2. Consider the integral

$$I \equiv \int_0^n x(x-1)(x-2)\ldots.(x-n)\,dx$$

$$= \left\{\int_0^{\frac{n}{2}} + \int_{\frac{n}{2}}^{n}\right\} x(x-1)(x-2)\ldots.(x-n)\,dx; \text{ (Put } y = n - x)$$

$$= \int_0^{\frac{n}{2}} x(x-1)\ldots.(x-n)\,dx - (-)^n \int_0^{\frac{n}{2}} y(y-1)\ldots(y-n)\,dy$$

$$= 0, \text{ if } n \text{ is even.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now let $x(x-1)\ldots.(x-n) = a_1 x + a_2 x^2 + \ldots. + a_{n+1} x^{n+1}$. Then if n is even, we have a consistent set of $(n+1)$ equations in the a's,

$$0 = a_1 + a_2 + \ldots. + a_{n+1}$$

$$0 = 2a_1 + 2^2 a_2 + \ldots. + 2^{n+1} a_{n+1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$0 = na_1 + n^2 a_2 + \ldots. + n^{n+1} a_{n+1},$$

and by (1), $0 = \frac{n}{2} a_1 + \frac{n^2}{3} a_2 + \ldots. + \frac{n^{n+1}}{n+2} a_{n+1}$,

and hence the determinant of the system is zero, which is Jung's result.

A. C. AITKEN.

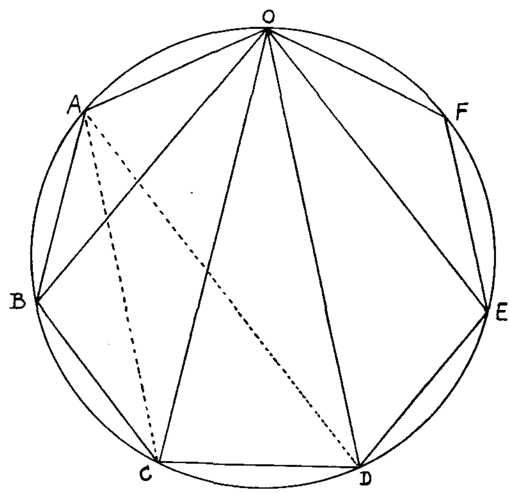
On the Roots of a Symmetrical Determinant.

The six values of x which make the determinant

$$\Delta = \begin{vmatrix} x & 1 & . & . & . & . \\ 1 & x & 1 & . & . & . \\ . & 1 & x & 1 & . & . \\ . & . & 1 & x & 1 & . \\ . & . & . & 1 & x & 1 \\ . & . & . & . & 1 & x \end{vmatrix}$$

vanish, are $-2\cos\frac{\pi}{7}$, $-2\cos\frac{2\pi}{7}$, ..., $-2\cos\frac{6\pi}{7}$. In general, the n values of x which make the corresponding determinant of order n vanish are given by $-2\cos\frac{r\pi}{n+1}$, $r = 1, 2, \ldots., n$. Each determinant has a diagonal filled with x's, bordered by adjacent parallels where each element is unity; and all other elements are zero.

The above result is no novelty, but the following geometrical proof may be of interest. Let $n+1$ be a prime number, such as 7, and let a, b, c, d, e, f denote the distances of a vertex O, of a regular heptagon $OABCDEF$, from the other six vertices, taken in cyclic order from O. Let α be the angle subtended at O by each of the equal sides OA, AB, etc., so that α is $\frac{\pi}{7}$, and in general is $\frac{\pi}{n+1}$.



Since $OA = CD$, it follows that AC is parallel with OD, and $AD = OC$. Hence, by projection,

$$(OC + AD)\cos\alpha = OD + AC = OD + OB,$$

or

$$OB - 2\,OC\cos\alpha + OD = 0.$$

By repeating this process we obtain the six equations

$$\begin{aligned} -2a\cos\alpha + b &= 0,\\ a - 2b\cos\alpha + c &= 0,\\ b - 2c\cos\alpha + d &= 0,\\ c - 2d\cos\alpha + e &= 0,\\ d - 2e\cos\alpha + f &= 0,\\ e - 2f\cos\alpha &= 0; \end{aligned}$$

whence, by elimination, $\Delta = 0$ when $x = -2\cos\alpha = -2\cos\frac{\pi}{7}$.

The same argument proves that $-2\cos\frac{2\pi}{7}$ also is a root of the equation $\Delta = 0$, if *OBDFACE* is taken as the heptagon, and the angle *BOD* is taken as α. Similarly the heptagon *OCFBEAD* leads to $-2\cos\frac{3\pi}{7}$, and so on, until six different cases have been identified.

Manifestly the process will apply, giving n roots of $\Delta = 0$, whenever $n+1$ is prime. If $n+1$ is composite the theorem is still true, as the argument remains applicable; but the proof is not so pretty, as several of the polygons degenerate.

The theorem may be stated otherwise:
The latent roots of the matrix of order n

$$\begin{bmatrix} . & 1 & . & . & . & \cdots \\ 1 & . & 1 & . & . & \cdots \\ . & 1 & . & 1 & . & \cdots \\ . & . & 1 & . & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \end{bmatrix}$$

are given by $\quad -2\cos\frac{r\pi}{n+1}, \quad r = 1, 2, \ldots, n.$

If the matrix is of infinite order its latent roots are all real, lying densely between -2 and $+2$.

H. W. Turnbull.