y] = [d_y, c_y] = 0, \quad [d_y, c(n')] = n'c(n'),
\end{aligned}
$$

for $m, n \in \mathbb{Z}$, $A \in M_N(\mathbb{C})$, $n', n'' \in \Lambda(q)$.

Next we extend the nondegenerate form on $\mathfrak{gl}_N(\mathbb{C}_q)$ to a symmetric bilinear form on $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$ as follows:

$$
(1.15)\qquad \big(A(a), B(b)\big) = \operatorname{tr}(AB)\chi(ab), \quad (c_y, d_y) = \big(c(n), d(-n)\big) = 1,
$$

all others are zero, for $A, B \in M_N(\mathbb{C})$, $a, b \in \mathbb{C}_q$, $n \in \Lambda(q)$. Then:

Lemma 1.16 *The form defined as in (1.15) is nondegenerate and invariant.*

Proof It is enough to show the form is invariant in the following case.

$$
(1.17)\qquad \big(d(n''), [A(x^{m_1} y^n), B(x^{m_2} y^{n'})]\big) = \big([d(n''), A(x^{m_1} t^n)], B(x^{m_2} y^{n'})\big),
$$

for $m_1, m_2, n, n' \in \mathbb{Z}$, $n'' \in \Lambda(q)$, $A, B \in M_N(\mathbb{C})$. Indeed, the left hand side of (1.17) is

$$
\begin{aligned}
&\big(d(n''), m_1 q^{m_2 n} \operatorname{tr}(AB)\kappa(x^{m_1+m_2} y^{n+n'})c(n+n')\big)\\
&\qquad = m_1 q^{m_2 n} \operatorname{tr}(AB)\delta_{m_1+m_2,0}\big(d(n''), \kappa(y^{n+n'})c(n+n')\big).
\end{aligned}
$$

The right hand side of (1.17) is

$$
\begin{aligned}
\big(m_1 A(x^{m_1} y^{n+n''}), B(x^{m_2} y^{n'})\big) &= m_1 q^{m_2(n+n'')} \operatorname{tr}(AB)\chi(x^{m_1+m_2} y^{n+n''+n'})\\
&= m_1 q^{m_2 n} \operatorname{tr}(AB)\delta_{m_1+m_2,0}\chi(y^{n+n''+n'}).
\end{aligned}
$$

Since $n'' \in \Lambda(q)$, we have

$$
\big(d(n''), \kappa(y^{n+n'})c(n+n')\big) = \chi(y^{n+n''+n'})
$$

and the proof is completed. ∎

If q is generic, $\widetilde{\mathfrak{gl}_1(\mathbb{C}_q)}$ is the so-called q-analog for two-dimensional Virasoro algebra [KPS]. Its vertex operator representation was given in [G-KL].

If $N \geq 2$, $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$ is an extended affine Lie algebra of type A_{N-1} with nullity 2. (See [AABGP] and [BGK] for definitions.) Some representations for those Lie algebras with generic q have been obtained in [BS], [G1] and [G2].

If $q = 1$, then

$$
\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)} \cong (\widetilde{gl}_N \otimes \mathbb{C}[y, y^{-1}]) \oplus \mathbb{C}c_y \oplus \mathbb{C}d_y,
$$

where

$$\widetilde{gl}_N = gl_N(\mathbb{C}[x, x^{-1}]) \oplus \mathbb{C}c_x \oplus \mathbb{C}d_x$$

is the affinization of the general linear Lie algebra gl_N.

Let E_{ij} be the matrix whose (i, j)-entry is 1 and 0 elsewhere. Then, in $\widehat{\mathfrak{gl}_N(\mathbb{C}_q)}$,

(1.18)

$$\begin{aligned}[E_{ij}(x^{m_1}y^{n_1}), E_{kl}(x^{m_2}y^{n_2})] = \delta_{jk}q^{n_1m_2}E_{il}(x^{m_1+m_2}y^{n_1+n_2}) - \delta_{il}q^{n_2m_1}E_{kj}(x^{m_1+m_2}y^{n_1+n_2}) \\ + m_1q^{n_1m_2}\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\kappa(y^{n_1+n_2})c(n_1+n_2) \\ + q^{n_1m_2}\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\delta_{n_1+n_2,0}n_1c_y\end{aligned}$$

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$.

2 Construction of Fermions and Bosons

In this section, we will give a unified treatment for both fermionic and bosonic constructions as were done in [FF]. However, for simplicity, we use a slightly different normal ordering from [FF] (see also [F2]).

Let $\mathcal{R}$ be an associative algebra. Let $\rho = \pm 1$. We define a ρ-bracket on $\mathcal{R}$ as follows:

$$\{r_1, r_2\}_\rho = r_1r_2 + \rho r_2r_1, \quad r_1, r_2 \in \mathcal{R}. \tag{2.1}$$

Then, one can easily see that

$$\{r_1, r_2\}_\rho = \rho\{r_2, r_1\}_\rho \text{ and } [r_1r_2, r_3] = r_1\{r_2, r_3\}_\rho - \rho\{r_1, r_3\}_\rho r_2 \tag{2.2}$$

for $r_1, r_2, r_3, r_4 \in \mathcal{R}$, where $[r_1, r_2] = \{r_1, r_2\}_{-1}$ is the Lie bracket.

Define $\mathfrak{a}(N, \rho)$ to be the unital associative algebra with infinitely many generators $a_i(n)$, $a_i^*(n)$, $n \in \mathbb{Z}$, $1 \le i \le N$, subject to relations

$$\begin{aligned}\{a_i(n), a_j(m)\}_\rho = \{a_i^*(n), a_j^*(m)\}_\rho = 0, \\ \{a_i(n), a_j^*(m)\}_\rho = \rho\delta_{ij}\delta_{n+m,0}.\end{aligned} \tag{2.3}$$

We now define the normal ordering as follows.

$$: a_i(n)a_j^*(m) : = \begin{cases} a_i(n)a_j^*(m) & \text{if } n \le m, \\ -\rho a_j^*(m)a_i(n) & \text{if } n > m, \end{cases} \tag{2.4}$$

for $n, m \in \mathbb{Z}$, $1 \le i, j \le N$. Set

$$\theta(n) = \begin{cases} 1, & \text{for } n > 0, \\ 0, & \text{for } n \le 0. \end{cases} \tag{2.5}$$

Then

$$a_i(n)a_j^*(m) = : a_i(n)a_j^*(m) : + \rho\delta_{ij}\delta_{n+m,0}\theta(n-m). \tag{2.6}$$

https://doi.org/10.4153/CMB-2002-057-3 Published online by Cambridge University Press

It follows from (2.2) that

$$
\begin{aligned}
[a_i(m)a_j^*(n), a_k(p)] &= \delta_{jk}\delta_{n+p,0}a_i(m),\\
[a_i(m)a_j^*(n), a_k^*(p)] &= -\delta_{ik}\delta_{m+p,0}a_j^*(n),
\end{aligned} \tag{2.7}
$$

for $m, n, p \in \mathbb{Z}$, $1 \le i, j, k \le N$.

Let $\mathfrak{a}(N,\rho)^+$ be the subalgebra generated by $a_i(n), a_j^*(m), a_k^*(0)$, for $n, m > 0$, and $1 \le i, j, k \le N$. Let $\mathfrak{a}(N,\rho)^-$ be the subalgebra generated by $a_i(n), a_j^*(m), a_k(0)$, for $n, m < 0$, and $1 \le i, j, k \le N$. Those generators in $\mathfrak{a}(N,\rho)^+$ are called annihilation operators while those in $\mathfrak{a}(N,\rho)^-$ are called creation operators. Let $V(N,\rho)$ be a simple $\mathfrak{a}(N,\rho)$-module containing an element v_0, called a "vacuum vector", and satisfying

$$
\mathfrak{a}(N,\rho)^+ v_0 = 0. \tag{2.8}
$$

So all annihilation operators kill v_0 and

$$
V(N,\rho) = \mathfrak{a}(N,\rho)^- v_0. \tag{2.9}
$$

Now we may construct a class of fermions (if $\rho = 1$) or bosons (if $\rho = -1$) on $V(N,\rho)$. For any $m, n \in \mathbb{Z}$, $1 \le i, j \le N$, set

$$
f_{ij}(m,n) = \sum_{s\in\mathbb{Z}} q^{-ns} \colon a_i(m-s)a_j^*(s) \colon. \tag{2.10}
$$

We also set

$$
D = \sum_{i=1}^{N}\sum_{s\in\mathbb{Z}} s \colon a_i(s)a_i^*(-s)\colon. \tag{2.11}
$$

Although $f_{ij}(m,n)$ and D are infinite sums, they are well-defined as operators on $V(N,\rho)$. Indeed, for any vector $v \in V(N,\rho) = \mathfrak{a}(N,\rho)^- v_0$, only finitely many terms in (2.10) and (2.11) can make a non-zero contribution to $f_{ij}(m,n)v$ and Dv.

Lemma 2.12 *We have*

$$
[f_{ij}(m,n), a_k(p)] = \delta_{jk}q^{np}a_i(m+p), \tag{2.13}
$$

$$
[f_{ij}(m,n), a_k^*(p)] = -\delta_{ik}q^{-n(m+p)}a_j^*(m+p), \tag{2.14}
$$

$$
[f_{ij}(m,n), a_k(p)a_l^*(s)] = \delta_{jk}q^{np}a_i(m+p)a_l^*(s) - \delta_{il}q^{-n(m+s)}a_k(p)a_j^*(m+s). \tag{2.15}
$$

for $m, n, p, s \in \mathbb{Z}$ and $1 \le i, j, k, l \le N$.

https://doi.org/10.4153/CMB-2002-057-3 Published online by Cambridge University Press

Proof

$$
\begin{aligned}
[f_{ij}(m,n), a_k(p)] &= \sum_{s\in\mathbb{Z}} q^{-ns}[: a_i(m-s)a_j^*(s) :, a_k(p)] \\
&= \sum_{s\in\mathbb{Z}} q^{-ns}[a_i(m-s)a_j^*(s), a_k(p)] \\
&= \sum_{s\in\mathbb{Z}} q^{-ns}\delta_{jk}\delta_{s+p,0}a_i(m-s) = q^{np}\delta_{jk}a_i(m+p).
\end{aligned}
$$

So (2.13) holds true. The proof of (2.14) is similar, and (2.15) follows from (2.13) and (2.14). ■

In what follows we shall mean $\frac{q^{mn}-1}{q^n-1} = m$ if $q^n = 1$. This will make our formula more concise.

Proposition 2.16

$$
\begin{aligned}
[f_{ij}(m_1,n_1), f_{kl}(m_2,n_2)] = {}& \delta_{jk}q^{n_1m_2} f_{il}(m_1+m_2, n_1+n_2) \\
& - \delta_{il}q^{n_2m_1} f_{kj}(m_1+m_2, n_1+n_2) \\
& + \rho\delta_{jk}\delta_{il}q^{n_1m_2}\delta_{m_1+m_2,0}\frac{q^{m_1(n_1+n_2)}-1}{q^{n_1+n_2}-1}q^{n_1+n_2}
\end{aligned}
$$

for all $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ *and* $1 \le i, j, k, l \le N$.

Proof It follows from (2.15) and (2.6) that

$$
\begin{aligned}
&[f_{ij}(m_1,n_1), q^{-n_2t}: a_k(m_2-t)a_l^*(t) :] \\
&\quad = \delta_{jk}q^{n_1m_2-(n_1+n_2)t}a_i(m_1+m_2-t)a_l^*(t) \\
&\qquad - \delta_{il}q^{n_2m_1-(n_1+n_2)(m_1+t)}a_k(m_2-t)a_j^*(m_1+t) \\
&\quad = \delta_{jk}q^{n_1m_2-(n_1+n_2)t}\big(: a_i(m_1+m_2-t)a_l^*(t) : + \rho\delta_{il}\delta_{m_1+m_2,0}\theta(m_1+m_2-2t)\big) \\
&\qquad - \delta_{il}q^{n_2m_1-(n_1+n_2)(m_1+t)}\big(: a_k(m_2-t)a_j^*(m_1+t) : \\
&\qquad\qquad + \rho\delta_{jk}\delta_{m_1+m_2,0}\theta(m_2-t-m_1-t)\big) \\
&\quad = \delta_{jk}q^{n_1m_2}q^{-(n_1+n_2)t}: a_i(m_1+m_2-t)a_l^*(t) : \\
&\qquad - \delta_{il}q^{n_2m_1}q^{-(n_1+n_2)(m_1+t)}: a_k(m_2-t)a_j^*(m_1+t) : \\
&\qquad + \rho\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}q^{n_1m_2}q^{-(n_1+n_2)t}\big(\theta(-2t) - \theta(-2m_1-2t)\big).
\end{aligned}
$$

https://doi.org/10.4153/CMB-2002-057-3 Published online by Cambridge University Press

Since

$$\begin{aligned}\sum_{t\in\mathbb{Z}} q^{-(n_1+n_2)t}\big(\theta(-2t)-\theta(-2m_1-2t)\big)\\ &= \begin{cases} 0, & \text{if } m_1=0,\\ \sum_{t=-m_1}^{-1} q^{-(n_1+n_2)t}, & \text{if } m_1>0,\\ \sum_{t=0}^{-m_1-1} q^{-(n_1+n_2)t}, & \text{if } m_1<0,\end{cases}\\ &= \frac{q^{m_1(n_1+n_2)}-1}{q^{n_1+n_2}-1}q^{n_1+n_2},\end{aligned}$$

we see that (2.16) holds true and the proof is completed. ■

Lemma 2.17

$$[D, a_k(p)] = pa_k(p), \tag{2.18}$$

$$[D, a_k^*(p)] = pa_k^*(p), \tag{2.19}$$

$$[D, f_{ij}(m,n)] = mf_{ij}(m,n), \tag{2.20}$$

for $p, m, n \in \mathbb{Z}$, $1 \le k, i, j \le N$.

Proof Let $D_i = \sum_{s\in\mathbb{Z}} s{:}\, a_i(s)a_i^*(-s)\,{:}$, for $1 \le i \le N$. As in (2.13), we may obtain that $[D_i, a_k(p)] = \delta_{ik} p a_k(p)$ and hence (2.18) follows. (2.19) can be proved similarly.

Now it follows from (2.18) and (2.19) that

$$\begin{aligned}[D, q^{-ns}{:}\, a_k(m-s)a_l^*(s)\,{:}] &= mq^{-ns}a_k(m-s)a_l^*(s)\\ &= mq^{-ns}\big({:}\, a_k(m-s)a_l^*(s)\,{:} + \rho\delta_{kl}\delta_{m,0}\theta(m-2s)\big)\\ &= mq^{-ns}{:}\, a_k(m-s)a_l^*(s)\,{:}\end{aligned}$$

which clearly yields (2.20). ■

Now we see that the operators $f_{ij}(m,n)$ for $m, n \in \mathbb{Z}$, $1 \le i, j \le N$, together with 1 and D, form a Lie algebra $\mathcal{L}$. Next we will consider two subalgebras $\mathcal{L}_{\mathrm{v}}$ and $\mathcal{L}_{\mathrm{h}}$ of $\mathcal{L}$. To this end, we first let

$$\mathcal{I} = \sum_{i=1}^{N} f_{ii}(0,0), \tag{2.21}$$

then one can easily show that

$$[\mathcal{I}, a_j(n)] = a_j(n) \quad \text{and} \quad [\mathcal{I}, a_j^*(n)] = -a_j^*(n), \tag{2.22}$$

for $n \in \mathbb{Z}$, $1 \le j \le N$. For any

$$v = a_{i_1}(n_1)\cdots a_{i_s}(n_s)a_{j_1}^*(m_1)\cdots a_{j_t}^*(m_t)v_0 \tag{2.23}$$

https://doi.org/10.4153/CMB-2002-057-3 Published online by Cambridge University Press

from $V(N,\rho)$, one has

$$\mathcal{I}v = (s-t)v. \tag{2.24}$$

Let $\mathcal{L}_{\rm v}$ be the subalgebra spanned by operators $f_{ij}(m,0)$ for $m \in \mathbb{Z}$, $1 \le i, j \le N$. Then $\mathcal{L}_{\rm v}$ is isomorphic to the affinization of the general linear Lie algebra $\mathfrak{gl}_N$. Moreover, as (5.56) in [FF], we have:

Proposition 2.25 *As $\mathcal{L}_v$-module, $V(N,\rho)$ is completely reducible. Moreover,*

$$V(N,\rho) = \sum_{k\in\mathbb{Z}} \oplus V_k,$$

where V_k is the k-eigenspace of the operator $\mathcal{I}$, and each V_k is an irreducible $\mathcal{L}_v$-module.

Remark 2.26 Let $\mathcal{L}_{\rm h}$ be the subalgebra spanned by operators $f_{ij}(0,n)$ for $n \in \mathbb{Z}$, $1 \le i, j \le N$. Then $\mathcal{L}_{\rm h}$ is isomorphic to the Lie algebra $\mathfrak{gl}_N\big(\mathbb{C}[\mathbb{Z}/\Lambda(0)]\big)$, where $\mathbb{C}[\mathbb{Z}/\Lambda(0)]$ is the group algebra. In particular, if q is generic, then $\mathcal{L}_{\rm h}$ is isomorphic to the loop algebra $\mathfrak{gl}_N(\mathbb{C}[\mathbb{Z}])$. Namely, each $\mathcal{L}_{\rm v}$-module V_k allows $\mathcal{L}_{\rm h}$-action. This shows that the level-ρ module for the affine Lie algebra $\hat{\mathfrak{gl}}_N$ allows a level-0 action for $\hat{\mathfrak{gl}}_N$.

3 Lifting to $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$

In this section, we shall obtain the $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$-module action by enlarging the Fock space $V(N,\rho)$.

From (2.16), we see that, if $n_1 + n_2 \in \Lambda(q)$,

$$\begin{aligned}[f_{ij}(m_1,n_1), f_{kl}(m_2,n_2)] = {}& \delta_{jk}q^{n_1m_2} f_{il}(m_1+m_2, n_1+n_2) \\ & - \delta_{il}q^{n_2m_1} f_{kj}(m_1+m_2,n_1+n_2) \\ & + \rho\delta_{jk}\delta_{il}q^{n_1m_2}\delta_{m_1+m_2,0}m_1,\end{aligned} \tag{3.1}$$

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, $1 \le i, j, k, l \le N$.

If $n_1 + n_2 \in \mathbb{Z} \setminus \Lambda(q)$, then

$$\begin{aligned}&[f_{ij}(m_1,n_1), f_{kl}(m_2,n_2)] \\ &\quad = \delta_{jk}q^{n_1m_2} f_{il}(m_1+m_2,n_1+n_2) - \delta_{il}q^{n_2m_1} f_{kj}(m_1+m_2,n_1+n_2) \\ &\qquad + \rho\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\frac{q^{(n_1+n_2)}}{q^{n_1+n_2}-1}(q^{n_2m_1}-q^{n_1m_2}) \\ &\quad = \delta_{jk}q^{n_1m_2}\left(f_{il}(m_1+m_2,n_1+n_2) - \rho\delta_{il}\delta_{m_1+m_2,0}\frac{q^{n_1+n_2}}{q^{n_1+n_2}-1}\right) \\ &\qquad - \delta_{il}q^{n_2m_1}\left(f_{kj}(m_1+m_2,n_1+n_2) - \rho\delta_{kj}\delta_{m_1+m_2,0}\frac{q^{n_1+n_2}}{q^{n_1+n_2}-1}\right),\end{aligned} \tag{3.2}$$

https://doi.org/10.4153/CMB-2002-057-3 Published online by Cambridge University Press

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$, $1 \le i, j, k, l \le N$. Therefore, if we define

$$F_{ij}(m,n) = \begin{cases} f_{ij}(m,n), & \text{for } n \in \Lambda(q) \\ f_{ij}(m,n) - \rho\delta_{ij}\delta_{m,0}\frac{q^n}{q^n-1}, & \text{for } n \in \mathbb{Z} \setminus \Lambda(q). \end{cases} \tag{3.3}$$

Then we have

$$\begin{aligned} [F_{ij}(x^{m_1}y^{n_1}), F_{kl}(x^{m_2}y^{n_2})] = \delta_{jk}q^{n_1m_2}F_{il}(x^{m_1+m_2}y^{n_1+n_2}) - \delta_{il}q^{n_2m_1}F_{kj}(x^{m_1+m_2}y^{n_1+n_2}) \\ + m_1q^{n_1m_2}\delta_{jk}\delta_{il}\delta_{m_1+m_2,0}\kappa(y^{n_1+n_2})\rho \end{aligned} \tag{3.4}$$

for $m_1, m_2, n_1, n_2 \in \mathbb{Z}$.

Set

$$W(N,\rho) = V(N,\rho) \otimes \mathbb{C}[y, y^{-1}], \tag{3.5}$$

and define operators

$$e_{ij}(m,n) = F_{ij}(m,n) \otimes y^n \tag{3.6}$$

which act on $W(N,\rho)$ as follows

$$e_{ij}(m,n)(v \otimes y^k) = F_{ij}(m,n)v \otimes y^{n+k}, \tag{3.7}$$

for $m, n, k \in \mathbb{Z}$, $1 \le i, j \le N$ and $v \in V(N,\rho)$. Then following from (1.18), (3.4) and Proposition 2.25 one can easily prove that:

Theorem 3.8 *$W(N,\rho)$ is a module for the Lie algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$ under the action given by*

$$\pi\big(E_{ij}(x^my^n)\big) = e_{ij}(m,n), \quad \text{for } 1 \le i, j \le N, m, n \in \mathbb{Z};$$
$$\pi\big(c(n)\big) = \rho 1 \otimes y^n, \quad \pi\big(d(n)\big) = D \otimes y^n, \quad \text{for } n \in \Lambda(q),$$
$$\pi(c_y) = 0, \quad \pi(d_y) = 1 \otimes d_y.$$

Moreover, $W(N,\rho)$ is completely reducible and each component $V_k \otimes \mathbb{C}[y, y^{-1}]$ is irreducible.

Remark 3.9 The above theorem gives us two families of irreducible representations for the extended affine Lie algebra $\widetilde{\mathfrak{gl}_N(\mathbb{C}_q)}$ of level $(\rho, 0)$, for $N \ge 2$. In particular, if q is not a root of unity (equivalently $\Lambda(q) = \{0\}$), then the center of the central extension in (1.10) is two dimensional. In this case, an irreducible vertex operator representation has been constructed in [BS], [G1] and [G2].

References

[AABGP] B. N. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, *Extended affine Lie algebras and their root systems.* Mem. Amer. Math. Soc. (605) **126**(1997).

[BGK] S. Berman, Y. Gao and Y. Krylyuk, *Quantum tori and the structure of elliptic quasi-simple Lie algebras.* J. Funct. Anal. **135**(1996), 339–389.

[BS] S. Berman and J. Szmigielski, *Principal realization for extended affine Lie algebra of type* sl_2 *with coordinates in a simple quantum torus with two variables.* Preprint.

[FF] A. J. Feingold and I. B. Frenkel, *Classical affine Lie algebras.* Adv. Math. **56**(1985), 117–172.

[F1] I. B. Frenkel, *Spinor representation of affine Lie algebras.* Proc. Natl. Acad. Sci. USA **77**(1980), 6303–6306.

[F2] ———, *Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory.* J. Funct. Anal. **44**(1981), 259–327.

[G1] Y. Gao, *Representations of extended affine Lie algebras coordinatized by certain quantum tori.* Compositio Math. **123**(2000), 1–25.

[G2] ———, *Vertex operators arising from the homogeneous realization for* $\widehat{gl}_N$. Comm. Math. Phys. **211**(2000), 745–777.

[H-KT] R. Høegh-Krohn and B. Torresani, *Classification and construction of quasi-simple Lie algebras.* J. Funct. Anal. **89**(1990), 106–136.

[JK] H. P. Jakobsen and V. G. Kac, *A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras. II.* J. Funct. Anal. **82**(1989), 69–90.

[KP] V. G. Kac and D. H. Peterson, *Spin and wedge representations of infinite-dimensional Lie algebras and groups.* Proc. Natl. Acad. Sci. USA **78**(1981), 3308–3312.

[KPS] E. Kirkman, C. Procesi and L. Small, *A q-analog for the Virasoro algebra.* Comm. Algebra **22**(1994), 3755-3774.

[G-KL] M. Golenishcheva-Kutuzova and D. Lebedev, *Vertex operator representation of some quantum tori Lie algebras.* Comm. Math. Phys. **148**(1992), 403–416.

[M] Y. I. Manin, *Topics in noncommutative geometry.* Princeton University Press, 1991.

[W] M. Wakimoto, *Extended affine Lie algebras and a certain series of Hermitian representations.* Preprint, 1985.

Department of Mathematics and Statistics
York University
Toronto, Ontario
M3J 1P3
email: ygao@yorku.ca

https://doi.org/10.4153/CMB-2002-057-3 Published online by Cambridge University Press