On a Set of Conform-Invariant Equations of the Gravitational Field

By H. A. Buchdahl

(*Received* 20*th July* 1949. *Read* 4*th November* 1949.)

1. Eddington[1] has considered equations of the gravitational field in empty space which are of the fourth differential order, viz. the sets of equations which express the vanishing of the Hamiltonian derivatives of certain fundamental invariants. The author has shown[2] that a wide class of such equations are satisfied by any solution of the equations

$$G_{\mu\nu} = \lambda g_{\mu\nu}, \tag{1.1}$$

where $G_{\mu\nu}$ and $g_{\mu\nu}$ are the components of the Ricci tensor and the metrical tensor respectively, whilst λ is an *arbitrary* constant. For a V_4 this applies in particular when the invariant referred to above is chosen from the set

$$\left.\begin{aligned} K_1 &= G^2 \,(= G^{\mu}_{\mu} G^{\nu}_{\nu}) \\ K_2 &= G_{\mu\nu} G^{\mu\nu} \\ K_3 &= B_{\mu\nu\sigma\rho} B^{\mu\nu\sigma\rho}, \end{aligned}\right\} \tag{1.2}$$

where $B_{\mu\nu\sigma\rho}$ is the covariant curvature tensor. K_3 has been included since, according to a result due to Lanczos[3], its Hamiltonian derivative $P_3{}^{\mu\nu}$ is a linear combination of $P_1{}^{\mu\nu}$ and $P_2{}^{\mu\nu}$, i.e. of the Hamiltonian derivatives of K_1 and K_2. In fact

$$P_3{}^{\mu\nu} = 4P_2{}^{\mu\nu} - P_1{}^{\mu\nu}. \tag{1.3}$$

It appears therefore that the most general invariant which will give rise to quasi-linear fourth order equations may be taken to be

$$aK_1 + bK_2, \tag{1.4}$$

where a and b are constants.

The question of the *general* solution of such equations seems to be as yet unsolved, even in the case of static spherically symmetric fields, which despite its relative simplicity presents great difficulties. In the present paper we shall be concerned with a special case of (1.4), viz. with the invariant

$$K = 3K_2 - K_1. \tag{1.5}$$

https://doi.org/10.1017/S001309150001419X Published online by Cambridge University Press

(This invariant is also mentioned in a different context by Gregory[4].) We shall show that if $P^{\mu\nu}$ is the Hamiltonian derivative of K, then *the equations*

$$P^{\mu\nu} = 0 \tag{1.6}$$

possess as a solution every line element representing a space conformal to an Einstein space.

Furthermore we shall show that the *general* solution of (1.6) in the case of the static spherically symmetric field may be written

$$ds^2 = \psi(r)\,[-\gamma^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + \gamma\, dt^2], \tag{1.7}$$

where $\gamma = 1 - 2m/r - \lambda r^2/3$ (m, λ arbitrary constants), and $\psi(r)$ is an *arbitrary* function of r. We are of course dealing with a V_4 throughout.

2. Let $C_{\mu\nu\sigma\rho}$ be the conformal curvature tensor[5]

$$C_{\mu\nu\sigma\rho} = B_{\mu\nu\sigma\rho} - 2g_{[\mu[\nu}(G_{\sigma]\rho]} - \tfrac{1}{6}g_{\sigma]\rho]}G). \tag{2.1}$$

(For the meaning of brackets enclosing indices, *vide* Schouten[6].) Consider the invariant

$$\bar{K} = C_{\mu\nu\sigma\rho}\, C^{\mu\nu\sigma\rho}. \tag{2.2}$$

Using (2.1) it is not difficult to show that we may write

$$\bar{K} = \tfrac{1}{3}K_1 - 2K_2 + K_3. \tag{2.3}$$

Now let

$$L = K_1 - 4K_2 + K_3. \tag{2.4}$$

Inserting this in (2.3) we obtain

$$\bar{K} = L - \tfrac{2}{3}K_1 + 2K_2. \tag{2.5}$$

In virtue of (1.3) the Hamiltonian derivative of L vanishes identically. Accordingly we simply consider the invariant K as given by (1.5). The Hamiltonian derivative $P^{\mu\nu}$ of K will then be the same as that of $\bar{K}$ except for a trivial numerical factor.

3. Consider the integral

$$J = \int \bar{K}\sqrt{-g}\, d\tau, \tag{3.1}$$

where $\bar{K}$ and g are formed with respect to a metrical tensor $g_{\mu\nu}$. In a conformal transformation in which the $g_{\mu\nu}$ are replaced by $\sigma g_{\mu\nu}$, where σ is an arbitrary function of the coordinates, $\bar{K}$ becomes multiplied by σ^{-2} and $\sqrt{-g}$ by σ^2, (in a V_4). Now $\bar{P}^{\mu\nu}$ is defined by the equation

$$\delta J = \int \bar{P}^{\mu\nu}\,\delta g_{\mu\nu}\sqrt{-g}\, d\tau, \tag{3.2}$$

where variations vanish on the boundary of the region of integration. It follows that in a conformal transformation the $\bar{P}^{\mu\nu}$ merely become multiplied by σ^{-3}. The set of equations

$$\bar{P}^{\mu\nu} \equiv P^{\mu\nu} = 0 \tag{3.3}$$

therefore transforms into itself, and we accordingly say that it is *conform-invariant*. (Strictly speaking $P^{\mu\nu}$ itself is "conform-covariant".) Obviously therefore, if $\bar{g}_{\mu\nu}$ is a particular solution of (3.3), then the product of $\bar{g}_{\mu\nu}$ with an arbitrary (sufficiently often differentiable) function of the coordinates is a more general solution. Making use of the known results stated in §1 we therefore have that if $\bar{g}_{\mu\nu}$ is the metrical tensor of an Einstein space, i.e. $\bar{g}_{\mu\nu}$ satisfies the equations (1.1), and $\Lambda(x_1, x_2, x_3, x_4)$ is an arbitrary function of the coordinates, then

$$\overset{*}{g}_{\mu\nu} = \Lambda(x)\,\bar{g}_{\mu\nu} \tag{3.4}$$

satisfies the set of equations (3.3); which proves our first assertion.

4. Although it is not essential for our purpose it may be of interest to write down the explicit form of $P^{\mu\nu}$. In fact, using some results due to the author[7], we find without difficulty that

$$\tfrac{1}{3}P^{\mu\nu} = 2C^{\mu\nu\sigma\rho}{}_{;\,\sigma\rho} - C^{\mu\nu\sigma\rho}\,G_{\sigma\rho}. \tag{4.1}$$

(4.1) may also be written

$$P^{\mu\nu} = S^{\mu\nu} - \tfrac{1}{4}g^{\mu\nu}\,S, \tag{4.2}$$

where $S^{\mu\nu} = 3\,\Box\,G^{\mu\nu} - G^{;\mu\nu} + 2GG^{\mu\nu} - 6B^{\mu\nu\sigma\rho}\,G_{\sigma\rho}$, $(S = S^{\mu}_{\mu})$. (4.3)

By (4.2) the spur of $P^{\mu\nu}$ vanishes identically. [This is a general property of the Hamiltonian derivatives of fundamental invariants K which are such that the corresponding scalar-densities $K\sqrt{-g}$ are conform-invariant. This is easily proved by considering the special variation

$$\delta g_{\mu\nu} = \epsilon g_{\mu\nu} \tag{4.4}$$

in the equation of definition of Hamiltonian derivatives (cf. (3.2)), where ϵ is an arbitrary infinitesimal function of the coordinates, vanishing on the boundary of the region of integration.]

5. We now come to the case of static spherically symmetric solutions of (3.3). If we disregard trivial arbitrary constants, the only Einstein spaces having the required property are[8].

$$\left.\begin{aligned} ds^2 = \bar{g}_{\mu\nu}\,dx^\mu\,dx^\nu \equiv -\gamma^{-1}\,d\rho^2 - \rho^2\,(d\theta^2 + \sin^2\theta\,d\phi^2) + \gamma dt^2, \\ \text{where } \gamma = 1 - 2m/\rho - \lambda\rho^2/3, \end{aligned}\right\} \tag{5.1}$$

m and λ being constants of integration. (We consider 'different' solutions obtainable from one another by coordinate transformations as constituting the same solution.)

By a suitable choice of coordinates every spherically symmetric static line element may be brought into the form

$$ds^2 = -e^{\mu(\rho)}\,d\rho^2 - \rho^2\,(d\theta^2 + \sin^2\theta\,d\phi^2) + e^{\nu(\rho)}\,dt^2. \tag{5.2}$$

In the present case we may further simplify (5.2) by carrying out first a conformal transformation in which the right-hand side of (5.2) is multiplied by $e^{-\nu}$ throughout, followed by a coordinate transformation $\rho = \rho(r)$ such that $\rho^2 \exp(-\nu(\rho)) = r^2$. We need then only consider line elements of the form

$$ds^2 = -(rw)^{-2}\,dr^2 - r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2) + dt^2. \tag{5.3}$$

It is not difficult to confirm that the components of the curvature tensor do not contain the second derivatives of w. Consequently the same is true of K, and in fact we find

$$K = 2(rww' + r^{-2})^2. \tag{5.4}$$

The condition that the Hamiltonian derivative of K vanish then yields the *second* order differential equation

$$\left(-\tfrac{1}{2}\frac{hK}{hw}r^4 w = \right) 2r^6w^3w'' + 6r^5w^3w' + r^6w^2w'^2 + 1 = 0. \tag{5.5}$$

If $r = \tau^{-1/2}$ and dots denote differentiation with respect to τ, (5.5) becomes

$$8w^3\ddot{w} + 4w^2\dot{w}^2 + 1 = 0, \tag{5.6}$$

which is easily integrated, its solution being

$$w^2 + 4mw^3 = \sigma - 27m^2\sigma^2, \tag{5.7}$$

where we have written $\sigma = \tau - \lambda/3$, and m, λ are constants of integration. In terms of a parameter ρ (5.7) may be given the equivalent form

$$\left.\begin{aligned} \sigma\rho^3 - \rho + 2m &= 0 \qquad &(5.81)\\ w = \rho^{-1} - 3m\rho^{-2}&. &(5.82)\end{aligned}\right\}$$

But if we now apply the transformations described above to the line element (5.1) then the relation between ρ and r is just that given by (5.81); and it may be confirmed that (5.82) in fact correctly represents the resulting function $w(r)$. It follows at once that all static spherically symmetric solutions of (3.3) can be written in the form (1.7); which was to be proved.

It may be noted that just the set of equations (3.3) is obtained in Weyl's theory [10] if we attempt to set up field equations by choosing

$$\delta \int \overline{C}_{\mu\nu\sigma\rho}\,\overline{C}^{\mu\nu\sigma\rho}\sqrt{-g}\,d\tau = 0 \tag{5.9}$$

as the determining gauge-invariant action principle. This is of course not surprising since Weyl's conformal curvature tensor[11] $\overline{C}_{\mu\nu\sigma\rho}$ does not involve the "electromagnetic potentials" at all. But it is interesting to observe that in this case we can at least obtain convergent solutions of the field equations (cf. Bergmann[12]). On the other hand the "unity" of gravitation and electromagnetism is then of an even more dubious kind.

In conclusion I wish to express my thanks to a referee who pointed out a missing link in the argument of the last section in the original draft of this paper.

REFERENCES.

1. Eddington, A. S., *The Mathematical Theory of Relativity* (2nd ed., Cambridge, 1930), §62, 141.

2. Buchdahl, H. A., *Proc. Nat. Acad.*, **38** (1948), 66.

3. Lanczos, C., *Annals of Math.* (2) **39** (1938), 842.

4. Gregory, C., *Phys. Rev.*, **72** (1947), 72.

5. Eisenhart, L. P., *Riemannian Geometry* (Princeton, 1926), Ch. II, 90.

6. Schouten, J. A., *Der Ricci-Kalkül* (Berlin, 1924), 25 and 31.

7. Buchdahl, H. A., *Oxford Quart. J. Math.*, 19 (1948), 150.

8. Eddington, A. S., Reference 1, §45, 100.

9. Eddington, A. S., Reference 1, §61, 140.

10. Weyl, H., *Space, Time, Matter* (London, 1922), §§35-36.

11. Weyl, H., *Math. Ztschr.*, 2 (1918), 404.

12. Bergmann, P. G., *Introduction to the Theory of Relativity* (New York, 1946), Ch. XVI, 253.

Note added in proof:

In the transformation of coordinates following (5.2) we have tacitly assumed that $\nu(\rho) \neq 2 \log \rho$. If this condition is not satisfied we need only consider a line element of the form

$$ds^2 = -e^{\lambda(\rho)}\, d\rho^2 - (d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2.$$

Then $K = 2$ and the Hamiltonian derivative of K is 1. The result of §5 therefore remains valid.

DEPARTMENT OF PHYSICS,
UNIVERSITY OF TASMANIA.