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CORRIGENDUM

The degree of knottedness of tangled vortex lines – CORRIGENDUM

H. K. Moffatt

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In a recent series of interesting papers, Bogoyavlenskij (2017*a*,*b*,*c*, and private communication) has drawn attention to an erroneous statement in the final paragraph of my 1969 *Journal of Fluid Mechanics* paper 'The degree of knottedness of tangled vortex lines' (Moffatt 1969), which calls for comment and correction.

This statement concerned the torus knots present among the vortex lines of a family of exact steady axisymmetric solutions of the Euler equations known as Hicks vortices, after Hicks (1899) (a paper that I was unaware of in 1969), which I discussed primarily as examples of flows having non-zero helicity. The structure of these flows is reproduced in figure 1. With spherical polar coordinates (R, θ, ϕ), the velocity is $\boldsymbol{u} = \boldsymbol{u}_P + \boldsymbol{u}_T$, where $\boldsymbol{u}_P = (u, v, 0)$ (the poloidal part) and $\boldsymbol{u}_T = (0, 0, w)$ (the toroidal part). Similarly, the vorticity $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u} = \boldsymbol{\omega}_P + \boldsymbol{\omega}_T$. The stream surfaces $\psi = \text{const.}$, where ψ is the Stokes streamfunction, are sketched in figure 1(*a*); inside the sphere, these form a family of nested tori, on which lie the streamlines and vortex lines, as illustrated in figure 1(*b*).

In the final paragraph of Moffatt (1969), I wrote as follows: 'If any one vortex line is followed in the direction of increasing ϕ, the value of z $[= R\cos\theta]$ on that line varies periodically; the pitch p of the vortex line may conveniently be defined as twice the increase in ϕ between successive zeros of z.' I then wrote 'This quantity clearly increases continuously from zero to infinity as ψ increases from zero (on $R = a$) to ψ_{max} (on the vortex axis)'. Bogoyavlenskij has shown that this statement is at best only partially correct: the variation of pitch is indeed continuous, but the limits 'zero and infinity' are both wrong, and this for quite subtle reasons.

What is clear is that, if the pitch of the streamlines on the (non-degenerate) torus sketched in figure 1(*b*) tends to zero, then the associated flow becomes purely poloidal; and if the pitch tends to infinity, then the associated flow becomes purely toroidal. So I jumped to the (false) conclusion that the limit of the pitch must be zero on the spherical surface $R = a$, $\psi = 0$ (where the flow is poloidal) and infinite on the vortex axis (where the flow is toroidal); this conclusion was, as Bogoyavlenskij has now shown, incorrect for the following reasons.

First, although the streamlines and vortex lines lie on the same family of toroidal surfaces, they do not have the same topology (as I had implicitly assumed), because the streamlines are determined by the dynamical system $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{u}(\boldsymbol{x})$, whereas the vortex lines are determined by the different dynamical system $\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t = \boldsymbol{\omega}(\boldsymbol{\xi})$. As already known to Hicks (1899), the vorticity, unlike the velocity, is not purely poloidal on $R = a$.

We focus here only on the vortex lines, and on solution $\boldsymbol{\xi}(t)$ of the dynamical system $\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t = \boldsymbol{\omega}(\boldsymbol{\xi})$; thus $\boldsymbol{\xi}(t)$ describes the trajectory of a 'vorticity particle'.

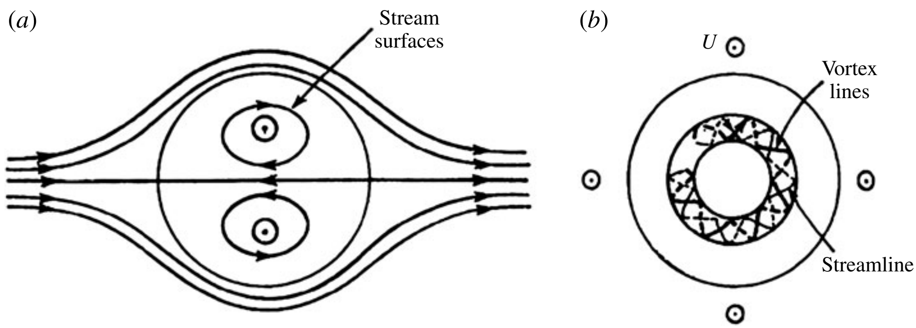

FIGURE 1. Sketched flow structure, reproduced from Moffatt (1969).

Considering first the flow structure at a small distance $O(\epsilon)$ from the vortex axis $\psi = \psi_m$, although the poloidal vorticity is indeed $O(\epsilon)$ in this neighbourhood, the time for a vorticity particle to make one complete turn round this axis remains $O(1)$ as $\epsilon \to 0$; hence the limit of the pitch of a vortex line as $\epsilon \to 0$ is not infinite, but some constant, p_m say, in general non-zero. (Indeed, the limit of any such torus knot of constant pitch as $\epsilon \to 0$ would be the circle $\psi = \psi_m$.) Bogoyavlenskij (2017*a*) has calculated this limit in a particular case to be $p_m = 5.1849$.

The situation near the spherical surface $R = a$, $\psi = 0$ is more delicate. Considering here the torus $\psi = \epsilon$, a vortex line on this torus passes near two 'vorticity stagnation points' (where $\boldsymbol{\omega} = 0$) on the axis of symmetry, and spends a time $t_\epsilon = O(\log(1/\epsilon))$ in the neighbourhood of each of these. With $r = R\sin\theta$, the toroidal vorticity $\omega_\phi = r\,\mathrm{d}\phi/\mathrm{d}t$ is $O(\epsilon)$ on this torus, but when a vorticity particle is near a vorticity stagnation point, $r = O(\epsilon)$ and so $\mathrm{d}\phi/\mathrm{d}t = O(1)$; hence the change in ϕ during the long time t_ϵ tends to $\pm\infty$ as $\epsilon \to 0$ (the sign depending on the sign of the helicity of the flow). So, remarkably, the limit of the pitch p as $\psi \to 0$ is not zero, as I asserted, but $\pm\infty$, depending on the handedness of the helices, the infinity arising entirely from the behaviour in the immediate neighbourhood of these zeros of $\boldsymbol{\omega}$ on the axis of symmetry of the flow.

Despite these important considerations concerning the limits on the pitch of the vortex lines, one implication of my 1969 conclusion is unaffected: namely that typically a countable infinity of topologically distinct torus knots $K_{m,n}$ (with m, n co-prime integers) are represented within the family of the vortex lines of each such flow.

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